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A SUBJECTIVE SPIN ON ROULETTE WHEELS

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Abstract

We provide a simple behavioral definition of 'subjective mixture' of acts. Subjective mixtures enjoy the same algebraic properties as the 'objective mixtures', which are used to great advantage in the decision setting introduced by Anscombe and Aumann [1].

Our construction allows one to formulate mixture-space axioms even in a fully subjective setting, without assuming the existence of randomizing devices. This simplifies the task of developing axiomatic models using only behavioral data. Moreover, it is effective also when decision makers treat the probabilities associated with randomizing devices in a nonlinear fashion.

For illustration, we present simple subjective axiomatizations of some models of choice under uncertainty, including the maxmin expected utility model of Gilboa and Schmeidler [13], and Bewley's model of choice with incomplete preferences [2]. Other applications are suggested.

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Introduction

The axiomatizations of subjective expected utility (SEU) of Savage [25] and Anscombe and Aumann [1, AA for short] are often contrasted in terms of their analytical complexity and behavioral content.

In particular, Savage explicitly aimed at constructing a theory that relies solely upon purely behavioral data—preferences among *acts*, i.e. maps assigning consequences to states. In contrast, the AA decision setting features pre-assigned, 'objective' probabilities. On the other hand, the latter is much more amenable to mathematical treatment than Savage's. This is especially apparent in Fishburn's [8] well-known reformulation and extension of AA's analysis, which employs familiar vector-space arguments.[1]

The main contribution of this paper is to show that it is possible to exploit all the advantages of the approach pioneered by AA and Fishburn ('AA approach' for short) in a fully behavioral setting. This places models that follow AA's approach on comparable epistemological footing with those that follow Savage's.

As noted above, Anscombe and Aumann modify Savage's setting and allow payoffs to be *lotteries* contingent on the output of an extraneous randomizing device, which they call a 'roulette wheel'. They observe that the existence of the roulette wheel enables one

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[1]AA assume that the state space is finite, and their original arguments employ the 'linearity' property of von Neumann-Morgenstern utility directly. Most textbook presentations of 'the AA axiomatization of SEU' are actually based on Fishburn's. See also Schmeidler [26, p. 578].

to define 'objective mixtures' of acts; this considerably simplifies the axiomatic derivation of the SEU model for two main reasons.

First, the set of acts, equipped with the objective mixture operation, can be viewed as a mixture set. Second, if the decision maker's preferences conform to expected utility (EU) over 'roulette-wheel acts',[2] the utility profile corresponding to the objective mixture of two acts (with weights α and $1-\alpha$) is the convex combination of the utility profiles of the latter (with the same weights).

The mixture-set structure of the collection of acts can be exploited to formulate axioms in the style of von Neumann and Morgenstern [27]; these imply, in particular, that the decision maker's preferences over roulette-wheel acts satisfy EU. The resulting isomorphism between objective mixtures of acts and convex combinations of utility profiles is key in the construction of the mathematical representation.

These features of the AA approach also simplify the development of models that address the well-known descriptive limitations of the SEU representation (see, e.g, Luce [18, Chap. 3]). Indeed, many successful extensions of SEU were first obtained in the AA 'roulette-wheel' setting: among others, we mention Schmeidler's 'Choquet expected utility' (CEU) model [26], Gilboa and Schmeidler's 'maxmin expected utility' (MEU) model [13], and Bewley's model [2] of choice with incomplete preferences.

However, reintroducing a non-behavioral element (the roulette wheel) in the decision model is generally perceived to detract from the conceptual appeal of the AA approach (see, e.g., the works cited in Sec. 4). Furthermore, the analytically convenient assumption that preferences over roulette-wheel acts conform to EU limits the scope of the models that adopt the approach outlined above (see our discussion of 'nonlinear risk preferences' below).

This paper shows that it is possible to define a mixture operation with convenient algebraic properties in a fully behavioral setting like Savage's; i.e., without the help of a randomizing device. Our construction requires that preferences satisfy some mild conditions, and that the set of possible outcomes be sufficiently rich (e.g., an interval in the real line). As we explain in detail below, if these conditions are met, we can use purely behavioral data to define a simple 'subjective mixture' operation. By construction, the utility profile of a subjective mixture of acts is the corresponding convex combination of their utility profiles. This enables us to readily extend AA-style axiomatics and techniques to a fully subjective environment.

For instance, we employ subjective mixtures to offer simple axiomatizations of the CEU and MEU models (of which SEU is a special case), as well as of Bewley's model. While previous axiomatizations of CEU and MEU in fully subjective settings exist, this is to the best of our knowledge the first such axiomatization of Bewley's model. Moreover, our explicit adoption of the AA approach results in more transparent axiomatics and

[2]The acts which only depend on the behavior of the roulette wheel.

analysis than earlier work (see Section 4 for discussion). Other fruitful applications are possible; as an example, we discuss 'menu choice' models in the style of Kreps [17].

Besides avoiding the philosophically debatable concept of objective probability — and extraneous devices in general — another advantage of our extension of the AA approach is the generality gained by accomodating nonlinearity in the decision maker's risk preferences.[3] There is strong evidence that experimental subjects violate EU when choosing among roulette-wheel acts. In such circumstances, the utility profile of an *objective* mixture of acts may not be equal to the corresponding convex combination of their utility profiles, thus eliminating one of the key advantages of working with mixtures mentioned earlier.

The definition of subjective mixture does not rely on the specific nature of the outcome space, as long as the latter is rich enough, and *by construction* preserves the isomorphism with convex combinations of utilities. Indeed, since the set of roulette-wheel acts satisfies our richness conditions, adopting our notion of subjective mixture in the AA 'roulette-wheel' setting makes it possible to formulate AA-style axiomatics without restricting risk preferences. We provide an example in Section 2.2.

Preference Averages and Subjective Mixtures of Acts

The key step in our construction of subjective mixtures of acts is the notion of a 'preference average' of two outcomes. In our interpretation, a *preference average* of two outcomes x and y is an outcome z with the following property: if x and y are possible results which play a symmetric role in the decision maker's evaluation of a bet, then replacing x and y with z leaves the decision maker indifferent. This appears to us to constitute a sensible behavioral definition of the midpoint of the preference interval between two outcomes.[4]

To see how a preference average is practically observed, fix an event E, and consider the decision maker's preferences over the bets 'on E', i.e., uncertain prospects of the form $x\,E\,y$ ='receive x if E obtains, and y otherwise', where outcome x is *at least weakly preferred* to outcome y.[5] Take E to be 'non-trivial'; that is, for some pair x and y such that x is strictly preferred to y, the bet $x\,E\,y$ is strictly between x and y in the preference ordering.

