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RESEARCH ARTICLE

Randomizing without randomness

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Abstract
We provide a methodology for eliciting utility midpoints from preferences, assuming that payoffs are consumption plans and that preferences satisfy a minimal form of additive separability. The methodology does not require any subjective or objective uncertainty. Thus, this construction of utility midpoints allows us to define mixtures of acts in a purely subjective fashion, without making any assumptions as to the decision maker's reaction to the uncertainty that may be present. This approach makes it possible to provide a simple and fully subjective characterization of the second-order subjective expected utility model, allowing a clear distinction of such model from subjective expected utility.

Keywords Subjective mixture space · Ambiguity · Second-order subjective expected utility

JEL Classification D81

1 Introduction

The success of the Anscombe–Aumann (1963) setting for decision models is due to the objective "mixture-space" structure on the choice set, which makes it easy to formulate axioms with a direct mathematical counterpart and to employ functional analysis. More precisely, letting X denote a convex set of consequences, $\alpha x + (1-\alpha) y$

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denotes a *mixture* of the payoffs $x, y \in X$ with weight $\alpha \in [0, 1]$, which also belongs to X. The preferences are assumed, *via* a property called Risk Independence,[1] to be consistent to such mixture. That is, given a preference representation U, its restriction to X satisfies the affinity property

$$U(\alpha x + (1 - \alpha)y) = \alpha U(x) + (1 - \alpha)U(y). \quad (1)$$

Under these conditions every choice option f (a function from a set S of states of the world to payoffs) can be transformed into a utility profile $U \circ f$, and the set of utility profiles inherits from X the mixture-space structure. As a consequence, standard functional-analytic techniques can be applied to model behavioral traits; for example, ambiguity sensitivity (see e.g. Schmeidler 1989; Gilboa and Schmeidler 1989).

On the other hand, the existence of the objective mixture-space structure and Risk Independence limit the scope of preferences that can be captured within an Anscombe-Aumann setting. For instance, Risk Independence rules out the possibility that, when the mixtures represent objective randomizations over payoffs, the decision maker has non-expected utility preferences. For this reason, Ghirardato and Pennesi (2020), extending Ghirardato et al. (2003), advocate using bets on a suitably chosen event $E \subseteq S$ to construct a *subjective* mixture-space structure for which the affinity property in Eq. (1) is automatically satisfied. Though more general, this approach still imposes restrictions on the decision maker's preferences; in particular, with respect to how she reacts to the uncertainty entailed by the event E.[2]

In this paper, we follow a different route. We show that when payoffs are consumption plans and preferences satisfy a separability requirement over such plans, it is possible to develop an alternative *subjective* mixture-space structure which is totally independent of the uncertainty in the problem (and thus of the decision maker's attitude towards such uncertainty). The separability requirement is mild; indeed, our analysis applies to the case in which the decision maker discounts future utilities geometrically or hyperbolically, as well as to more general forms of discounting and nonseparability (e.g. Uzawa 1968). We develop a procedure for eliciting "utility midpoints" from preferences: for any two consumption plans a and b, we show how to find a third consumption plan c, the utility of which is halfway between the utility of a and the utility of b.[3] This procedure requires only *finitely* many preference statements. Utility midpoints are then used recursively to super-impose a mixture-space structure on the set of consumption plans, which is clearly subjective.

As in Ghirardato and Pennesi (2020), the subjective mixture structure can be used to provide a general framework for preference models.[4] However, an advantage of

[1] The Risk Independence axiom posits that if $x, y \in X$ then, $x \succcurlyeq y$ if and only if $\alpha x + (1 - \alpha)z \succcurlyeq \alpha y + (1 - \alpha)z$, for all $z \in X$ and all $\alpha \in [0, 1]$.

[2] Loosely, the decision maker's preferences over bets on the event E must have a Choquet-like (locally biseparable) representation; see Ghirardato and Pennesi (2020) for details. A similar comment can be made about alternative approaches to the fully subjective axiomatizations presented by Gul (1992), Nau (2006) and Ergin and Gul (2009). See the discussion in Sect. 1.1.

