Problem Set 4 – Solution

Solution of Exercise 1:

a) Since the agent is risk averse, when effort is observable the optimal contract will feature a constant wage paid to the agent. In addition to that, we know that the participation constraint must bind in the optimal contract. Hence, the wage w_i that is compatible with effort level e_i when effort is observable solves:

$$\sqrt{w(e_i)} - g(e_i) = 0, \quad \text{for } i = 1, 2, 3. \tag{1}$$

Equation (1) implies that: for e_1 the principal must pay $w_1 = 25/9$ to the agent regardless of the (profit) outcome; for e_2 the principal must pay $w_2 = 64/25$ to the agent regardless of the outcome; for e_3 the principal must pay $w_3 = 16/9$ to the agent regardless of the outcome.

Replacing those wages into their respective expected profits functions leads to:

$$\begin{aligned}
\Pi(e_1) &= \frac{2}{3} \times 10 - \frac{25}{9} = 3.89 \\
\Pi(e_2) &= \frac{1}{2} \times 10 - \frac{64}{25} = 2.44 \\
\Pi(e_3) &= \frac{1}{3} \times 10 - \frac{16}{9} = 1.55
\end{aligned}$$

As a result, the optimal contract when effort is observable is: $[e_1, w_1 = 25/9]$, which yields 3.89 as expected profit for the principal.

b) Since e_2 is *not* the lowest possible effort level, then the principal will need to pay a higher wage in the high-profit outcome (π_H) than in the low-profit one (π_L). Denote by w_H the former and by w_L the latter. For e_2 to be implementable we need to satisfy all incentive compatibility constraints and the participation constraint. All this requires:

$$\frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - \frac{8}{5} \geq \frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{5}{3} \tag{2}$$

$$\frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - \frac{8}{5} \geq \frac{1}{3}\sqrt{w_H} + \frac{2}{3}\sqrt{w_L} - \frac{4}{3} \tag{3}$$

$$\frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - \frac{8}{5} \geq 0 \tag{4}$$

Equation (1) implies $\sqrt{w_H} \leq \sqrt{w_L} + 2/5$, while (2) leads to $\sqrt{w_H} \geq \sqrt{w_L} + 8/5$. Since those two conditions cannot be simultaneously satisfied, e_2 is not implementable under asymmetric information.

To find for which levels of $g(e_2)$ we can implement e_2 when effort is unobservable, re-write the first two equations above as follows:

$$\begin{aligned}
\frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - g(e_2) &\geq \frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{5}{3} \\
\frac{1}{2}\sqrt{w_H} + \frac{1}{2}\sqrt{w_L} - g(e_2) &\geq \frac{1}{3}\sqrt{w_H} + \frac{2}{3}\sqrt{w_L} - \frac{4}{3}
\end{aligned}$$

from where we may obtain, respectively:

$$\sqrt{w_H} \leq 10 - 6g(e_2) + \sqrt{w_L} \quad \text{and} \quad \sqrt{w_H} \geq 6g(e_2) - 8 + \sqrt{w_L}.$$

Combining these two conditions it follows that e_2 is implementable if and only if $g(e_2) \leq \frac{3}{2}$.

c) From point b) we already know that e_2 cannot be implemented when effort is unobservable. Thus, the relevant comparison is between e_1 and e_3.

To implement e_3 we know that we can simply replicate the full-information contract; that is, we pay a constant wage $w_3 = 16/9$. This will allow the principal to obtain expected profits equal to 1.55.

Finally, to implement e_1 we will have to satisfy the participation constraint and the incentive compatibility constraint relative to e_3, both binding. Mathematically,

$$\begin{aligned} \frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{5}{3} &= 0, \quad \text{PC (binding)} \\ \frac{2}{3}\sqrt{w_H} + \frac{1}{3}\sqrt{w_L} - \frac{5}{3} &= \frac{1}{3}\sqrt{w_H} + \frac{2}{3}\sqrt{w_L} - \frac{4}{3}, \quad \text{ICC relative to } e_3 \text{ (binding)} \end{aligned}$$

Solving those two equations we obtain: $w_H = 4$ and $w_L = 1$. Replacing these two values into the profit function, we get:

$$\Pi(e_1) = \frac{2}{3}(10 - 4) + \frac{1}{3}(0 - 1) = \frac{11}{3}.$$

The optimal contract when effort is unobservable is then $[w_H = 4, w_L = 1]$, which induces the agent to choose e_1.