Assume further that the decision maker's preferences satisfy the following condition: there exist a cardinal, convex-ranged utility u on the set of outcomes and a number

[3] See Machina and Schmeidler [21] for a different approach to modelling nonlinear risk preferences in the AA 'roulette-wheel' setting.

[4] The following analogy may be helpful: if α, β are real numbers, then $\frac{1}{2}\alpha + \frac{1}{2}\beta = \frac{1}{2}\beta + \frac{1}{2}\alpha$, and their (arithmetic) average is the number ζ such that $\frac{1}{2}\alpha + \frac{1}{2}\beta = \frac{1}{2}\zeta + \frac{1}{2}\zeta$.

[5] Conversely, when y is weakly preferred to x we call $x\,E\,y$ a bet 'against E' (equivalently, a bet 'on E^c').

$\rho(E) \in (0, 1)$ such that $x\,E\,y$ is preferred to $x'\,E\,y'$ if and only if

$$u(x)\,\rho(E) + u(y)\,(1 - \rho(E)) \geq u(x')\,\rho(E) + u(y')\,(1 - \rho(E)). \tag{1}$$

A consequence of this assumption is that every bet $x\,E\,y$ has a certainty equivalent c_{xEy}. We emphasize that the above representation can be derived from behavioral axioms: see e.g. Ghirardato and Marinacci [11]. It is assumed directly in the sole interest of conciseness.[6]

Now consider two outcomes x and y such that x is weakly preferred to y, and suppose that z is a third outcome that satisfies the following condition:

$$x\,E\,y \quad \text{is indifferent to} \quad c_{xEz}\,E\,c_{zEy}. \tag{2}$$

Under our assumption on the decision maker's preference, such a z exists for every x and y. We now argue that z can also be interpreted as a 'preference average' of x and y.

First, since $x\,E\,y$ is trivially indifferent to $c_{xEx}\,E\,c_{yEy}$, we have that

$$c_{xEx}\,E\,c_{yEy} \quad \text{is indifferent to} \quad c_{xEz}\,E\,c_{zEy}. \tag{3}$$

In words, substituting z for the inner x and y in the 'compound' act on the left-hand side leaves the decision maker indifferent.

Second, our preference assumption implies that the inner outcomes in the 'compound' bets in Eq. (3) have the same impact on the decision maker's evaluation. Formally, if z' and z'' are both between x and y in preference, then Eq. (1) implies that

$$c_{xEz'}\,E\,c_{z''Ey} \quad \text{is indifferent to} \quad c_{xEz''}\,E\,c_{z'Ey}.$$

In words, permuting z' and z'' does not affect the decision maker's preferences: the two inner outcomes play a symmetric role in his evaluation of these bets.

We conclude that the inner x and y in the compound bet $c_{xEx}\,E\,c_{yEy}$ play a symmetric role in the evaluation of the latter, *and* they can be replaced with z without changing the decision maker's evaluation of the bet. Therefore, z fits our intuitive description of a preference average of x and y.

To see this from a different perspective, it is simple to show using Eq. (1) that z satisfies Eq. (2) if and only if

$$u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y).$$

[6] Notice that the representation in Eq. (1) only holds for bets *on E*. As the results in Sections 2.2 and 3 indicate, this allows considerable flexibility in the evaluation of all other acts.

The key step in the proof is the quantitative counterpart to the intuition provided above: The inner outcomes x and y in the act $c_{xEx}\,E\,c_{yEy}$ and the z's in the act $c_{xEz}\,E\,c_{zEy}$ all receive the same 'weight' according the evaluation functional described by Eq. (1). We also show that the notion of preference average is independent of the 'non-trivial' event E used to assess it.

The preference average described above corresponds to the $\frac{1}{2} : \frac{1}{2}$ mixture of two outcomes. It is clear that by iterating our definition, arbitrary dyadic mixtures may be defined. Finally, *subjective mixtures* of acts are defined state-by-state, as in the standard AA setting.

Organization of the Paper

The paper proceeds as follows. Section 1 introduces the required terminology and notation. Section 2 is the core of the paper. Subsection 2.1 introduces the notion of subjective mixture formally, as well as preference conditions that make it consistent. Subsection 2.2 presents the axiomatizations of the CEU, SEU and MEU models, and Subsection 2.3 sketches an application to unforeseen contingencies in the spirit of Dekel, Lipman and Rustichini [6]. Section 3 is devoted to Bewley's model. Section 4 compares this paper to the existing literature. The appendices contain the proofs of the results, as well as additional technical material.

1 Preliminaries

The Savage-style setting we use consists of a non-empty set S of **states of the world**, an algebra Σ of subsets of S called **events**, and a non-empty set X of **consequences**. We denote by $\mathcal{F}$ the set of all the **simple acts**: finite-valued functions $f : S \to X$ which are measurable with respect to Σ. For $x \in X$ we define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. So, with the usual slight abuse of notation, we identify X with the subset of the constant acts in $\mathcal{F}$. Also, given $x, y \in X$ and $A \in \Sigma$, we use $x\,A\,y$ to denote the binary act which yields x if $s \in A$ and y otherwise; $\mathcal{F}_A$ is the set of all such acts.

We model the decision maker's preferences on $\mathcal{F}$ by a binary relation $\succcurlyeq$. A functional $V : \mathcal{F} \to \mathbb{R}$ such that $V(f) \geq V(g)$ if and only if $f \succcurlyeq g$ is called a **representation of** $\succcurlyeq$. Clearly, a necessary condition for $\succcurlyeq$ to have a representation is that it be a **weak order** — a complete and transitive relation — and, as customary, we denote by $\sim$ and $\succ$ its symmetric and asymmetric components respectively. V is **monotonic** if $V(f) \geq V(g)$ whenever $f(s) \succcurlyeq g(s)$ for all $s \in S$; it is **nontrivial** if $V(f) \neq V(g)$ for some $f, g \in \mathcal{F}$.

