Copula-based default dependence modelling: where do we stand?*

Elisa Luciano†

University of Turin, Collegio Carlo Alberto

and International Center for Economic Research, Turin

Abstract

Copula functions have proven to be extremely useful in describing joint default and survival probabilities in credit risk applications. We overview the state of the art and point out some open modelling issues. We discuss first joint default modelling in diffusion based structural models, then in intensity based ones, focusing on the possibility - and the dynamic inconsistency - of re-mapping a model of the second type into one of the first. For both types of models, we discuss calibration issues under the risk neutral measure, using the factor copula device.

The survey leads us to focus on a non-diffusive structural model, which can be re-mapped in a dynamic consistent intensity-based one, and which can be calibrated under a risk neutral measure without assuming equicorrelation.

JEL classification number: G32, G12

Keywords: default dependence, copula functions, risk neutral versus historical dependence

* This paper has been prepared for publication in Wagner N. (ed.) (2008): Credit Risk – Models, Derivatives and Management, Financial Mathematics Series Vol. 6, Chapman & Hall/CRC, Boca Raton, London, New York

† Contact address: Prof. Elisa Luciano, ICER, Villa Gualino, V. S. Severo, 63 I-10133 Torino, Italy, tel.+39-011-6604828, fax + 39-011-6600082, luciano@econ.unito.it

The assessment of the joint default probability of groups of obligors, as well as related notions, such as the probability that the n-th one of them defaults, is a crucial problem in credit risk. In order to solve it, both the Industry and the Academia have extensively relied on copula methods. These allow to split any joint default probability into the marginal ones and a function, the copula itself, which represents only the association or dependence between defaults. Essentially, the splitting up makes both default modelling and calibration much easier, since it permits separate fitting at the univariate and joint level. At present, the use of copula techniques is a well established fact in risk modelling and credit derivative pricing. However, the choice of the copula and its calibration is still an open issue: model risk indeed exists at both stages, and its understanding relies on a deep discussion of the current practices and model limitations.

The first part of the paper reviews the theoretical background for different copula choices in default-no default models and discusses its static and dynamic consistency, when switching from structural to intensity based approaches. The review proceeds as follows: we first resume some basic facts about structural and intensity-based models, and how copulas are introduced and calibrated in the two settings. We then recall how and under which conditions two multivariate models of these classes can be statically re-mapped into each other, keeping dependence fixed. This remapping is indeed exploited in most of the industry applications of intensity-based models, starting from the well known Li's model (Li, 2000). However, it suffers from dynamic inconsistency, in that structural models with diffusive asset values do not have an intensity-based representation. In addition, re-mapping allows calibration under the historical measure only, since the historical equity returns are used as proxies for the asset ones. In order to calibrate under the/a risk neutral measure, factorization, i.e. the so-called factor copula, is introduced. We discuss factorization in joint default modelling. The survey will put into evidence the lack of dynamically consistent foundations for inferring the/a risk-neutral dependence of survival times for more than two obligors, without assuming equicorrelation.

The second part of the paper suggests a possible solution to this problem: structural models, in which the asset value is a pure jump process, such as the Merton's restatement of Madan (2000), or the corresponding early default model. These models allow for an intensity interpretation, and are therefore dynamically consistent. They fit data nicely at the univariate level (Fiorani and Luciano (2006))

- can be extended to the multivariate case (Luciano and Schoutens (2006), Fiorani, Luciano, Semeraro (2007)),
- can be calibrated under the risk neutral measure and without equicorrelation,
- and have proven to fit credit data very satisfactorily (ibidem)

1 Copula modelling in structural models

This section makes an overview of basic copula uses in structural and models. We will briefly recall how copulas are introduced, how historical calibration is usually done, for a given copula choice, and finally how the “best" copula can (or cannot) be chosen.

We consider a portfolio of n obligors, with random default times $\tau_1, \tau_2, ...\tau_n$. The indicator variable of default of obligor i, $i = 1, 2, .., n$, at time t, $t \in \mathbb{R}^+$, is denoted as $Y_i(t) := 1_{\{\tau_i \leq t\}}$. Let ($\Omega, \mathcal{H}$, P) be the probability space over which the times to default are defined. The space is endowed with the filtrations associated the single stochastic processes $Y_i(t)$, $\mathbb{G}_i := \{\mathcal{G}_{it}; t \geq 0\}$:

$$\mathcal{G}_{it} := \sigma \left(Y_i(u); 0 \leq u \leq t\right) \tag{1}$$

as well as with the joint filtration $\mathbb{G} := \vee_{i=1}^n \mathbb{G}_i$, with the usual properties of right continuity and completeness. Initially, all obligors are alive, so that $Y_i(0) = 0$ for all i. The default probability by time t, as determined at time 0, $P(Y_i(t) = 1 \mid \mathcal{G}_{i0})$, will be denoted as $F_i(t)$, while $S_i(t)$ will denote the corresponding survival probability, $S_i(t) := 1 - F_i(t)$. Analogously, we will denote the joint default probability at time t, evaluated at time 0, as $F(t)$

$$F(t) := P(Y_1(t) = 1, Y_2(t) = 1, ..., Y_n(t) = 1 \mid \mathcal{G}_0) \tag{2}$$

and the corresponding survival probability as $S(t)$

$$S(t) := P(Y_1(t) = 0, Y_2(t) = 0, ..., Y_n(t) = 0 \mid \mathcal{G}_0) \tag{3}$$

In describing the marginal and joint default or survival functions, we will omit, from now on, the filtration.

