Game Theory, Spring 2021
Final Exam - Solutions

You have two hours to complete this exam. Please answer the following three questions. Be sure to allocate your time in proportion to the points. Always justify your answers by providing a formal proof or a detailed argument. Good luck.

1. **[30 points]** A principal has the authority to make a decision. However, he can delegate the decision to an informed agent.

   There are two states of the world, 0 and 1. The two states are equally likely. If the principal makes the decision, then he chooses an action $a \in \mathbb{R}$ (without observing the state). If the principal delegates the final decision, then the agent chooses an action $a \in \mathbb{R}$ after observing the state.

   Suppose that the state is $\omega \in \{0, 1\}$ and the chosen action is a. Then the agent's payoff is $-(\omega - a)^2$ while the principal's payoff is $-(\omega + \beta - a)^2$ for some $\beta \in \mathbb{R}$.

   Find the values of the parameter β for which the principal delegates the decision to the agent.

Suppose that the principal decides to make the decision. The optimal action solves the following problem:

$$\max_{a \in \mathbb{R}} -\frac{1}{2}(\beta - a)^2 - \frac{1}{2}(1 + \beta - a)^2$$

It follows from the first order conditions that the optimal action is $a^* = \mathbb{E}(\omega) + \beta = \frac{1}{2} + \beta$. Thus, the principal's expected payoff is equal to $-\frac{1}{4}$.
Suppose now that the principal delegates the choice to the agent. Clearly, the agent will choose the action 0 when the state is 0 and the action 1 when the state is 1. Thus, the principal's expected payoff is equal to $-\beta^2$.
The principal prefers to delegation the decision to the agent if $-\beta^2 \geq -\frac{1}{4}$, i.e. if $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

2. **[30 points]** Let G be the following two-person normal-form game

	A	B
A	$4, 4$	$3, 6$
B	$6, 3$	$0, 0$

   Consider the repeated game $G(\infty, \delta)$ and the strategy profile (s_1, s_2) which be described as follows. There are two phases, 1 and 2. The initial history h^0 belongs to phase 1. In phase 1 the players play A. The play remains in phase 1 unless there is a unilateral deviation from (A, A). If there is a unilateral deviation, the play goes to phase 2. In phase 2 the players play (B, B). If there are no deviations or two deviations from

(B, B) the play returns to phase 1 (i.e., phase 2 lasts one period). If there is a unilateral deviation from (B, B) then phase 2 restarts.

Find the values of the discount factor δ for which (s_1, s_2) is a subgame perfect equilibrium of $G(\infty, \delta)$.

We apply the one-shot deviation principle. In phase 1 a one-shot deviation is not profitable if

$$4 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots \geq 6 + 0 + 4\delta^2 + 4\delta^3 + \ldots$$

i.e., if $\delta \geq \frac{1}{2}$;
In phase 2 a one-shot deviation is not profitable if

$$0 + 4\delta + 4\delta^2 + 4\delta^3 + \ldots \geq 3 + 0 + 4\delta^2 + 4\delta^3 + \ldots$$

i.e., if $\delta \geq \frac{3}{4}$.
We conclude that (s_1, s_2) is a subgame perfect equilibrium of $G(\infty, \delta)$ if and only if $\delta \geq \frac{3}{4}$.

3. [**40 points**] An agent has to decide whether to implement a change or to keep the status quo. The optimal decision depends on the state of the world which is either g or b. The probability of the state g is $\frac{3}{10}$. The agent's payoff is equal to 1 if he implements the change and the state is g and equal to zero if he implements the change and the state is b. The agent obtains a payoff of 1/2 in both states if he keeps the status quo.

The agent does not observe the state. However, he can rely on the report of an expert who observes the state. There are two possible reports: γ and β. The expert *commits* to a reporting strategy $(\mu_g, \mu_b) \in [0,1]^2$ with $\mu_g \geq \mu_b$. When the state is $\omega \in \{g, b\}$, the expert sends the report γ with probability μ_ω and the report β with probability $1 - \mu_\omega$. In both states the expert's payoff is equal to 1 if the agent implements the change and equal to 0 if the agent keeps the status quo (notice that the expert's payoff does not depend on the state).

The timing of the game is as follows. First, the expert chooses the reporting strategy (μ_g, μ_b). Then he observes the state and sends a report according to (μ_g, μ_b). The agent observes the reporting strategy (μ_g, μ_b) and the report (but not the actual state) and makes the final decision.

Construct a perfect Bayesian equilibrium.

(Hint: Can the expert induce the agent to implement the change after receiving both the report γ and the report β? Let $\Pr(g|\gamma)$ and $\Pr(g|\beta)$ denote the agent's belief that the state is g after receiving the report γ and β, respectively. It is easy to check that the expected value of $\Pr(g|\gamma)$ and $\Pr(g|\beta)$ is equal to the prior $\frac{3}{10}$.)

Let p denote the agent's belief that the state is g. The agent's expected payoff is equal to p if he changes the status quo and equal to $\frac{1}{2}$ if he keeps the status quo. Thus, the agent is willing to change the status quo if and only if $p \geq \frac{1}{2}$.

Fix $\left(\mu_g, \mu_b\right) \in [0,1]^2$. The agent's belief that the state is g after receiving the report γ is equal to

$$\Pr\left(g|\gamma\right) = \frac{\frac{3}{10}\mu_g}{\frac{3}{10}\mu_g + \frac{7}{10}\mu_b}$$

Similarly, the agent's belief that the state is g after receiving the report β is equal to

$$\Pr\left(g|\beta\right) = \frac{\frac{3}{10}\left(1-\mu_g\right)}{\frac{3}{10}\left(1-\mu_g\right) + \frac{7}{10}\left(1-\mu_b\right)}$$

Notice that the agent receives the report γ with probability $\frac{3}{10}\mu_g + \frac{7}{10}\mu_b$ and the report β with probability $\frac{3}{10}\left(1-\mu_g\right) + \frac{7}{10}\left(1-\mu_b\right)$. Therefore, the expected value of $\Pr\left(g|\gamma\right)$ and $\Pr\left(g|\beta\right)$ is equal to the prior $\frac{3}{10}$.
Clearly, the expert cannot make both $\Pr\left(g|\gamma\right)$ and $\Pr\left(g|\beta\right)$ larger than $\frac{1}{2}$ as their expectation is $\frac{3}{10}$. Recall that $\mu_g \geq \mu_b$. This implies $\Pr\left(g|\gamma\right) \geq \Pr\left(g|\beta\right)$. Thus, $\Pr\left(g|\beta\right) < \frac{1}{2}$ and the agent must keep the status quo after receiving the report β.
The expert can induce the agent to change the status quo only if $\Pr\left(g|\gamma\right) \geq \frac{1}{2}$ which implies $\mu_b \leq \frac{3}{7}\mu_g$. Clearly, the expert is interested in maximizing the probability that the agent receives the report g. Thus, he solves the following problem

$$\max_{\mu_g,\mu_b} \frac{3}{10}\mu_g + \frac{7}{10}\mu_b$$

$$\text{s.t. } 1 \geq \mu_g \geq \frac{7}{3}\mu_b \geq 0$$

The objective function is increasing both in μ_g and μ_b. Thus, the solution is $\mu_g = 1$, $\mu_b = \frac{3}{7}$. We conclude that in equilibrium the expert chooses the reporting strategy $\mu_g = 1$, $\mu_b = \frac{3}{7}$. The agent changes the status quo if and only if the belief that the state is g is weakly larger than $\frac{1}{2}$.