A set-function $\rho : \Sigma \to \mathbb{R}_+$ is a **capacity** if it is monotone and normalized; that is, $\rho(A) \leq \rho(B)$ if $A \subseteq B$, $\rho(S) = 1$ and $\rho(\emptyset) = 0$. A capacity is called a **probability** if it is additive; that is $\rho(A \cup B) = \rho(A) + \rho(B)$ if $A \cap B = \emptyset$.

1.1 Biseparable Preferences

We first define subjective mixtures for the class of biseparable preferences introduced by Ghirardato and Marinacci [11, henceforth GM].[7] As we argue below, we do not need all the structure entailed by that model, but this simplifies the exposition.

Given a binary relation $\succcurlyeq$, we say that a functional $V : \mathcal{F} \to \mathbb{R}$ is a **canonical representation** of $\succcurlyeq$ if it is a nontrivial and monotonic representation of $\succcurlyeq$ and moreover, letting $u(x) \equiv V(x)$ for all $x \in X$, there is a set-function $\rho : \Sigma \to [0,1]$ such that

$$V(x\, A\, y) = u(x)\, \rho(A) + u(y)\, (1 - \rho(A)) \tag{4}$$

for all consequences $x \succcurlyeq y$ and all $A \in \Sigma$. For instance, given a MEU preference $\succcurlyeq$ with nonconstant utility u and set of priors $\mathcal{D}$, the functional $V(f) = \min_{P \in \mathcal{D}} \int u(f)\, dP$ is a canonical representation of $\succcurlyeq$.

A biseparable preference is a relation that has canonical representations, but not too many. To be more precise, we need to introduce the 'nontriviality' property for events mentioned in the Introduction. We say that $E \in \Sigma$ is an **essential** event if $x \succ x\, E\, y \succ y$ for some $x, y \in X$ — i.e., if $0 < \rho(E) < 1$ for every canonical representation.

Definition 1 *A binary relation $\succcurlyeq$ on $\mathcal{F}$ is a **biseparable preference** if: 1) it has a canonical representation; 2) in case that $\succcurlyeq$ has at least one essential event, then such representation is unique up to a positive affine transformation.*

GM argue that biseparable preferences are the weakest model achieving a separation of cardinal state-independent utility and a unique representation of beliefs. It encompasses CEU, MEU (hence SEU) and other well-known decision models, like Gul's 'disappointment aversion' [14] model.[8] They also show that if $\succcurlyeq$ is a biseparable preference with some essential event, then: 1) u is cardinal; 2) ρ is a capacity and it is unique. We follow them in calling u and ρ respectively the **canonical utility index** and **willingness to bet** of $\succcurlyeq$. Further discussion of the properties of biseparable preferences is found in their paper.

The necessity of cardinality of u for our purposes is easy to motivate. Recall that our chief objective is to identify for every pair of consequences x and y a third consequence z, located at the midpoint of the preference interval between x and y. Such preference average should arguably be independent of the normalization of utility. Thus, if u and v both represent $\succcurlyeq$ on X and z satisfies

$$u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y),$$

[7] A similar representation appears as an intermediate result in Nakamura [22] and also in Casadesus-Masanell *et al.* [3, 4].

[8] On the other hand, the preferences of Section 3 are not biseparable, and 'probabilistically sophisticated' preferences in the sense of Machina and Schmeidler [20] may not be as well (see GM [11, Remark 5.1]).

we should also obtain the equality

$$v(z) = \frac{1}{2}v(x) + \frac{1}{2}v(y).$$

This is easily seen to imply that v is an affine transformation of u, provided the set X is rich.

As noted above, subjective mixtures can be defined and fruitfully employed in more general circumstances; this is illustrated in Sections 2.3 and 3, and also discussed in Remark 8. In particular, the following generalization of biseparability often suffices. Given an event $E \in \Sigma$, let $\mathcal{F}_E$ denote the set of all the acts of the form $x\,E\,y$, with $x, y \in X$. We say that the binary relation $\succcurlyeq$ is a **E-locally biseparable preference** if its restriction to $\mathcal{F}_E$ is a biseparable preference.

2 Subjective Mixtures

In this section we introduce the key notion of 'subjective mixture' of two acts for a biseparable preference relation $\succcurlyeq$ on $\mathcal{F}$ that admits an essential event. Formally, the following assumption is maintained *throughout this section:*

Preference Assumption A *The relation $\succcurlyeq$ is a biseparable preference with some essential event and a canonical utility index u such that the range $u(X)$ is convex.*

We refer the reader to GM for an axiomatization of preferences that satisfy this assumption. Notice that Preference Assumption A implies that the set of consequences X is infinite, and that every act $f \in \mathcal{F}$ (in particular the bets in $\mathcal{F}_E$) has a **certainty equivalent** c_f, i.e. an arbitrarily chosen element of the set $\{x \in X : x \sim f\}$.

2.1 Definition and Properties

We begin with the definition of preference average of two consequences.

Definition 2 *Given $x, y \in X$ such that $x \succcurlyeq y$, we say that a consequence $z \in X$ is a **preference average of** x **and** y **(given** E**)** if $x \succcurlyeq z \succcurlyeq y$ and*

$$x\,E\,y \sim c_{xEz}\,E\,c_{zEy}. \tag{5}$$

We have explained in the Introduction why we interpret such a z as a preference average of x and y. Notice that this definition is well-posed provided that $\succcurlyeq$ has some essential event E and that the certainty equivalents for bets 'on' the event E exist. That is, the definition of preference average does *not* require the full strength of Preference Assumption A.

On the other hand, the following Proposition (see Appendix B for the proof) shows that, if the decision maker's preferences satisfy Preference Assumption A, then the definition enjoys several properties of interest:

Proposition 3 *For each $x, y \in X$ and each essential event $E \in \Sigma$, a consequence $z \in X$ is a preference average of x and y given E if and only if*

$$u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y).$$

In particular, preference averages of x and y given E exist for every essential $E \in \Sigma$, they do not depend on (the choice of) E, and they form an indifference class.

In light of this result, we can denote equivalently by $(1/2)x \oplus (1/2)\, y$ or by $(1/2)y \oplus (1/2)\, x$ *a representative* of the class of preference averages of x and y.