As for the copula tool, we will denote an n-dimensional copula, defined on $I^n := [0,1] \times ... \times [0,1]$, as $C(u_1, ..., u_n)$, while its survival counterpart will be $\breve{C}(u_1, ..., u_n)$. We take for granted the copula definition as a joint distribution function with uniform margins, which implies that C is always between the bounds $\max(u_1 + u_2 + ... + u_n - 1, 0)$ and $\min(u_1, u_2, ..., u_n)$. We will also take for granted the fundamental Sklar's theorem, which, under continuity of the margins, allows to represent any joint distribution function of n random variates (rvs), $F(t)$, in terms of a copula C and the marginal distribution functions, $F_i(t), i = 1, 2, .., n$:

$$F(t) = C(F_1(t), F_2(t), ..., F_n(t)) \tag{4}$$

Analogously, any joint survival probability $S(t)$ can be represented in terms of a so-called survival copula, $\breve{C}$, and the marginal survival functions, $S_i(t) := 1 - F_i(t)$,

$$S(t) = \breve{C}(S_1(t), S_2(t), ..., S_n(t)) \tag{5}$$

This is the reason why the copula is called also dependence function, in that it isolates dependence from the margin values: independence for instance is represented by the so-called product or factor copula $C^{\perp}$, which is simply the product of its arguments:

$$C^{\perp}(u_1, ..., u_n) = u_1 \times u_2 \times .. \times u_n. \tag{6}$$

Copulas can be parametrized by the linear correlation coefficient as well as by more general association measures: in the sequel, we will use for instance the so called Gaussian and Student's t copulas, which are characterized by linear correlation and - the second only - by the degrees of freedom parameter; they are linked by the fact that the former is the "limit" of the latter, when the number of degrees of freedom diverges. For further details, as well as for an overview of other common specifications, we refer the reader to either Nelsen (1999) or Cherubini, Luciano and Vecchiato (2004).

In the credit risk case, since the variables τ_i are default times, the copula represents their dependence, and we could denote it as C^{τ}, where τ is the vector formed by the rvs τ_i :

$$F(t) = C^{\tau}(F_1(t), F_2(t), ..., F_n(t)) \tag{7}$$

$$S(t) = \breve{C}^{\tau}(S_1(t), ...S_n(t)) \tag{8}$$

The specification of the copula depends on whether we rely on a structural approach or an intensity based one. In the seminal structural model of Merton (1974), to start with, it is well known that default of firm i can occur only at debt maturity. Let the latter be t for all the firms in the portfolio: default is triggered by the fact that the firm's asset value $V_i(t)$ falls to the liability one, $K_i(t)$. The distribution of the time to default, τ_i, is therefore

$$\tau_i = \begin{cases} t & P(V_i(t) \leq K_i(t)) \\ +\infty & P(V_i(t) > K_i(t)) \end{cases} \tag{9}$$

while the default probability at time t is

$$F_i(t) = P(V_i(t) \leq K_i(t)) \tag{10}$$

If the asset value follows a Geometric Brownian motion, as assumed by Merton, and the standard notation of the Black-Scholes framework is adopted, the marginal default probability can be easily computed to be

$$F_i(t) = \Phi(-d_{2i}(t)) \tag{11}$$

where Φ is the distribution function of a standard normal, while

$$d_{2i} := \frac{\ln\left(V_i(0)/K_i(t)\right) + (m - \sigma^2/2)t}{\sigma\sqrt{t}} \tag{12}$$

and m is the instantaneous return on assets, which equates the riskless rate r under the risk neutral measure. By assuming that log assets are not only normally distributed at the individual level, but also jointly normally distributed with correlation matrix R, it follows that the joint default probability of the n assets is

$$F(t) = P(V_1(t) \leq K_1(t), ...V_n(t) \leq K_n(t) \mid \mathcal{G}_0) = \Phi_R\left(-d_{21}(t), ..., -d_{2n}(t)\right) \tag{13}$$

where Φ_R is the distribution function of a standard normal vector with correlation matrix R. From the fact that the arguments of the distribution Φ_R can be written as the inverses, according to the univariate normal density, of the marginal default probabilities, i.e. $-d_{2i}(t) = \Phi^{-1}(F_i(t))$, it follows, by simple substitution, that

$$F(t) = \Phi_R\left(\Phi^{-1}(F_1(t)), ..., \Phi^{-1}(F_n(t))\right) \tag{14}$$

The latter is the copula representation of the joint default probability for the case at hand, since F is represented in terms of the marginal distributions. The copula which we have obtained is the so-called Gaussian copula:

$$C(u_1, ..., u_n) := \Phi_R\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)\right) \tag{15}$$

Instead of joint normality, we can keep the margins fixed and assume another type of dependence, which means another copula. For instance, we can assume a Student t copula, defined as

$$C(u_1, ..., u_n) := t_{R,\nu}(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2), ..., t_\nu^{-1}(u_n)) \tag{16}$$

where $t_{R,\nu}$ is the standardized multivariate Student's t distribution with correlation matrix R and ν degrees of freedom., while t_ν^{-1} is the inverse of the corresponding margin. By so doing, we essentially incorporate fat tails in the asset return joint distribution. The joint default probability immediately becomes

$$F(t) = t_{R,\nu}(t_\nu^{-1}(F_1(t)), t_\nu^{-1}(F_2(t)), ..., t_\nu^{-1}(F_n(t))) \tag{17}$$

In general, starting from a marginal probability of the type (10), and therefore from a marginal model of the Merton's type, we can write the joint default probability as

$$F(t) = C^V\left(F_1(t), ..., F_n(t)\right) \tag{18}$$

where C^V is the copula of the asset values. Let us recall however that the above expression applies only at debt maturity, and assuming that the latter is common to all firms in the basket.

Starting with Black and Cox (1976), the structural model of Merton has been extended by introducing the possibility of early distress. Default occurs as soon as the asset value $V_i(t)$ falls below a (deterministic) level $K_i(t)$, which can be, for instance, the present value of debt. As a consequence, the time to default is

$$\tau_i := \inf\left\{t : V_i(t) \leq K_i(t)\right\} \tag{19}$$

and the cumulated default probability is

$$F_i(t) = P(\min_{0 \leq u \leq t} V_i(u) \leq K_i(u)) \tag{20}$$

In this model, even if V_i follows a Geometric Brownian motion, its minimum does not, and the departure from the plain vanilla option pricing at the individual level is immediate. Other departures have been caused, during the development of the literature on structural models, by introducing more realistic assumptions, such as the stochastic interest rate one of Longstaff-Schwartz (1995). However, the use of a Gaussian or Student version at the multivariate level has been widespread, mostly for analytical convenience.

