

## Solutions to Problem Set 2

### Question 1

Let  $p$  denote the price of the coconuts and  $w$  denote the price of Robinson's time. Normalise  $p = 1$ .

Given the production function, the producer will choose  $l$  to maximise  $p\sqrt{l} - wl$ .

Hence, the producer's labour demand

$$l^p = \left(\frac{1}{2w}\right)^2$$

and coconut supply

$$y^p = \frac{1}{2w}$$

The producer's profit is

$$\frac{1}{2w} - \left(\frac{1}{2w}\right)^2 w = \frac{1}{4w}$$

The consumer's problem is

$$\max x^{\frac{1}{3}}h^{\frac{2}{3}} \quad \text{s.t.} \quad px + wh = 24w + \frac{1}{4w}$$

Hence, the consumer's coconut demand

$$x^c = \frac{24w + \frac{1}{4w}}{3}$$

and leisure demand

$$h^c = \frac{48w + \frac{1}{2w}}{3w}$$

Hence, the consumer's labour supply is

$$24 - h^c = 24 - \frac{48w + \frac{1}{2w}}{3w}$$

For the coconut market to clear,

$$y^p = x^c \implies \frac{1}{2w} = \frac{24w + \frac{1}{4w}}{3} \implies w = \frac{\sqrt{30}}{24}$$

Hence, the Walrasian equilibrium prices are

$$w = \frac{\sqrt{30}}{24} \quad \text{and} \quad p = 1$$

In the coconut market,  $y^p = x^c = \frac{2}{5}\sqrt{30} = 2.1909$

Also, labour supply = labour demand =  $24 - \frac{96}{5} = 4.8$ .

The consumer's leisure consumption =  $\frac{96}{5} = 19.2$

### Question 2

Denote  $p_1, p_2, w$  as the prices for good 1, good 2 and labour, respectively. Normalise  $w = 1$ .

Firm 1's problem is

$$\max_{\{l_1\}} p_1 \sqrt{l_1} - w l_1$$

F.O.C.

$$\frac{p_1}{2\sqrt{l_1}} - w = 0$$

Hence, firm 1's labour demand is  $\frac{(p_1)^2}{4}$  and the supply of  $y_1$  is  $\frac{p_1}{2}$ .

Firm 1's profit

$$\pi_1 = \frac{p_1}{2} \cdot p_1 - \frac{(p_1)^2}{4} \times 1 = \frac{(p_1)^2}{4}$$

Similarly, firm 2's problem is to choose  $l_2$  to maximise  $(p_2 - w)l_2$ .

When  $p_2 > 1$ , firm 2's demand for labour is  $\infty$  and this is impossible in an equilibrium.

When  $p_2 < 1$ , supply of  $y_2 = 0$ .

When  $p_2 = 1$ , supply of  $y_2$  can be anything within the resource constraint.

Hence, firm 2's profit  $\pi_2 = 0$ .

Consumer 1's problem is

$$\max x_1 x_2 \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 = 6 + \frac{(p_1)^2}{8}$$

Hence, consumer 1's demands are

$$x_1^1 = \frac{6 + \frac{(p_1)^2}{8}}{2p_1}$$
$$x_2^1 = \frac{6 + \frac{(p_1)^2}{8}}{2p_2}$$

Similarly, we find that consumer 2's demands are

$$x_1^2 = \frac{6 + \frac{(p_1)^2}{8}}{2p_1}$$
$$x_2^2 = \frac{6 + \frac{(p_1)^2}{8}}{2p_2}$$

Since the demand for good 2 is positive, it has to be the case that  $p_2 = 1$  for the market of good 2 to clear.

For the market of good 1 to clear,

$$x_1^1 + x_1^2 = \frac{p_1}{2} \implies \frac{6 + \frac{(p_1)^2}{8}}{p_1} = \frac{p_1}{2} \implies p_1 = 4$$

In summary, the Walrasian equilibrium prices are  $p_1 = 4, p_2 = 1, w = 1$ .

Firm 1's labour demand  $l_1 = 4$ , supply of good 1  $y_1 = 2$ .

Firm 2's labour demand  $l_2 = 8$ , supply of good 2  $y_2 = 8$ .

Each consumer supplies 6 units of labour and each consumes 1 unit of good 1 and 4 units of good 2.