Remark 4 *The condition $x \succcurlyeq z \succcurlyeq y$ in Definition 2 is necessary for Proposition 3 to hold. Consider, for example, the CEU preference on $S = \{0, 1\}$ and $X = \mathbb{R}$ with $\rho(0) = 0.8$, $\rho(1) = 0$, and linear utility. Let $E = \{0\}$, $x = y = 1$, and $z = 2$. Then $xEy \sim c_{xEz}\, E\, c_{zEy}$ but $u(z) \neq \frac{1}{2}u(x) + \frac{1}{2}u(y)$.*

For some applications, the notion of 'arithmetic' preference average introduced above suffices; see for instance the axiomatization of the CEU and SEU models below. However, it is natural to ask whether our construction can be extended to deliver 'weighted' preference averages. This is obtained by considering iterated averages such as

$$\frac{1}{2}\, x \oplus \frac{1}{2}\left[\frac{1}{2}\, x \oplus \frac{1}{2}\, y\right],$$

which is tantamount to an average of x and y with weights 3/4 and 1/4.

As we explain in full detail in Appendix C, proceeding along these lines and using a standard continuity argument, it is possible to identify, for any $\alpha \in [0, 1]$ and every x and y in X, the weighted preference averages z characterized by

$$u(z) = \alpha u(x) + (1 - \alpha)u(y). \tag{6}$$

We denote by $\alpha\, x \oplus (1 - \alpha)\, y$ or $(1 - \alpha)\, y \oplus \alpha\, x$ an arbitrary element of the equivalence class of such preference averages.

In other words, if X is sufficiently rich, it is possible to endow it with a subjective mixture structure *that respects the algebraic structure of the interval $u(X)$*. To formalize this, consider the following abstract definition of mixture set, proposed by Fishburn [9]:

Definition 5 *A triple $(M, \widehat{=}, \widehat{+})$ is a* ***generalized mixture set*** *if $\widehat{=}$ is an equivalence relation on M and $\widehat{+}$ satisfies: for all $x, y \in M$ and all $\alpha, \beta \in [0, 1]$, $\alpha x \,\widehat{+}\, (1 - \alpha)y \in M$ and*

M1($\widehat{=}$) $1x \mathbin{\widehat{+}} 0y \mathrel{\widehat{=}} x$,

M2($\widehat{=}$) $\alpha x \mathbin{\widehat{+}} (1-\alpha)y \mathrel{\widehat{=}} (1-\alpha)y \mathbin{\widehat{+}} \alpha x$,

M3($\widehat{=}$) $\alpha[\beta x \mathbin{\widehat{+}} (1-\beta)y] \mathbin{\widehat{+}} (1-\alpha)y \mathrel{\widehat{=}} \alpha\beta x \mathbin{\widehat{+}} (1-\alpha\beta)y$.

Going back to our setting, our findings can be summarized as follows:

Proposition 6 *The triple $(X, \sim, \oplus)$ is a generalized mixture set; furthermore, for all $x, y, z \in X$ and $\alpha \in [0,1]$, $z \sim \alpha\, x \oplus (1-\alpha)\, y$ if and only if $u(z) = \alpha\, u(x) + (1-\alpha)\, u(y)$.*

It is now immediate to define subjective mixtures of acts.

Definition 7 *Given $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, a* ***subjective mixture*** *of f and g with weight α is any act $h \in \mathcal{F}$ such that such that $h(s) \sim \alpha f(s) \oplus (1-\alpha)\, g(s)$ for every $s \in S$.*

Clearly all subjective mixtures of f and g with weight α are state-wise indifferent, hence indifferent. Therefore, we denote by $\alpha f \oplus (1-\alpha)\, g$ any one of them. It follows from the definition and Eq. (6) that for every $f, g \in \mathcal{F}$ and $s \in S$,

$$u((\alpha f \oplus (1-\alpha) g)(s)) = \alpha\, u(f(s)) + (1-\alpha)\, u(g(s)).$$

That is, $\alpha f \oplus (1-\alpha)\, g$ is an act whose utility profile *mixes* with weights α and $(1-\alpha)$ those of f and g — as does the 'objectively mixed' act $\alpha f + (1-\alpha)\, g$ for a preference that satisfies the von Neumann-Morgenstern axioms over consequences in an AA framework.

Remark 8 *Most of the results seen thus far do not require the full structure of biseparable preferences. Given an essential event E, a 'local' version of Proposition 3 holds for any preference $\succcurlyeq$ that has certainty equivalents and is E-locally biseparable (with $u(X)$ convex). Though preference averages may then depend on the essential event E, the independence of E is re-established if $\succcurlyeq$ further satisfies the axiom of Certainty Independence (see Section 2.2) with respect to preference averages constructed using some E. Thus, for most applications of interest, requiring E-local biseparability is essentially sufficient to obtain an appropriate mixture-set structure.*

2.2 Application: SEU, CEU and MEU

It is simple to use the notion of subjective mixture to provide an axiomatic foundation to some popular models of decision making under uncertainty in a fully subjective environment.

We begin with the generalization of the CEU model of Schmeidler [26], and hence of the classical SEU model of Anscombe and Aumann [1]. This has the advantage of just requiring the notion of $\frac{1}{2} : \frac{1}{2}$ preference average of Definition 2.

As it is well-known, the CEU model hinges on a weakening of the classical independence axiom which imposes the independence restriction only for acts which are 'commonly monotonic', in the following sense: $f, g \in \mathcal{F}$ are **comonotonic** if there are no $s, s' \in S$ such that $f(s) \succ f(s')$ and $g(s') \succ g(s)$.

Axiom A 1 (Comonotonic Independence) *For every $f, g, h \in \mathcal{F}$ such that h is comonotonic with both f and g,*

$$f \succcurlyeq g \Longrightarrow \frac{1}{2} f \oplus \frac{1}{2} h \succcurlyeq \frac{1}{2} g \oplus \frac{1}{2} h.$$

The following Proposition states that a preference satisfying Preference Assumption A and this property is represented by a Choquet integral with respect to the decision maker's willingness to bet ρ.[9] The proof can be omitted as it amounts to restating Schmeidler's [26] original argument, after replacing objective AA-type mixtures with subjective mixtures.

Proposition 9 *The preference $\succcurlyeq$ satisfies axiom A1* if and only if

$$f \succcurlyeq g \Longleftrightarrow \int_S u(f)\, d\rho \geq \int_S u(g)\, d\rho.$$

Clearly, strengthening Comonotonic Independence by asking that the implication hold for *every* f, g and h yields an axiomatization of SEU that corresponds to that of Anscombe and Aumann [1].