More recently, default has been supposed to be triggered by the fall of a “credit worthiness index" under an empirically calibrated barrier: see for instance Hull and White (2000). The credit index does not necessarily admit a transform which behaves as a Geometric Brownian motion. Because of the lack of marginal normality, it is not natural any more to assume joint normality and to collect the univariate probabilities in a Gaussian or an elliptic copula, such as the Student: the bivariate version was outlined in Hull and White (2000), while multivariate versions are manageable through a purely numerical fit of the copula, similar to the implied volatility derivation (see the "perfect copula" of Hull and White (2005)).

1.1 Calibration (under the historical measure) and "best" copula choice

The calibration of structural models is straightforward, especially if one refers to the seminal Merton one, as we will do. Under a Gaussian copula, as in the natural multivariate version of Merton's work, only the linear correlation coefficients between the asset values are needed. In case the Student copula is adopted, also the degrees of freedom parameter is required.

Since asset values are generally unobservable, the corresponding equity ones are generally used for calibration. This is the approach taken - for instance - by Mashal and Naldi (2003), together with the Student assumption: they suggest a joint maximum likelihood estimation of the degrees of freedom parameter and the correlation matrix. Obviously, the calibration is done under the historical measure: under such a measure, as Mashal, Naldi and Zeevi (2003) have shown, the dependence of assets is well proxied by the equity one. However, the coincidence between historical and risk neutral correlation requires either no premium for default risk or particular assumptions, such as the ones listed by Rosenberg (2000).In general, if we exit the strict Merton model, in which asset returns are jointly normally distributed, it is not guaranteed.

It also follows from the meaning of the copula in the structural approach that the best copula is the one which best describes equity dependence. The current literature tends to agree on the fact that - at least for extreme comovements - the Student dependence is more appropriate, since it encapsulates tail dependence: evidence is reported for instance in Mashal and Zeevi (2002), Mashal, Naldi and Zeevi (2003).

2 Copula modelling in intensity-based models

In the intensity-based models of Lando (1998), Duffie and Singleton (1997, 1999), default of the obligor i occurs as soon as a counting process, which is assumed to be of the Cox or doubly stochastic type, jumps for the first time. Omitting the technical details, for which the reader is referred to the probabilistic set up of Brémaud (1981), or to the credit applications just mentioned, occurrence of default is linked to the event that the so called compensator of the Cox process, denoted as Λ_i, be greater than or equal to a stochastic barrier. The latter is exponentially distributed, with parameter one:

$$\tau_i := \inf \{t : \Lambda_i(t) \geq \theta_i\} \tag{21}$$

with $\theta_i \sim \exp(1)$. Equivalently, using the change of variable $\theta_i := -\ln U_i$:

$$\tau_i := \inf \{t : \exp(-\Lambda_i(t)) \leq U_i\} \tag{22}$$

where $U_i \sim U(0,1)$. The compensator is usually assumed to have an intensity, which means to admit the representation

$$\Lambda_i(t) = \int_0^t \lambda_i(s) ds \tag{23}$$

where λ_i is a non negative process, satisfying $\int_0^t \lambda_i(s) ds < \infty$ almost surely for all t. Given a filtration $\mathbb{F} := \{\mathcal{F}_t; t \geq 0\}$ on Ω, the intensity $\lambda_i(s)$ is assumed to be predictable, while the threshold θ_i is independent of $\mathbb{F}$. We also need to define the filtration $\mathbb{H} := \mathbb{F} \vee \mathbb{G}$, or $\mathbb{H} := \{\mathcal{H}_t = \mathcal{F}_t \vee \mathcal{G}_t; t \geq 0\}$.

The default probability of the above firm at time zero turns out to be

$$F_i(t) = 1 - E\left[\exp\left(-\int_0^t \lambda_i(s) ds\right) \mid \mathcal{F}_0\right] \tag{24}$$

If, in particular, the intensities are deterministic, the Cox processes describing default arrival become inhomogeneous Poisson ones and

$$F_i(t) = 1 - \exp\left(-\int_0^t \lambda_i(s) ds\right) \tag{25}$$

Under the additional requirement that intensities are constant, $\lambda_i(s) = \lambda_i$ for every s, they become homogeneous Poisson processes, with $F_i(t) = 1 - \exp(-\lambda_i t)$.

When extending the Cox framework to multiple firms, i.e. taking $i = 1, ..n$, the previous framework naturally leads to conditionally independent defaults, or a multivariate Cox Process, as in Duffie (1998) and the literature thereafter: the intensity processes are liable to be correlated, while the thresholds θ_i, or their transform U_i, are assumed to be independent, and therefore have the product copula. It follows that the joint default probability of the n firms by time t, evaluated at time 0, is

$$S(t) = E\left[\exp\left(-\int_0^t (\lambda_1(s) + \lambda_2(s) + .. + \lambda_n(s))\, ds\right) \mid \mathcal{F}_0\right] \tag{26}$$

This extension usually produces low levels of both (linear) correlation between the default indicators of different firms, $\rho(Y_i(t), Y_j(t))$, for $i \neq j$ and all t, and (linear) correlation of survival times, $\rho(\tau_i, \tau_j)$, $i \neq j$. This happens, as is intuitive, as long as the intensities are diffusion-driven.

An alternative approach, due to Schönbucher and Schubert (2001), to whom we refer for the technical details, allows defaults not to be conditionally independent. The thresholds θ_i (or U_i) are linked by a copula which is not the product copula of Duffie (1998). With the usual notation, let C^U be the copula of the thresholds U_i, which is also the survival copula[1] of θ_i, $C^U = \breve{C}^\theta$.The construction of Schönbucher and Schubert (2001) gives as joint survival probability

$$S(t) =$$

$$E\left[C^U\left(\exp\left(-\int_0^t \lambda_1(s)ds\right), \exp\left(-\int_0^t \lambda_2(s)ds\right), .., \exp\left(-\int_0^t \lambda_n(s)ds\right)\right) \mid \mathcal{F}_0\right] \tag{27}$$

which reduces to (26), as due, in the Duffie (1998) framework.