Solution of Exercise 2:

a) Since $w(0) = 0$ and $p(0|\, e_L) > 0$, then the expected utility to the agent when exerting low effort equals $-\infty$. In addition, the participation constraint to the agent when exerting high effort is satisfied. As a result, the agent will accept the contract and put e_H.

b) Using the fact that the IC is immediately satisfied when setting $w(0) = 0$, we can find the optimal contract that leads to high effort by solving the following optimisation problem (NB: below we're already using the fact that the participation constraint (2) must bind in the optimum):

$$\max_{w(2), w(4)} \; : \; \frac{1}{2}(2 - w(2)) + \frac{1}{2}(4 - w(4)), \tag{1}$$

$$\text{st} \; : \; \frac{1}{2}\ln(w(2)) + \frac{1}{2}\ln(w(4)) - e_H = 0 \tag{2}$$

The FOC will lead to (I leave to *you* working this out): $w(2) = w(4) = w^*$. Hence, replacing the constant value w^* into (2), implies $\ln(w^*) = e_H$, from where $w^* = \exp(e_H)$ immediately obtains.

The contract $[w(0) = 0, w(2) = w(4) = \exp(e_H)]$ induces high effort and provides full insurance to the agent, *given* the fact that he chooses to exert high effort (although $w(0) = 0 < w(2) = w(4)$, the wage $w(0) = 0$ never realises on the equilibrium path).

Solution of Exercise 3:

a) Note first that the negative dependence of expected utility on the income variance implies that the manager is *risk averse* (the mean-variance utility function is a particular example of preferences with risk-aversion).

Replacing $w(\pi) = \alpha + \beta\pi$ into the expected utility function, we get:

$$E(u\,|e) = E(\alpha + \beta\pi\,|e) - \phi Var(\alpha + \beta\pi\,|e) - g(e).$$

Since $E(\alpha + \beta\pi\,|e) = \alpha + \beta E(\pi\,|e) = \alpha + \beta e$, and $Var(\alpha + \beta\pi\,|e) = \beta^2 Var(\pi\,|e) = \beta^2\sigma^2$, the claimed result immediately obtains.

b) Suppose the outside option available to the agent is given by $\underline{U} = 0$. Under full information, the optimal contract will provide full insurance to the risk averse agent. This means that the principal will offer a fixed wage w, and thus $Var(w\,|e) = 0$. As a result, the participation constraint in the problem will read as follows:

$$E(w\,|e) - g(e) \geq 0. \tag{1}$$

By using the fact that (1) will necessarily bind in the optimum, and that $E(\pi\,|e) = e$, we can write the optimisation problem faced by the principal as follows:

$$\begin{aligned} \max_{e} \quad &: \quad e - E(w\,|e) \\ \text{subject to} \quad &: \quad E(w\,|e) = g(e). \end{aligned}$$

Which can be restated as:

$$\max_{e} : e - g(e)\,. \tag{2}$$

Problem (2) yields the following FOC –which is also sufficient given the assumptions on the function $g(e)$–:

$$g'(e^*) = 1.$$

Hence, optimally, the principal makes the agent exert effort up to the point in which the marginal disutility of effort equals the marginal (expected) productivity of effort.

Next, using again (1), we can observe that: $E(w\,|e) = g(e^*)$ for the agent to accept the contract. In fact, since the wage is constant in the optimum, this means that the principal sets: $\alpha^* = g(e^*)$ and $\beta^* = 0$.

c) When effort is unobservable, we must take into account the incentive-compatibility constraint. Plus, we no longer can restrict the attention to fixed wages, as they would lead to the lowest level of effort by the agent. Recalling that we are just looking for *linear contracts* (i.e., $w(\pi) = \alpha + \beta\pi$), the problem for the principal can be written as follows:

$$\max_{e,\alpha,\beta} : E\left[\pi - (\alpha + \beta\pi)\,|e\right] \tag{3}$$

subject to:

$$\alpha + \beta e - \phi\beta^2\sigma^2 - g(e) = 0. \quad \text{(participation constraint)}; \tag{4}$$

$$e = \arg\max_{\widetilde{e}} \left\{\alpha + \beta e - \phi\beta^2\sigma^2 - g(e)\right\} \quad \text{(IC constraint)}. \tag{5}$$