Next, we offer a subjective axiomatization of Gilboa and Schmeidler's **maxmin expected utility** model [13]. First, we need to impose a further generalization of the independence axiom, requiring that an indifference be unaffected by arbitrary mixtures with a *constant* act (observe that constants are comonotonic with respect to any act):

Axiom A 2 (Certainty Independence) *For every $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in [0, 1]$,*

$$f \succcurlyeq g \Longrightarrow \alpha\, f \oplus (1-\alpha)\, x \succcurlyeq \alpha\, g \oplus (1-\alpha)\, x.$$

Second, we add the following axiom, which is analogous to the axiom Gilboa and Schmeidler call 'uncertainty aversion'.[10]

Axiom A 3 (Ambiguity Hedging) *For every $f, g \in \mathcal{F}$,*

$$f \sim g \Longrightarrow \frac{1}{2} f \oplus \frac{1}{2} g \succcurlyeq f.$$

[9] See Appendix A for a definition, and notice that a Choquet integral with respect to a probability is a standard integral.

[10] See Ghirardato and Marinacci [12] for a discussion of such terminology, motivating our departure from it.

Intuitively, a decision maker displays ambiguity hedging if he prefers the 'even mixture' of indifferent acts (which possibly hedges ambiguity) to either of the 'pure' acts (which certaintly do not).

We can now state the characterization result, again omitting the simple proof.

Proposition 10 *The preference* $\succcurlyeq$ *satisfies axioms A2 and A3* if and only if *there exists a unique nonempty, weak* compact and convex set* $\mathcal{D}$ *of probabilities on* Σ *such that*

$$f \succcurlyeq g \Longleftrightarrow \min_{P \in \mathcal{D}} \int_S u(f)\, dP \geq \min_{P \in \mathcal{D}} \int_S u(g)\, dP.$$

As mentioned in the introduction, another advantage of the extension of the AA approach outlined here is that the decision maker's risk preferences are not required to be linear in objective probabilities when the latter are part of the model. Suppose that Z is a finite set of prizes and let $X = \Delta(Z)$, the set of lotteries over Z. Then, given $z, z' \in Z$ and the even-chance lottery $[z, 1/2;\, z', 1/2]$, it is possible that

$$\frac{1}{2}\, u(z) + \frac{1}{2}\, u(z') = u\left(\frac{1}{2}\, z \oplus \frac{1}{2}\, z'\right) \neq u\left(\left[z, \frac{1}{2};\, z', \frac{1}{2}\right]\right);$$

that is, the decision maker's preference over $\Delta(Z)$ is nonlinear in probabilities. For instance, let $v : Z \to \mathbb{R}$ and P be a probability on Σ. Consider a decision maker whose preferences over $\Delta(Z)$ are represented by $u(p) = \int_Z v(z)\, d(p(z))^2$ and whose preferences over $\mathcal{F}$ are represented by $V(f) = \int_S u(f(s))\, d(P(s))^2$, where both integrals are taken in the sense of Choquet. Because of the nonlinearity of $u(\cdot)$, this decision maker does not satisfy the axioms of Schmeidler [26], whereas he satisfies the axioms given above for the CEU model.

2.3 Application: Preference for Flexibility and Related Models

The notion of subjective mixture can be employed also to model phenomena that are not necessarily related to the presence of ambiguity. As an example, we now briefly sketch an application to unforeseen contingencies, in the spirit of Dekel, Lipman and Rustichini [6]. Specifically, we argue that, if the 'menu choice' framework introduced by Kreps [17] is augmented with a set S of 'objective' states, then it is possible to introduce a subjective mixture structure on the choice set without invoking the notion of lotteries. Thus, our analysis indicates that, in settings where uncertainty and unforeseen contingencies both play a role, the appropriate notion of convexity of the choice set follows from mild assumptions on preferences and consequences, and does not require the introduction of extraneous elements.

We build on the extension of model of Dekel *et al.* due to Ozdenoren [23]. This author considers an AA setting, and assumes that the decision maker has preferences over **opportunity acts**; i.e., maps from the objective state space S into the set of nonempty bundles of *lotteries over a set of consequences*. As in Dekel *et al.*, the latter set is thus endowed with an (objective) mixture-set structure; therefore, so is the set of opportunity acts.

In our Savage-style setting, opportunity acts are instead maps from S into the set $2^X \setminus \emptyset$ of the nonempty bundles of *consequences*. If the restriction of the preference to single-valued opportunity acts satisfies Preference Assumption A, then the technique outlined in Section 2.1 can be used to endow the set of opportunity acts with a subjective mixture-set structure. Thus, we obtain a counterpart to Ozdenoren [23]'s results in a fully subjective framework.[11]

We expect that an analogous treatment can be provided to obtain a fully subjective version of the related 'temptation and self-control' model of Gul and Pesendorfer [16].

3 A Subjective Axiomatization of Bewley's Model

In this final section, we modify the analysis of Section 2 in order to provide a subjective foundation to Bewley's [2] model of choice with incomplete preferences.

For this section only, we add further requirements to the sets S and X, analogous to those in Bewley [2]:

Structural Assumption *The set X is a connected and compact topological space with topology τ. The set S is finite.*

It is also necessary to relax our Preference Assumption A, which entails completeness of $\succcurlyeq$. As announced earlier, our technique works under the following weaker assumption (that holds throughout this section).

Preference Assumption B *The relation $\succcurlyeq$ is a E-locally biseparable preference with respect to an essential event E and it has a τ-continuous canonical utility index u.*

An axiomatizatic characterization of such relations can be easily obtained from either GM or Chew and Karni [5]. Notice that the convexity of $u(X)$ is now a consequence of the connectedness of X and of the continuity of u. It follows that every act $f \in \mathcal{F}_E$ has a certainty equivalent (clearly, this need not be true of acts outside $\mathcal{F}_E$), so that we can apply Definition 2 to define preference averages, and thus define subjective mixtures of acts. Such mixtures will in general enjoy all the properties stated in Proposition 3 except for the possibility of dependence of the notion of mixture on the event E. As we prove later, such dependence is excluded by the following axioms.

[11]The line of argument is similar to the one used in the proof of Proposition 11 below; the details are available on request.

We require that $\succcurlyeq$ satisfies four axioms in addition to Preference Assumption B. The first axiom is the natural weakening of the weak order assumption in this context, and the second is a statewise dominance condition.