Whenever the n firms are driven by inhomogeneous Poisson processes, since the intensities are deterministic, the survival probability in (27) becomes

$$S(t) = C^U\left(\exp\left(-\int_0^t \lambda_1(s)ds\right), \exp\left(-\int_0^t \lambda_2(s)ds\right), .., \exp\left(-\int_0^t \lambda_n(s)ds\right)\right) =$$

$$= C^U\left(S_1(t), ..., S_n(t)\right) \tag{28}$$

which further reduces to the product

$$S(t) = \exp\left(-\int_0^t \left(\lambda_1(s) + \lambda_2(s) + .. + \lambda_n(s)\right) ds\right) =$$

$$= S_1(t) \times ... \times S_n(t) \tag{29}$$

with independent thresholds.

2.1 Calibration (under the historical measure) and "best" copula choice

In the general case of stochastic intensities, by choosing independent thresholds one restricts C^U to be the product copula, as in expression (26). Apart from this case, calibration is a very delicate issue, since the thresholds U_i or θ_i are not observable: the literature has concentrated on copulas which produce a shift of

[1]The copula of the thresholds U_i and the one of their transforms θ_i are linked by the fact that each U_i is a decreasing function of the corresponding θ_i.

the remaining intensities after default of one obligor, such as the Archimedean copulas (see Schönbucher and Schubert (2001)). The calibration is usually done restricting to a one-parameter copula family, such a Gumbel one, a Clayton, or a Gaussian with equicorrelation. Once such a family has been chosen, the parameter value is perfectly fitted using for instance the diversity score produced by Moody's. The perfect fitting, which is illustrated for instance in Jouanin et alii (2001), does not permit to select a "best" copula: with respect to the structural approach then, in which there is scope for a copula selection procedure, taking into consideration the best fit of asset or equity returns, here there is no room for a best-fit selection.

As long as intensities are deterministic, calibration at a single point in time is much easier, since, as we will now explain, the copula C^U is nothing else than the survival copula of the “associated" asset values. This is the approach first taken by Li (2000), made rigorous in Frey and McNeil (2003), and extremely popular in practice, together with its “best" copula selection. It is based on the observation that structural models with a diffusion-driven asset or credit index process on the one side and reduced form models with deterministic intensities on the other are both latent variable models: for any fixed horizon t, there exist n rvs $Z_i(t)$ and n thresholds $D_i(t) \in \mathbb{R}$ such that default occurs if and only if the random variables are smaller than or equal to the corresponding thresholds:.

$$Y_i(t) = 1 \;\Leftrightarrow\; Z_i(t) \leq D_i(t) \tag{30}$$

Indeed, in the structural case, one has simply to choose either the asset values or their minimum and the liabilities ones as rvs and thresholds respectively:.

$$Z_i = V_i, D_i = K_i \tag{31}$$

In the intensity based case, one has to replace the asset/index values V_i and the threshold ones, K_i, with the stochastic thresholds and the compensators respectively:

$$Z_i = \theta_i, D_i = \Lambda_i \tag{32}$$

or with the uniform thresholds and the negative exponential of the compensators:

$$Z_i = -U_i = -\exp(-\theta_i), D_i = -\exp\left(-\Lambda_i\right) \tag{33}$$

Having made this simple remark, we can exploit the equivalence of latent variable models in order to re-map intensity-based models into structural ones, for a fixed horizon. Indeed, two latent variable models,

$$(V_i(t), K_i(t))_{1\leq i\leq n}\,, (\theta_i(t), \Lambda_i(t))_{1\leq i\leq n} \tag{34}$$

are equivalent if the corresponding default indicator vectors are equal in distribution: this allows us to assess that, at any fixed time t, the copula for the thresholds θ_i, $C^{\theta}-$ which, as said above, is also the survival copula of the uniform thresholds, $\breve{C}^U$ – is the same as the asset value copula, C^V. The estimate of the last one obtained from equity returns, as justified in structural models, can therefore be directly used to construct joint default probabilities, starting from marginal intensity-based ones. Formally, we can then write the joint default probability as

$$F(t) = C^V(F_1(t), F_2(t), ..., F_n(t)) =$$

$$C^V\left(1-\exp\left(-\int_0^t \lambda_1(s)ds\right), 1-\exp\left(-\int_0^t \lambda_2(s)ds\right), ..., 1-\exp\left(-\int_0^t \lambda_n(s)ds\right)\right) \tag{35}$$

However, this remapping has two problems. First, it is not dynamically consistent, since in structural models with a diffusive asset value, such as the ones described above and used in common applications and commercial softwares, the times to default are not totally unpredictable stopping times and, as a consequence, there does not exist an intensity which can represent their arrival. Second, it leads to an historical calibration of default dependency, via equity returns, which again has no reason to coincide with the risk neutral one.

3 Factor models

While remapping, as described in the previous section, permits to reconnect the copula choice to the asset one, while paying the price of dynamic inconsistency and historical calibration, another popular approach to default modelling allows us to switch to the so called product copula. The reduction technique, which is widely adopted for the evaluation of losses in high-dimensional portfolios, with hundreds of obligors (see for instance Laurent and Gregory (2003)), is the standard approach of (linear) factorization, or transformation into a Bernoulli factor model.