[3] Our mild separability requirement implies the cardinal uniqueness of the utility function, which guarantees that utility midpoints are meaningfully defined.

[4] Subjective mixtures could be also of use beyond axiomatic endeavors, see e.g. Webb (2013).

the alternative route taken here is that by dispensing with uncertainty and with the restrictions of the decision maker's reaction to such uncertainty, we are able to capture differences between preference models that cannot be captured with the previous "subjective" constructions. For example, in our context it is possible to provide a simple axiomatic characterization of the Second-Order Subjective Expected Utility (SOSEU) model, as distinct from Subjective Expected Utility (SEU). In other settings this is impossible without ancillary assumptions (see Strzalecki 2011).

To provide intuition on our analysis, consider the case in which there are only two periods, preferences are additively separable and there is no discounting: $U(d) = u(x_0) + u(x_1)$ for all $d = (x_0, x_1) \in X \times X$ and for some $u : X \to \mathbb{R}$. Consider two constant consumption plans $a = (x, x)$ and $b = (y, y)$. In this case, the plan $c = (x, y)$ provides an intuitive "mixture" of the plans $a = (x, x)$ and $b = (y, y)$. And indeed:

$$\begin{aligned} U(c) = u(x) + u(y) &= \left(\frac{1}{2}u(x) + \frac{1}{2}u(x)\right) + \left(\frac{1}{2}u(y) + \frac{1}{2}u(y)\right) \\ &= \frac{1}{2}U(a) + \frac{1}{2}U(b) \end{aligned} \tag{2}$$

So $c = (x, y)$ is also a "midpoint" in terms of utility of the plans a and b. Notice that a similar property is satisfied by the mirror-image plan $c' = (y, x)$, so that $c = (x, y) \sim (y, x) = c'$. So, in this case, the utility midpoint of $a = (x, x)$ and $b = (y, y)$, though not unique, can be directly observed from preferences.

Consider now two non-constant consumption plans $a = (z_0, z_1)$ and $b = (y_0, y_1)$, and the left panel of Fig. 1 which depicts indifference curves on $X \times X$ for a given U. Neither (z_0, y_1) nor (y_0, z_1) provide a "midpoint" of a and b. The reason is that $(z_0, y_1) \nsim (y_0, z_1)$, and indifference has to hold when either pair is a midpoint of a and b.[5] However, this does not mean that we cannot find a midpoint of a and b. We can do so by moving along the indifference curves A and B that respectively include a and b. Consider the right panel of Fig. 1, where we keep the plan b fixed and find a plan $a' = (x_0, x_1)$ belonging to A such that $(y_0, x_1) \sim (x_0, y_1)$, which implies $U(y_0, x_1) = \frac{1}{2}U(y_0, y_1) + \frac{1}{2}U(x_0, x_1)$. That is, (y_0, x_1) is a midpoint of a' and b, and therefore also a midpoint of a and b. Thus, "mixing" payoffs may work also for non-constant consumption plans in $X \times X$.[6]

This approach extends to more general preference models that satisfy our separability condition over discrete (but not necessarily finite) time. Once midpoints are defined for a generic pair of plans a and b, a recursive application of midpoints allows us to define the $\gamma : (1 - \gamma)$ mixture of a and b for any dyadic rational $\gamma \in [0, 1]$. As in Ghirardato et al. (2003), we then use this mixture notion to provide a subjective structure on the set of consumption plans which allows, *à la* Anscombe-Aumann, to define "subjective mixtures" over the set of state-contingent consumption plans in problems of choice under uncertainty.

5 Indeed, because of additive separability and analogously to Eq. (2) above, $U(y_0, z_1) = \frac{1}{2}U(y_0, y_1) + \frac{1}{2}U(z_0, z_1)$ if and only if $U(z_0, y_1) = \frac{1}{2}U(y_0, y_1) + \frac{1}{2}U(z_0, z_1)$ implying $U(y_0, z_1) = U(z_0, y_1)$.