Axiom B 1 (Preorder) *(a) For all $f \in \mathcal{F}$, $f \succcurlyeq f$. (b) For all $f, g, h \in \mathcal{F}$, if $f \succcurlyeq g$ and $g \succcurlyeq h$, then $f \succcurlyeq h$.*

Axiom B 2 (Dominance) *For every $f, g \in \mathcal{F}$, if $f(s) \succcurlyeq g(s)$ for every $s \in S$ then $f \succcurlyeq g$.*

The third axiom is a standard continuity requirement (cf. Dubra, Maccheroni and Ok [7]). To state it, we first observe that the topology τ on X induces the product topology on the set X^S of all functions from S into X. In this topology, a net $\{f_\alpha\}_{\alpha\in D} \subseteq X^S$ converges to $f \in X^S$ if and only if $f_\alpha(s) \xrightarrow{\tau} f(s)$ for all $s \in S$. For this reason it is also called the topology of **pointwise convergence**.

Axiom B 3 (Continuity) *Let $\{f_\alpha\}_{\alpha\in D} \subseteq \mathcal{F}$ be a net that pointwise converges to $f \in \mathcal{F}$ and $\{g_\alpha\}_{\alpha\in D} \subseteq \mathcal{F}$ be a net that pointwise converges to $g \in \mathcal{F}$. If $f_\alpha \succcurlyeq g_\alpha$ for all $\alpha \in D$, then $f \succcurlyeq g$.*

Finally, we have the independence axiom, which is stronger than the versions seen in the previous section, as it applies to every triple of acts and arbitrary mixtures.

Axiom B 4 (Independence) *For every $f, g, h \in \mathcal{F}$ and every $\alpha \in [0, 1]$,*

$$f \succcurlyeq g \Longrightarrow \alpha\, f \oplus (1-\alpha)\, h \succcurlyeq \alpha\, g \oplus (1-\alpha)\, h.$$

Our last result (see Appendix D for the proof) shows that a decision maker satisfying Preference Assumption B and these axioms has a *family* of probabilities as beliefs, and prefers act f to act g when the expected utility of f is *unanimously* greater than the expected utility of g.

Proposition 11 *The preference $\succcurlyeq$ satisfies axioms B1–B4* if and only if *there exist a unique nonempty, closed and convex set $\mathcal{C}$ of probabilities on Σ such that for all $f, g \in \mathcal{F}$,*

$$f \succcurlyeq g \Longleftrightarrow \int_S u(f)\, dP \geq \int_S u(g)\, dP \quad \text{for all } P \in \mathcal{C}. \tag{7}$$

Notice that it follows from Eq. (7) and Preference Assumption B that the set $\mathcal{C}$ satisfies $P(E) = P'(E)$ for all $P, P' \in \mathcal{C}$. It also follows that the notion of subjective mixture is independent of the essential event E used to construct it, in the following sense: If $F \neq E$ is an essential event such that $\succcurlyeq$ is complete on $\mathcal{F}_F$, then $x\, F\, y \sim c_{xFz}\, F\, c_{zFy}$ if and only if $x\, E\, y \sim c_{xEz}\, F\, c_{zEy}$. Therefore, as long as F is essential and such that the decision maker can rank any bet on or against F, it will induce the same mixture structure $(X, \sim, \oplus)$.

4 The Related Literature

Ever since AA's seminal paper [1], it has been well-understood that introducing a mixture operation on the choice set makes it possible to provide simple and intuitive axiomatizations of preferences under uncertainty. To the best of our knowledge, the present paper is the first attempt to provide a generalization of AA-style axiomatics and techniques that applies to a wide range of preference models (all previous papers we are aware of focus on specific preference models).

The paper that is closest to ours is Casadesus-Masanell, Klibanoff and Ozdenoren [3], which provides an axiomatization of the MEU model in a Savage-style setting. Their axioms employ standard sequences (a tool from measurement theory) to identify acts whose utility profiles are convex combinations of the utility profiles of other acts. Thus, altough these authors do not explicitly define a mixture operation, their approach is similar in spirit to ours. Since standard sequences are relatively more complex constructs than our notion of preference average, the statements and interpretations of their versions of axioms A2 and A3 are more involved than those of the axioms presented in this paper. However, their characterization result is essentially equivalent to Proposition 10.

Other papers employ an 'event-based' notion of mixture of acts for axiomatic purposes — but without deriving from it a mixture-set structure with the algebraic properties we are interested in. Such notion of mixture is due to Gul [15], and may be briefly described as follows. Given an event A, call a certainty equivalent of the bet $x\,A\,y$ the 'A-preference average' of x and y. Then define the **eventwise (A-) mixture** of two acts as the act whose payoff at any given state is the A-preference average of the outcomes assigned at that state by the original acts.[12]

Gul introduces eventwise mixtures to provide an axiomatization of SEU which does not require the state space to be infinite. He assumes the existence of an 'ethically neutral' event A such that the preference over bets on A have an SEU representation; in our notation, A satisfies $\rho(A) = \rho(A^c) = 1/2$. He then uses A-mixtures to formulate an independence axiom that implies our axiom A1, duly extended to any triple of acts. His result can thus be seen as a special case of Proposition 9.

In a companion to the paper discussed above, Casadesus *et al.* [4] use eventwise mixtures to obtain a different axiomatization of the MEU model. They assume the existence of an event A such that the bets on A have a SEU representation (i.e., such that $\rho(A)+\rho(A^c) = 1$), and use A-mixtures for their key axioms. While their ambiguity hedging axiom is analogous to axiom A3, their certainty independence axiom is significantly different from axiom A2. The preferences they describe are a subset of those described by Proposition 10.

Chew and Karni [5] generalize Gul [15] in two respects. First, they characterize CEU preferences (the preferences they describe coincide with those described by Proposition 9).

[12]Formally, the A-mixture of acts f and g is the act $f\,A\,g$ defined by $f\,A\,g(s) = c_{f(s)\,A\,g(s)}$ for every $s \in S$.