Notice now that (5) is a strictly concave function of e –this is because of the assumption $g''(e) > 0$–. As a result, we can (safely) use the *first-order approach* to solve the problem. That is, we can replace (5) by its (much simpler) first-order condition:

$$\beta - g'(e) = 0. \tag{6}$$

So the problem reads as:

$$\begin{gathered} \max_{e,\alpha,\beta} : E\left[\pi - (\alpha + \beta\pi)\,|e\right] \\ \text{s. t.: } \alpha + \beta e - \phi\beta^2\sigma^2 - g(e) = 0. \quad \text{(PC)} \\ \beta = g'(e). \quad \text{(ICC)}. \end{gathered}$$

Notice now that we can thus replace $\beta = g'(e)$ into (4), to obtain:

$$\alpha = -g'(e)e + \phi\left[g'(e)\right]^2\sigma^2 + g(e) = 0. \tag{7}$$

And we can finally replace the right-hand side of (7) and $\beta = g'(e)$ into (3), to clear out α and β from the expression. By doing so, and using again the fact that $E[\pi\,|e] = e$, we obtain the following (reduced-form) maximisation problem for the principal:

$$\max_{e} : e - g(e) - \phi\sigma^2\left[g'(e)\right]^2. \tag{8}$$

Remark: the problem (8) essentially has got implicit into it both the Participation Constraint and the Incentive-Compatibility Constraint (using its first-order condition version).

From (8), we obtain the FOC:

$$g'(e) - 2\phi\sigma^2 \left[g'(e)\right] g''(e) = 1;$$

which leads to the optimal $g'(e^*_{inc})$ under incomplete information, and also equals the optimal β^*_{inc} due to (6):

$$\beta^*_{inc} = g'(e^*_{inc}) = \frac{1}{1 + 2\phi\sigma^2 g''(e^*_{inc})}. \tag{9}$$

Note the following:

1. $0 < \beta^*_{inc} < 1$: That means the wage of the agent increases with output (although less that one-by-one). In other words, part of the risk is now transferred to the (risk-averse) agent, so that to provide him with incentives to exert (unobservable) effort.

2. $e^*_{inc} < e^*$, where e^* is the full-information solution: That means that effort will be inefficiently low under moral hazard.

3. $\dfrac{\partial \beta}{\partial \phi} < 0$ and $\dfrac{\partial \beta}{\partial \sigma^2} < 0$: the more risk-averse the agent is (i.e., higher ϕ), or the riskier production is (i.e., higher σ^2), the more insurance that must be provided (and the less powerful the incentives will be). You can prove by yourselves that this, in turn, implies $\dfrac{\partial e^*_{inc}}{\partial \phi} < 0$ and $\dfrac{\partial e^*_{inc}}{\partial \sigma^2} < 0$.

Solution of Exercise 4:

a) The expected return to entrepreneurs for project a and b, are given respectively by:

$$\begin{aligned} E\left(\Pi_a\right) &= 0.7 \times (2 - R), \\ E\left(\Pi_b\right) &= 0.1 \times (7 - R). \end{aligned}$$

Hence, $E\left(\Pi_a\right) \geq E\left(\Pi_b\right) \Longleftrightarrow R \leq 1.167$.

b) From the bank's viewpoint, the only project that could in principle be profitable financing is project a. The expected return to the bank of financing project a is then:

$$E(\Pi^{bank}) = 0.7\,R - 1, \quad \text{if } 0 \leq R \leq 1.167. \tag{1}$$

Since (1) is strictly increasing in R, it reaches a maximum at the upper bound $R = 1.167$. Replacing this value into (1), we obtain: $0.7 \times 1.167 - 1 = -0.1831$. As a result, it is neither profitable for the bank to finance project a, and the bank chooses not to lend to entrepreneurs at all and to lend only to the government).