6 As we discuss in detail in Sect. 3, there might be still pairs of plans for which the "mixing" technique does not yield a specific midpoint. However, in those cases, an appropriate (finite) iteration of the procedure just sketched can be used to elicit midpoints.

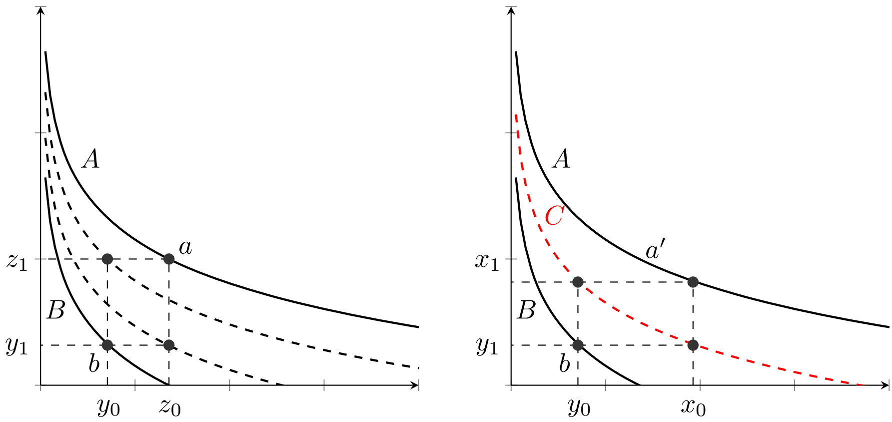

Fig. 1 Looking for preference midpoints

As mentioned above, the notion of subjective mixture allows us to offer fully subjective axiomatizations of very general preference models. For instance, we characterize the Monotone, Bernoullian and Archimedean (MBA) preferences of Cerreia-Vioglio et al. (2011), which include as special cases most of the (static) models of choice under uncertainty. We are also able to characterize the SOSEU model without imposing any restriction on the DM's attitudes toward ambiguity (as described by the second-order utility function ϕ, see Definition 4) or "risk" (as described by the utility function U). Indeed, in our most general characterization the function ϕ is increasing but not necessarily concave or convex. Analogously, the function U is arbitrary and it does not have to reflect any structure on the space of consequences, as would be entailed by properties such as by Risk Independence. We also show that a number of interesting special cases of SOSEU, including for instance Multiplier Preferences (Hansen and Sargent 2001), follow by adding the natural counterpart in our setting of well-known preference axioms; e.g., Weak Certainty Independence.

The paper is structured as follows: Sect. 2 spells out the basic assumptions on preferences, the definition of utility and preference midpoint, and it also contains our main characterization result. Section 3 provides intuition for the existence and elicitation of utility midpoints. Section 4 concludes by providing the main decision-theoretic applications with the axiomatizations of MBA preferences and of the SOSEU model. The appendix contains a few additional results and all the proofs of the results in the paper.

1.1 Related literature

The three works that are most directly related to this paper are: Kochov (2015), Bastianello and Faro (2020) and Vind and Grodal (2003). The works of Kochov (2015) and Bastianello and Faro (2020) use separability of preferences over state-contingent consumption plans to provide a fully subjective axiomatization of some preference models. In particular, Kochov (2015) axiomatizes the MaxMin Expected Utility model of Gilboa and Schmeidler (1989) and the Variational Preference model of Maccheroni et al. (2006) in a context in which acts are functions from states of the world to infinite-horizon consumption plans. Bastianello and Faro (2020) analogously axiomatize the Choquet Expected utility. Like us, these papers do not require the existence of an objective randomization device, but differently from us their approach crucially depends on the infinity of the time horizon and on geometric discounting of future utilities. Instead, we derive our results for a possibly finite horizon (more generally, we allow for a finite or infinite product of potentially non-homogeneous sets, see Remark 1), and we dispense with the stationarity assumption implied by geometric discounting.[7]

The work of Vind and Grodal (2003) is directly related to our main theoretical contribution (Theorem 1). We adopt their notion of preference midpoint and show that it can be used to provide a foundation to the notion of utility midpoints. As we explain in detail in Sect. 3, preference midpoints may not always exist. In contrast, we prove the general existence of utility midpoints and we present a finite algorithm for their elicitation. Another difference is that Vind and Grodal (2003) do not define midpoints with the objective of building a mixture-space structure, but only as a tool to provide conditions under which preferences have an additively separable representation.