Second, they show that, for this purpose, it is enough to use eventwise A-mixtures with respect to an arbitrary essential event A; i.e., the bets on/against A need not have an SEU representation. As a result of this generalization, their comonotonic independence axiom is quite dissimilar from axiom A1.[13]

Machina [19] moves along an altogether different direction from the papers discussed thus far in showing how to construct 'almost-objective' events in a Savage-style setting. He assumes that the state space has a Euclidean structure, and that preferences satisfy an 'event smoothness' condition; he then constructs sequences of events that, in the limit, are treated by the decision-maker 'as if' they had an 'objective' (agreed upon) probability. He investigates the properties of almost-objective events, and also suggests constructing 'almost-objective mixtures' of acts. The latter differ from our subjective mixtures, in that they do not yield a preference average of the outcomes in every state, so that their utility profile is not the convex combination of the utility profiles of the acts being mixed.

Finally, we note that there exist characterizations of SEU that are similar in spirit to AA's original result, but do not require postulating the existence of lotteries with objective probabilities. In particular, Pratt, Raiffa and Schleifer [24] assume that the state space is the Cartesian product of a finite set E of 'real-world elementary events', and the set $[0,1]\times[0,1]$, viewed as the collection of outcomes of a 'hypothetical experiment'. Their axioms imply that the decision maker's subjective beliefs about events in $[0,1]\times[0,1]$ are uniform. They show that such hypothetical experiment considerably simplifies the development of the SEU model. However, they do not use this device to derive a notion of mixture, and their analysis does not employ the vector-space techniques that distinguish the AA approach.

[13] Furthermore, it can be shown that this generalization actually makes it impossible to construct a mixture-set structure over the set $\mathcal{F}$ that preserves the isomorphism with convex combinations of utilities.

Appendix A An Integral for Capacities

The notion of integral used for capacities is the **Choquet integral**: For a given Σ-measurable function $\varphi : S \to \mathbb{R}$, the Choquet integral of φ with respect to a capacity ρ is defined as:

$$\int_S \varphi \, d\rho = \int_0^\infty \rho(\{s \in S : \varphi(s) \geq \alpha\}) d\alpha + \int_{-\infty}^0 [1 - \rho(\{s \in S : \varphi(s) \geq \alpha\})] d\alpha \tag{8}$$

where the r.h.s. is a Riemann integral (which is well defined because ρ is monotone).

Appendix B Proof of Proposition 3

Let $E \in \Sigma$ be an essential event and $x \succcurlyeq y$ two consequences. A consequence z is such that

$$x \succcurlyeq z \succcurlyeq y \quad \text{and} \quad x \, E \, y \sim c_{xEz} \, E \, c_{zEy} \tag{9}$$

iff $u(x) \geq u(z) \geq u(y)$ and $V(x \, E \, y) = V(c_{xEz} \, E \, c_{zEy})$. Setting $r = \rho(E) \in (0, 1)$ we obtain

$$\begin{array}{rcl} V(x \, E \, y) & = & u(x)r + u(y)(1 - r), \\ V(x \, E \, z) & = & u(x)r + u(z)(1 - r), \\ V(z \, E \, y) & = & u(z)r + u(y)(1 - r), \end{array}$$

so that, in particular, $c_{xEz} \succcurlyeq c_{zEy}$. Using these equations, we have

$$\begin{array}{rcl} V(c_{xEz} \, E \, c_{zEy}) & = & u(c_{xEz})r + u(c_{zEy})(1 - r) \\ & = & V(xEz)r + V(zEy)(1 - r) \\ & = & [u(x)r + u(z)(1 - r)]r + [u(z)r + u(y)(1 - r)](1 - r) \\ & = & u(x)r^2 + u(y)(1 - r)^2 + 2u(z)r(1 - r). \end{array}$$

Thus, a consequence z satisfies Eq. (9) iff $u(x) \geq u(z) \geq u(y)$ and

$$u(x)r + u(y)(1 - r) = u(x)r^2 + u(y)(1 - r)^2 + 2u(z)r(1 - r).$$

The last equation is in turn equivalent to

$$u(z) = \frac{u(x) + u(y)}{2} \tag{10}$$

(which also implies $u(x) \geq u(z) \geq u(y)$).

Since $u(X)$ is convex, for all $x \succcurlyeq y$ there exists a $z \in X$ such that Eq. (10) is satisfied. This concludes the proof, since the case $y \succcurlyeq x$ is dealt with just exchanging the names of the outcomes, and the other statements follow immediately.

Appendix C Weighted Preference Averages

A **dyadic rational** is a number $\gamma \in (0,1)$ of the form

$$\gamma = \sum_{i=1}^{\infty} \frac{a_i}{2^i} = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + ...,$$

where $a_i \in \{0,1\}$ for every i and $a_i = 1$ for finite indexes i. The **length** of γ is $\ell(\gamma) = \max\{i \geq 1 : a_i = 1\}$. If $\ell(\gamma) = 1$, then $\gamma = \frac{1}{2}$. If $\ell(\gamma) > 1$ and

$$\gamma' = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i} = \frac{a_2}{2} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + ...,$$

then $\ell(\gamma') = \ell(\gamma) - 1$ and $\gamma = (a_1 + \gamma')/2$.

Notice that if $x \sim \bar{x}$ and $y \sim \bar{y}$, then $(1/2)\, x \oplus (1/2)\, y \sim (1/2)\, \bar{x} \oplus (1/2)\, \bar{y}$. In particular, if $x \in X$ and w is an arbitrary element of an indifference class $\mathbf{w}$, then $z \sim (1/2)\, x \oplus (1/2)\, w$ defines in turn an indifference class.

Let $x, y \in X$ be such that $x \succcurlyeq y$. If $\ell(\gamma) = 1$ we say that $z \in X$ is a **γ-preference average** of x and y if z is a preference average of x and y: i.e. $z \sim (1/2)\, x \oplus (1/2)\, y$. If $\ell(\gamma) > 1$, we say that $z \in X$ is a **γ-preference average** of x and y if

$$z \sim \begin{cases} \frac{1}{2}x \oplus \frac{1}{2}w & \text{if } a_1 = 1 \\ \frac{1}{2}\, y \oplus \frac{1}{2}w & \text{if } a_1 = 0 \end{cases} \tag{11}$$

where w is a **γ'-preference average** of x and y.

Notice that for all dyadic rationals γ, γ-preference averages form an indifference class.[14] Therefore, we can denote by $\gamma\, x \oplus (1-\gamma)\, y$ or by $(1-\gamma)\, y \oplus \gamma\, x$ a representative of the class.