Given a vector of random factors,

$$(X_1(t), X_2(t), ..., X_p(t)) \tag{36}$$

the first step in reducing it consists in assuming that there exist n (probability) functions $p_i^t : \mathbb{R}^p \to [0,1]$, $i = 1, .., n$, such that, conditional on a specific realization of the factors $X_1 = x_1, X_2 = x_2, ..., X_p = x_p$, the default indicators $Y_i(t)$ of the n obligor are independent Bernoulli variables, with probability :

$$Y_i(t) = \begin{cases} 1 & p_i^t(x_1, .., x_p) \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

In the sequel we will work with one or two factors. In the former case, if the (unique) factor X has a density $f(x)$ on the real line $\mathbb{R}$, it follows from the definition of conditional probability that the marginal (unconditional) default probabilities can be written as

$$F_i(t) = \int_{\mathbb{R}} p_i^t(x) f(x) dx \tag{38}$$

while the joint one can be represented through a (conditional) factor copula $C^{\perp}$, as desired:

$$F(t) = \int_{\mathbb{R}} \prod_{i=1}^{n} p_i^t(x) f(x) dx = \int_{\mathbb{R}} C^{\perp} \left(p_1^t(x), .., p_n^t(x) \right) f(x) dx \tag{39}$$

Let us discuss the factorization separately for structural, intensity-based with non stochastic intensity, and stochastic intensity with independent and dependent thresholds ones: without much loss of generality, we will use one-factor models, in which the factor is standard Gaussian.

Structural models: the key assumption is that the $i-th$ asset value or credit worthiness indicator, properly normalized, can be factorized as

$$V_i = \rho_i X + \sqrt{1-\rho_i^2} \varepsilon_i \tag{40}$$

where $\rho_i \in \mathbb{R}$, X and $\varepsilon_1, \varepsilon_2, ..\varepsilon_n$ are independent standard Gaussian, so that V_i is standard normal too. The common factor is evidently X, while the ε_i are to be interpreted as the idiosyncratic ones: it follows from (40) not only that the V_is are independent, conditionally on X, but also that the unconditional linear correlation coefficient between V_i and V_j is $\rho_i \rho_j$. The conditional marginal default probabilities, $p_i^t(x)$, if K_i is the properly normalized liability or barrier level, are easily calculated:

$$p_i^t(x) = P\left(V_i \leq K_i \mid x\right) = \Phi\left(\frac{K_i - \rho_i x}{\sqrt{1-\rho_i^2}}\right) \tag{41}$$

The expressions for the unconditional ones, (38), and the joint unconditional ones, (39), easily follow.

Intensity-based models with non stochastic intensity: In order to adopt a factor model, starting from a uniform threshold, i.e. from the representation (25) of the unconditional marginal default probability, one must factorize a transform of it, which is standard Gaussian by definition: $W_i := \Phi^{-1}(1-U_i)$. By imposing

$$W_i = \rho_i X + \sqrt{1-\rho_i^2}\varepsilon_i \tag{42}$$

one gets as conditional marginal default probability, with a little bit of algebra

$$p_i^t(x) = P(\theta_i \le \Lambda_i \mid x) = \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Lambda_i)) - \rho_i x}{\sqrt{1-\rho_i^2}}\right) \tag{43}$$

Again, the unconditional marginal probabilities, (38), and the joint unconditional ones, (39), easily follow[2].

Stochastic intensity models with independent thresholds: in a multivariate Cox framework, à la Duffie (1998), the variable which can be factorized is a transform of the compensator, with no factorization of the thresholds, which are already independent. Indeed, if we denote as G_i the distribution function of the compensator at time t, so that $1 - G_i(\Lambda_i)$ is uniform, we can define

$$W_i := \Phi^{-1}(1 - G_i(\Lambda_i)) \tag{44}$$

and the latter can easily be proven to be standard normal. Let us impose on W_i the factor structure (42), as above. Noting that $\Lambda_i = G_i^{-1}(1 - \Phi(W_i))$, the conditional marginal default probability

$$p_i^t(x) = P(\theta_i \le \Lambda_i \mid x) = 1 - E[\exp(-\Lambda_i) \mid \mathcal{F}_0, x] \tag{45}$$

can be written as

$$p_i^t(x) = 1 - E\left[\exp\left(-G_i^{-1}\left(1 - \Phi(\rho_i x + \sqrt{1-\rho_i^2}\varepsilon_i)\right)\right) \mid \mathcal{F}_0, x\right] \tag{46}$$

The expression for the unconditional marginal probability, (38), is straightforward.

As for the joint unconditional default probabilities, they can still be written as (39), as follows. From the fact that for the case at hand the representation (26) of the joint survival probability holds, it follows that the joint default is

$$F(t) = E[(1-\exp(-\Lambda_1))(1-\exp(-\Lambda_2)) \mathinner{..} (1-\exp(-\Lambda_n)) \mid \mathcal{F}_0] \tag{47}$$

[2]More interestingly, the factorized models "confirm" the correspondence between structural and intensity based discussed in section 2.1, since the two models give the same conditional probabilites (and therefore unconditional ones) whenever

$$K_i = \Phi^{-1}(1-\exp(-\Lambda_i))$$

which is the condition which characterizes the equivalence between latent variable models of the structural and intensity type.

Conditioning it on $X = x$, we get

$$F(t) = \int_{\mathbb{R}} E\left[(1 - \exp(-\Lambda_1))(1 - \exp(-\Lambda_2)) .. (1 - \exp(-\Lambda_n)) \mid \mathcal{F}_0, x\right] f(x) dx \tag{48}$$

Finally, recalling that the compensators are - by assumption - independent for given x, we have

$$F(t) = \int_{\mathbb{R}} \prod_{i=1}^{n} (1 - E\left[\exp(-\Lambda_i) \mid \mathcal{F}_0, x\right]) f(x) dx \tag{49}$$

which is nothing else than the Bernoulli representation of the default processes, (39) above.

Stochastic intensity models with dependent thresholds: in this case not only we need to perform the factorization for the multivariate Cox case at the univariate level, but we also need to impose a factor structure on the thresholds, in order to guarantee the conditional independence at the multivariate level.

At the univariate step, nothing changes with respect to the independence case just discussed, and the marginal conditional default probabilities (46) still hold.

At the multivariate level, Rogge and Schönbucher (2003) propose to assume that the copula of the uniform thresholds is an Archimedean one:

$$C^U(u_1, u_2, ..., u_n) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + ... + \varphi(u_n)) = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi(u_i)\right) \tag{50}$$

where $\varphi(u) : [0,1] \to [0, \infty]$ is a continuous, strictly decreasing and completely monotonic function, and its generalized inverse can be chosen to be the Laplace transform of a positive rv, Z.