### Question 3

(a) To find the PE allocations mathematically, we maximise the utility of one consumer holding that of the other fixed, while respecting the overall endowment constraints. The maximisation problem is therefore:

$$\max_{x_1^A, x_2^A, x_1^B, x_2^B} u^B(x_1^B, x_2^B, x_2^A)$$

subject to

$$u^A(x_1^A, x_2^A) \geq \bar{u}$$

$$x_1^A + x_1^B = e_1^A + e_1^B = 15$$

$$x_2^A + x_2^B = e_2^A + e_2^B = 20$$

We can formulate the Lagrangian:

$$L = x_1^B + 2 \ln x_2^B - \ln(20 - x_2^B) + \lambda[-\bar{u} + (15 - x_1^B) + 2 \ln(20 - x_2^B)]$$

The first order conditions are:

$$1 = \lambda$$

$$\frac{2}{x_2^B} + \frac{1}{20 - x_2^B} - \frac{2\lambda}{20 - x_2^B} = 0$$

$$\bar{u} - (20 - x_1^B) - 2 \ln(15 - x_1^B) = 0$$

Solving the first two equations yields  $x_2^B = \frac{40}{3}$  and therefore  $x_2^A = \frac{20}{3}$ .

(b) We now find the Walrasian equilibrium price and allocations. Let the prices be  $(p_1, p_2) = (1, p)$ . Consumer A's Marshallian demands are derived from the following maximisation problem:

$$\max_{x_1^A, x_2^A} x_1^A + 2 \ln(x_2^A)$$

subject to

$$x_1^A + px_2^A = 12 + 15p$$

The first order conditions from this problem yield:

$$\frac{2}{p} = x_2^A$$

$$x_1^A = 10 + 15p$$

Consumer B solves the following problem:

$$\max_{x_1^B, x_2^B} x_1^B + 2 \ln(x_2^B) - \ln(x_2^A)$$

subject to

$$x_1^B + px_2^B = 3 + 5p$$

The first order conditions yield:

$$\frac{2}{p} = x_2^B$$

$$x_1^B = 1 + 5p$$

In equilibrium,  $x_1^A + x_1^B = 20$ , that is  $\frac{2}{p} + \frac{2}{p} = 20$ , and so this implies that  $p = \frac{1}{5}$ . The equilibrium allocations are therefore  $x^A = (13, 10)$  and  $x^B = (2, 10)$ . Using the result from part (a) we know that this allocation is not PE. Consumer A does not take into account the negative externality he produces when consuming good 2, therefore ending up consuming too much of it in equilibrium ( $10 > \frac{20}{3}$ ).

(c) This time, consumer A solves the following problem:

$$\max_{x_1^A, x_2^A} x_1^A + 2 \ln(x_2^A)$$

subject to

$$x_1^A + (p + t)x_2^A = 12 + 15p + T_A$$

Again, the first-order conditions yield:

$$\frac{2}{p + t} = x_2^A$$

$$x_1^A = 10 + 15p + T_A$$

Consumer B solves the following problem:

$$\max_{x_1^B, x_2^B} x_1^B + 2 \ln(x_2^B) - \ln(x_2^A)$$

subject to

$$x_1^B + px_2^B = 3 + 5p + T_B$$

The first-order conditions yield:

$$\frac{2}{p} = x_2^B$$

$$x_1^B = 1 + 5p + T_B$$

Since we must achieve the allocation  $((13, \frac{20}{3}), (2, \frac{40}{3}))$ , we must have:

$$\begin{aligned} \frac{2}{p} &= \frac{40}{3} \implies p = \frac{3}{20} \\ \frac{2}{p+t} &= \frac{20}{3} \implies t = \frac{3}{20} \\ 2 &= 1 + \frac{15}{20} + T_B \implies T_B = \frac{1}{4} \\ 13 &= 10 + \frac{9}{4} + T_A \implies T_A = \frac{3}{4} \end{aligned}$$

The government's revenue from taxes is  $tx_2^A = 1$  which it redistributes entirely through lump-sum subsidies  $T_A + T_B = 1$ . Thus the government's budget is balanced. We can also see that the government's budget must balance in equilibrium by considering the budget constraints of the two consumers:

$$x_1^A + (p+t)x_2^A = e_1^A + pe_2^A + T_A$$

$$x_1^B + x_2^B = e_1^B + pe_2^B + T_B$$

Adding these two equations gives us

$$(x_1^A + x_1^B - e_1^A - e_1^B) + p(x_2^A + x_2^B - e_2^A - e_2^B) + (-T_A - T_B + tx_2^A) = 0$$

The first term represents the excess demand for commodity 1 and the second term is the excess demand for commodity 2. In equilibrium, these must each be 0, and hence the third term, which is the government's net revenue, must be 0 as well.