Lemma 12 *For any dyadic rational $\gamma \in (0,1)$, $z \in X$ is a γ-preference average of x and y iff*

$$u(z) = \gamma\, u(x) + (1-\gamma)\, u(y).$$

Proof: Let $x \succcurlyeq y$. The proof is by induction on $\ell(\gamma)$. If $\ell(\gamma) = 1$, then the result follows from Proposition 3. Suppose that the result holds for all γ such that $\ell(\gamma) \leq n$, and let

[14] If $\ell(\gamma) = 1$, γ-preference averages of x and y are preference averages, hence an indifference class. Suppose that the result holds for all γ such that $\ell(\gamma) \leq n$, and let $\ell(\gamma) = n+1$. Then the claim follows from Eq. (11).

$\ell(\gamma) = n+1$. If $a_1 = 1$ (so that $\gamma = (1+\gamma')/2$),

$$\begin{aligned} u(z) &= u\left(\frac{1}{2}\,x \oplus \frac{1}{2}\,w\right) = \frac{1}{2}\,u(x) + \frac{1}{2}\,u(w) \\ &= \frac{1}{2}\,u(x) + \frac{1}{2}\,(\gamma'\,u(x) + (1-\gamma')\,u(y)) \\ &= \left(\frac{1}{2} + \frac{1}{2}\,\gamma'\right) u(x) + \left(\frac{1}{2} - \frac{1}{2}\,\gamma'\right) u(y) \\ &= \gamma\,u(x) + (1-\gamma)\,u(y). \end{aligned}$$

The case of $a_1 = 0$ is dealt with symmetrically. *Q.E.D.*

For any $x \succcurlyeq y$ in X and any $\beta \in (0,1)$ which is not a dyadic rational, there exist two sequences of dyadic rationals $\beta_n^- \uparrow \beta$ and $\beta_n^+ \downarrow \beta$. Call a β-**preference average** of x and y a consequence $z \in X$ such that

$$\beta_n^+ x \oplus (1-\beta_n^+) y \succcurlyeq z \succcurlyeq \beta_n^- x \oplus (1-\beta_n^-) y$$

for all $n \in \mathbb{N}$. Again it is easy to verify that β-preference averages form an indifference class, defined by the equality $u(z) = \beta\,u(x) + (1-\beta)\,u(y)$. Finally, call a 1-**preference average** (resp. 0-**preference average**) of x and y any consequence $z \sim x$ (resp. $z \sim y$).

We conclude that for all $\alpha \in [0,1]$, $z \in X$ is an α-preference average of x and y iff $u(z) = \alpha\,u(x) + (1-\alpha)\,u(y)$. The collection of such z is an indifference class, so that we denote indifferently by $\alpha x \oplus (1-\alpha)\,y$ or $(1-\alpha)\,y \oplus \alpha x$ a representative of it.

Appendix D Proof of Proposition 11

Necessity is straightforward. We prove sufficiency. Let u and V be respectively the utility and functional from Preference Assumption B. Since u is nonconstant (recall that a biseparable preference is nontrivial), we can choose u s.t. $u(X) = [0,1]$. We define a binary relation on $B(\Sigma, [0,1])$ as follows: For all $f, g \in \mathcal{F}$, we let $u \circ f \succsim_1 u \circ g$ iff $f \succcurlyeq g$.[15] The following are then true:

- $\succsim_1$ is well-defined and monotonic: Assume $u \circ f = u \circ f'$ and $u \circ g = u \circ g'$. By monotonicity $f \sim f'$ and $g \sim g'$. Then $f \succcurlyeq g$ iff $f' \succcurlyeq g'$. The proof of monotonicity is obvious.
- $\succsim_1$ is reflexive: for all $f \in \mathcal{F}$, $f \sim f$, hence $u \circ f \sim_1 u \circ f$.
- $\succsim_1$ is transitive: If $u \circ f \succsim_1 u \circ g$ and $u \circ g \succsim_1 u \circ h$, then $f \succcurlyeq g$ and $g \succcurlyeq h$, hence $f \succcurlyeq h$ implying $u \circ f \succsim_1 u \circ h$.

[15]For the rest of this appendix we denote the utility profile induced by act f by $u \circ f$ rather than by $u(f)$.

- $\succsim_1$ is continuous: Assume that $u \circ f_\alpha \to u \circ f$ and $u \circ g_\alpha \to u \circ g$ and $u \circ f_\alpha \succsim_1 u \circ g_\alpha$, implying $f_\alpha \succcurlyeq g_\alpha$, for all $\alpha \in D$. Taking subnets we can assume that $f_\beta \to f'$ and $g_\beta \to g'$ (as $\mathcal{F} = X^S$ is compact). By axiom B3 $f' \succcurlyeq g'$, whence $u \circ f' \succsim_1 u \circ g'$. Finally, we observe that $u \circ f' = u \circ f$ and $u \circ g' = u \circ g$ since $u \circ f_\beta \to u \circ f$ and $u \circ g_\beta \to u \circ g$. This shows that $u \circ f \succsim_1 u \circ g$.

- $\succsim_1$ is independent: Let $f, g, h \in \mathcal{F}$ and suppose that $u \circ f \succsim_1 u \circ g$. Then $f \succcurlyeq g$ implies $\alpha f \oplus (1-\alpha)h \succcurlyeq \alpha g \oplus (1-\alpha)h$ for all $\alpha \in [0,1]$, by axiom B4. This implies

$$\begin{aligned} \alpha\, u \circ f + (1-\alpha)\, u \circ h &= u \circ (\alpha\, f \oplus (1-\alpha)\, h) \\ &\succsim_1 u \circ (\alpha\, g \oplus (1-\alpha)\, h) \\ &= \alpha\, u \circ g + (1-\alpha)\, u \circ h. \end{aligned}$$

The statement in the proposition now follows from a standard result, whose proof we omit (see, e.g., Ghirardato, Maccheroni, Marinacci [10, Prop. 42]):

Lemma 13 *$\succsim$ is a nontrivial, independent, continuous and monotonic preorder on $B(\Sigma, [0,1])$* if and only if *there exists a nonempty, closed and convex set C of probability measures on Σ such that*

$$\varphi \succsim \psi \Longleftrightarrow \int_S \varphi\, dP \geq \int_S \psi\, dP \quad \textit{for all } P \in C.$$

Moreover, such C is unique.

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