We show in the Appendix that, under this choice, a sufficient condition for obtaining a factor model is to assume that the variable Z, is independent of the factor driving the intensities. As proven in Rogge and Schönbucher (2003), the previous copula choice is indeed equivalent to assuming a non linear factor structure for the thresholds:

$$U_i = \varphi^{-1}\left(-\frac{\ln \Phi(\zeta_i)}{Z}\right) \tag{51}$$

in which the ζ_i are independent standard normal, which are also independent of the $\varepsilon_{i,}$, X and Z.

Summing up, both in structural and intensity based models, factorization permits to substitute the original copulas with the product one, according to (39), something which makes computations and analytical pricing results easier to obtain.

In particular, assuming, in addition to a unique risk factor, also equicorrelation of the single obligors with it, i.e. $\rho_i = \rho$ for all obligors i, calibration of default dependence boils down to finding an appropriate value for the parameter ρ: this is practically done by calibrating to the observed fees of liquid CDO tranches. However, practical applications of the methodology lead to the so-called correlation skew phenomenon, which is inconsistent with the theoretical set up: for a survey and discussion see Andersen and Sidenius (2005). In turn, this has lead to a number of calibration-oriented extensions, mainly related to stochastic correlation. Their full insertion in the intensity based framework is certainly worth high consideration, while not being completed yet: however, it is far beyond the scope of this paper.

4 A structural model with remapping and risk neutral calibration diffusion - based

Given that the credit applications of copula functions are still work in progress, the models reviewed in the previous sections present a number of limitations, as concerns either dynamic consistency or dependency calibration.

Intensity based models with deterministic intensity in particular do not have a dynamic consistent structural representation, and an asset copula consistent over time, since intensities exist if and only if the time to default is not a totally predictable stopping time. This condition, as mentioned above, is violated by diffusion based structural models.

When we enlarge our consideration to stochastic intensities, we encounter two problems. The limited range of dependency which can be reached via intensity-based models with stochastic intensity and independent thresholds renders their usefulness quite limited. Using stochastic intensities and allowing for dependent thresholds one runs into the problem of unobservability of the thresholds themselves. These two difficulties justify the success of factor models, which span a wide range of dependency, while allowing easy risk neutral calibration, at the price of equicorrelation and correlation skews. Or the use of a purely empirical approach to copula calibration, similar to the implied volatility one, as in Hull and White (2005).

Recently it has been stressed that the lack of total predictability - and therefore, of intensity representation - of structural models can be overcome by assuming that the information flow on asset values is not continuous over time: at the opposite, since accounting data are revealed only at discrete tenors, the wedge between the filtration of investors and the actual one renders default times totally inaccessible. The seminal paper in the field, which is for the time being developed only at the single obligor level, is Duffie and Lando (2001).

As an alternative, the total predictability can be eliminated by assuming that asset values are not described by a diffusion process. In order to introduce this approach we refer to a Merton-like default model, as originally introduced by Madan (2000), which can be extended to the case of several obligors. Let us consider a default mechanism as the one described in section 1, and assume that asset returns, as described by the logarithm of the asset value, $ln(V_i/V_0)$, follow a pure jump process instead of a diffusion one. The simplest way to introduce the pure jump process in this case is as a transform of a diffusion, such as a Brownian motion with drift, via a stochastic transform of time:

$$\ln\left(\frac{V_i(t)}{V_i(0)}\right) = mG(t) + \sigma W(G(t)) \tag{52}$$

where m is the (constant) drift of returns under the historical measure, σ is their instantaneous (constant) variance, W is a standard Brownian motion, while $G(t)$ is the value at time t a so called subordinator, namely of a real increasing Lévy process, with no diffusion component and finite variation. The effect of such a time transform on time, via G, is that of making log returns belong to the realm of Lévy processes: in particular, if G is a Gamma process, the corresponding returns are Variance Gamma, while if the subordinator is Generalized Inverse Gaussian we get a Generalized Hyperbolic Model. The amount of time change can be easily interpreted as a "stochastic clock", which in turn can be related to the amount of trade in a given market: see for instance Geman (2005). The usual restriction is that the random time change is equal to the effective time in expectation:

$$E(G(t)) = t \tag{53}$$

Let us denote with $p_i^t(x)$ the marginal default probability of obligor i, conditional on the time change being equal to x: given that we have a Merton default mechanism, if $H_i^t(u \mid x)$ is the asset value distribution of obligor i at time t, conditional on the change x, we have

$$p_i^t(x) = H_i^t(K_i \mid x) \tag{54}$$

The remarkable property of describing individual assets as above is that, conditional on the time change, these values are independent and therefore their (conditional) copula is the product one. If we denote with f_t is the density of the time change at t (with support $\mathbb{R}^+$), we have a representation analogous to the one in section 3:

$$F_i(t) = \int_{\mathbb{R}^+} p_i^t(x) f_t(x)dx = \int_{\mathbb{R}^+} H_i^t(K_i \mid x) f_t(x)dx \tag{55}$$

while the joint probability can still be represented through a (conditional) factor copula $C^{\perp}$:

$$F(t) = \int_{\mathbb{R}^+} \prod_{i=1}^{n} H_i^t(K_i \mid x) f_t(x) dx = \int_{\mathbb{R}^+} C^{\perp} \left(H_1^t(K_1 \mid x), .., H_n^t(K_n \mid x) \right) f_t(x) dx \tag{56}$$

Luciano and Schoutens (2006) study the case in which the time change is Gamma distributed, so that asset returns are Variance Gamma: this process indeed has been repeatedly used in the equity literature. Under this choice, taking into consideration the restriction (53), and focusing on time one, we have

$$f_1(x) = \frac{\nu^{1/\nu}}{\Gamma(1/\nu)} x^{1/\nu - 1} \exp(-x/\nu) \tag{57}$$

where Γ is the usual gamma function. At the same time, since - conditionally on the time change - returns are normally distributed, we can easily derive the conditional probabilities $p_i^t(x)$. Focusing again on their value at time one, $p_i^1(x)$

$$p_i^1(x) = H_i^1(K_i \mid x) = \Phi \left(\frac{z_i - m_i x - \theta_i}{\sigma_i \sqrt{x}} \right) \tag{58}$$

where $\theta_i = r + w_i$ and the shift w_i is used so as to obtain a risk neutral measure:

$$w_i = \frac{1}{\nu} \log \left(1 - \sigma_i^2 \nu / 2 - m_i \nu \right) \tag{59}$$

In this framework, the correlation coefficient between the returns of obligors i and j is

$$\rho_{ij} = \frac{m_i m_j \nu}{\sqrt{\sigma_i^2 + m_i^2 \nu} \sqrt{\sigma_j^2 + m_j^2 \nu}} \tag{60}$$

This shows that there is no equicorrelation.

>From conditional normality it follows that the marginal and joint default probabilities at time 1 are respectively

$$F_i(1) = \int_{\mathbb{R}_+} \Phi \left(\frac{z_i - m_i x - \theta_i}{\sigma_i \sqrt{x}} \right) \frac{\nu^{1/\nu} x^{1/\nu - 1} \exp(-x/\nu)}{\Gamma(1/\nu)} dx \tag{61}$$

$$F(1) = \int_{\mathbb{R}_+} \prod_{i=1}^{n} \Phi \left(\frac{z_i - m_i x - \theta_i}{\sigma_i \sqrt{x}} \right) \frac{\nu^{1/\nu} x^{1/\nu - 1} \exp(-x/\nu)}{\Gamma(1/\nu)} dx \tag{62}$$

where the threshold $z_i = \ln(K_i / V_i(0))$ is obtained from the inequality $V_i(1) < K_i$ (see Luciano and Schoutens 2006)).

The copula representation of the last probability can be obtained only by numerical inversion; however, the model can be easily calibrated, under the risk neutral measure, so as to be able to evaluate actual default probabilities. Luciano and Schoutens extract all the model parameters (single and joint) from CDS quotes. For each obligor they get the standard variance gamma parameters, namely m_i, σ_i; the third parameter (ν) is common to all of the obligors. With respect to the standard calibration of univariate Variance Gamma processes therefore they impose a constraint on the last parameter: the fit they obtain at the marginal level is quite good. They have a risk neutral joint default probability, since all the parameters are extracted from market prices, with no need for equity (historical) correlation.

The model just described therefore overcomes two main problems evidenced in the previous approaches: lack of structural models which can be used to map intensity-based ones and lack of risk neutral calibration without equicorrelation.

Fiorani, Luciano and Semeraro (2007) provide an even more general model of joint default with VG margins and without equicorrelation. The model is based on an extended multivariate VG process in which each asset is subject to a common and an idiosyncratic time change. As a result, the constraint on the common gamma parameter (ν) can be eliminated. The calibration of this model to credit market data is performed in Fiorani, Luciano, Semeraro (2007), via equity correlation and without equicorrelation. The Authors also justify the fact that the historical correlation coincides with the risk neutral: the reason for this is that the change of measure adopted, quite standard in the field, does not modify the correlation.

In Fiorani and Luciano (2006) it was shown that the univariate VG model was able to overcome the spread underprediction typical of the Merton approach and the prediction biases of more sophisticated diffusive structural models, since it works well both on high yield and investment grade names. Since the multivariate models in Fiorani, Luciano, Semeraro (2007) builds on the univariate calibration as in Luciano and Fiorani (2006), the marginal fit is guaranteed.

The explicit representation of the previous models as intensity ones is under investigation: for the time being, the existence result is guaranteed, and the representation is not needed in order to calibrate the model neither at the individual, nor at the joint level.

5 Summary and concluding remarks

Default dependence, most of the times represented by default (linear) correlation, is an important feature of credit derivatives pricing and hedging. Copula functions have proven to be extremely useful in describing joint default and survival dependency - and probabilities - in credit risk applications.

In the first part of the paper we have overviewed the state of the art and pointed out some open modelling issues. We have discussed joint default modelling first in diffusion-based structural models, then in intensity-based ones. We have focused on the possibility - and the dynamic inconsistency - of re-mapping a model of the second type into one of the first. For both types of models, we have discussed calibration issues under the risk neutral measure, using the factor copula device.

The survey has led us to present a structural model, based on a non diffusive asset value, which can be re-mapped in a dynamic consistent intensity-based one, and which can be calibrated under the risk neutral measure without assuming equicorrelation. Its copula properties and calibration have been investigated in Luciano and Schoutens (2006). Fiorani, Luciano and Semeraro (2007) extend further both the multivariate credit model and the calibration. They fit default correlation via equity correlation. It turns out that the model fits actual data quite satisfactorily, when applied to the CDS spreads of both investment grade and high yield names. In particular, it overcomes the underprediction of the diffusion based Merton case and the prediction biases of more sophisticated diffusion structural models.

Evidently, it is just one possible rigorous solution to the currently immature state of the research in the field. Models who are dynamically consistent in the sense adopted here and liable of calibration under the/a risk neutral measure, for pricing and hedging purposes, are very welcome.

Appendix

In this Appendix we briefly review the factor model with stochastic intensities and non independent thresholds. Having assumed that the copula of the uniform thresholds is Archimedean, as in (50), $\varphi^{-1}(u)$ can be chosen to be the Laplace transform of a positive rv, Z, i.e. $\varphi^{-1}(u) = E\left[\exp(-uZ)\right]$, and the copula (assuming a density exists for Z) can be written as

$$C^U(u_1, u_2, ..., u_n) = E \exp\left(-Z\sum_{i=1}^{n}\varphi(u_i)\right) = \int_{\mathbb{R}+} \exp\left(-z\sum_{i=1}^{n}\varphi(u_i)\right) h(z)dz \tag{63}$$

With the Archimedean assumption, the joint survival probability can then be manipulated as

$$S(t) = E\left[C^U\left(\exp\left(-\Lambda_1\right), \exp\left(-\Lambda_2\right), .., \exp\left(-\Lambda_n\right)\right) \mid \mathcal{F}_0\right] = \tag{64}$$

$$= E\left[\varphi^{-1}\left(\sum_{i=1}^{n}\varphi(\exp\left(-\Lambda_i\right))\right)\right] =$$

$$= E\left[\int_{\mathbb{R}+} \exp\left(-z\sum_{i=1}^{n}\varphi(\exp\left(-\Lambda_i\right))\right) h(z)dz\right] \tag{65}$$

Let us change the order of integration and condition the inner expectation with respect to $X = x$, which is the factor used in the marginal factorization (the one for $W := \Phi^{-1}(1 - G_i(\Lambda_i))$), and is assumed to be independent of Z :

$$S(t) = \int_{\mathbb{R}+}\int_{\mathbb{R}} E\left[\exp\left(-z\sum_{i=1}^{n}\varphi(\exp\left(-\Lambda_i\right))\right) \mid x\right] f(x)dxh(z)dz \tag{66}$$

Recognizing that the compensators are independent, conditionally on $X = x$, we have

$$S(t) = \int_{\mathbb{R}+}\int_{\mathbb{R}} \prod_{i=1}^{n} E\left[\exp\left(-z\varphi(\exp\left(-\Lambda_i\right))\right) \mid x\right] f(x)dxh(z)dz \tag{67}$$

which is of the form corresponding to (39), since

$$E\left[\exp\left(-z\varphi(\exp\left(-\Lambda_i\right))\right) \mid x\right] =$$

$$E\left[\exp\left(-z\varphi\left(\exp\left(-G_i^{-1}\left(1 - \Phi(\rho_i x + \sqrt{1-\rho_i^2}\varepsilon_i)\right)\right)\right)\right)\right] \tag{68}$$

is the marginal survival probability conditional on x and z, $1 - p_i^t(x, z)$.

References

[1] Andersen, L. , Sidenius, J. (2005): CDO pricing with factor models: survey and comments, *Journal of Credit Risk* 1: 71-88.

[2] Black F., Cox J.C. (1976): Valuing Corporate Securities: Some Effects of Bond Indenture Provisions, *Journal of Finance* 31: 351-367.

[3] Brémaud P. (1981): Point Processes and Queues: Martingale Dynamics, New York: Springer.

[4] Cherubini U., Luciano E., Vecchiato W. (2004): Copulas methods in finance, Chichester, UK. J. Wiley.

[5] Duffie D. (1998): First to Default Valuation, working paper, Institut de Finance, University of Paris, Dauphine, and Graduate School of Business, Stanford University.

[6] Duffie D., Lando D. (2001): Term Structure of Credit Spreads with Incomplete Accounting Information, *Econometrica* 69 633:664.

[7] Duffie D., Singleton K. (1997): An econometric model of the term structure of interest rate swap yields, *The Journal of Finance* 52 1287:1322.

[8] Duffie D., Singleton K. (1999): Modelling term structure of defaultable bonds, *The Review of Financial Studies* 12 687:720.

[9] Fiorani, F., Luciano, E. (2006): Credit risk in pure jump structural models: an empirical analysis, Working paper 6/06, ICER.

[10] Fiorani, F., Luciano, E., Semeraro P. (2007): Single and joint default in a structural model with purely discontinuous assets, Working paper, Collegio Carlo Alberto

[11] Frey R., Mc Neil A. J. (2003): Dependent defaults in models of portfolio risk, *Journal of Risk* 6 59:62.

[12] Geman, H. (2005): From measure changes to time changes, *Journal of Banking and Finance* 29 2701:22

[13] Hull J.,White A. (2000): Valuing credit default swaps I: no counterparty default risk, *Journal of Derivatives* 8 (1) 29:40

[14] Hull J.,White A. (2001): Valuing credit default swaps II: modelling default correlation, *Journal of Derivatives* 8 (3) 12:22

[15] Hull J.,White A. (2000): The perfect copula, Working Paper, University of Toronto

[16] Jouanin J.F., Riboulet G., Roncalli T. (2001): Modelling dependence for credit derivatives with copulas, Working paper, GRO, Credit Lyonnais, Paris.

[17] Lando D. (1998): On Cox processes and credit risky securities, *Review of Derivatives Research* 2 99:120

[18] Laurent J. P., Gregory J. (2003): Basket Default Swaps, CDOs and Factor Copulas, Working paper, BNP Paribas and University of Lyon

[19] Li D. X. (2000): On default correlation: a copula function approach, *Journal of Fixed Income* 9 43:54.

[20] Longstaff F. A., Schwartz E. S. (1995): A simple approach to valuing risky fixed and floating rate debt, *Journal of Finance* 50 789:819.

[21] Luciano, E., Schoutens, W. (2006): A multivariate jump-driven financial asset model, *Quantitative Finance*, 6 (5), 385:402.

[22] Madan D.B. (2000): Pricing the Risk of default, Working Paper, University of Maryland

[23] Mashal R., Naldi M. (2003): Extreme events and Default Baskets, in: Gordy, M. (eds): Credit Risk Modelling, London: Risk Books, pp. 243-50.

[24] Mashal, R., Naldi M. and Zeevi, A. (2003):Comparing the dependence structure of equity and asset returns, Lehman Brothers Quantitative Credit Research.

[25] Mashal R., Zeevi A. (2002): Beyond correlation: Extreme co-movements between financial assets, Working paper, Columbia Business School.

[26] Merton R. C. (1974): On the pricing of corporate debt: The risk structure of interest rates, *Journal of Finance* 29 449:470.

[27] Nelsen R.B. (1999): An introduction to copulas, Lectures Notes in Statistics, New York: Springer-Verlag.

[28] Rogge E., Schönbucher P. (2003): Modelling Dynamic Portfolio Credit Risk, Working paper, Dept. of Mathematics ETH Zurich.

[29] Rosenberg J. V. (2000): Nonparametric pricing of multivariate contingent claims, Working paper, Stern School of Business.

[30] Schönbucher P., Schubert D. (2001): Copula-dependent default risk in intensity models, Working paper, University of Bonn, Germany.