Finally, as mentioned earlier, our construction of subjective mixtures is complementary to those that propose subjective mixture operators using bets on "special" events; see Footnote 2 and the related discussion. As the title of this paper suggests, we are able to elicit utility midpoints even in the absence of any randomness.

2 Midpoints

In this section, we present our main definition and its behavioral characterization. We consider the case in which the consequences are consumption plans, hence elements of a homogeneous product space $V = X^{T+1}$, with $T \in \mathbb{N} \cup \{\infty\}$. We denote by a a generic consumption plan in V.

2.1 Minimal additive separability

The following basic assumption on preferences, which is necessary to obtain a well-defined notion of midpoint in terms of utility in our setting, is crucial for our analysis:

Definition 1 *(MAS)* We say that $U : V \to \mathbb{R}$ is a *Minimally Additively Separable (MAS)* utility function over the set of consumption plans $V := X^{T+1}$ where $T \in \{1, 2, \ldots\} \cup \infty$, if $U : V \to \mathbb{R}$ is a convex-ranged mapping for which there exist a proper partition $\{I_0, I_1\}$ of $\{0, 1, \ldots, T\}$ and two functions[8] $u_0 : X^{I_0} \to \mathbb{R}$ and $u_1 : X^{I_1} \to \mathbb{R}$ such that for all $a \in V$

$$U(a) = u_0(a_{I_0}) + u_1(a_{I_1}) \tag{3}$$

[7] Another difference is that Kochov (2015) provides an extension of his model to an explicitly dynamic setting, which we do not attempt in this paper.

[8] It is shown by standard arguments that the range convexity of a U satisfying Eq. (3) is equivalent to the range convexity of u_0 and u_1 separately.

where $a_K \in \prod_{k \in K} X$, $K \in \{I_0, I_1\}$.

The axioms characterizing preferences with a MAS utility representation are standard (see Appendix A). Notice that a MAS representation is necessarily cardinally unique.

Any additively separable representation, $U(a_0, \ldots, a_T) = \sum_{t=0}^{T} u_t(a_t)$ is MAS. This includes the case of discounted utility representations, $U(a_0, \ldots, a_T) = \sum_{t=0}^{T} D(t)u(a_t)$, *regardless* of the shape of the discount function (e.g. geometric, hyperbolic, quasi-hyperbolic).

On the other hand, requiring just the existence of a MAS representation is much weaker than requiring additive separability. This is illustrated by the following examples.

Example 1 Consider the preference for increasing consumption profiles of De Waegenaere and Wakker (2001) (see also Gilboa 1989) represented by

$$U(a) = \lambda_0 a_0 + \sum_{t=1}^{T} [\lambda_t a_t - \tau_t (a_{t-1} - a_t)^+]$$

where $X \subseteq \mathbb{R}$, $T \in \mathbb{N} \cup \{\infty\}$ and for a vector of non-negative τ_t, and where $(r)^+ = \max\{r, 0\}$. If there is at least one t such that $\tau_t = 0$, U is a MAS utility representation with $I_0 = \{t - 1\}$.

Example 2 A classical example of non-separable discounted utility model is Uzawa's consumption-dependent discounting (Uzawa 1968): $U(a) = u(a_0) + \sum_{t=1}^{T} \prod_{s=0}^{t-1} \delta(a_s) u(a_t)$, where $T \in \mathbb{N} \cup \{\infty\}$. However, if $\delta(a_0) = \delta$ for all $a_0 \in X$, so that

$$U(a) = u(a_0) + \delta u(c_1) + \sum_{t=2}^{T} \prod_{s=1}^{t-1} \delta(a_s) u(a_t)$$

then U is a MAS utility representation with $I_0 = \{0\}$. More generally, consider a preference represented by an "initial additive" utility, such as

$$U(a) = u_0(a_0) + f(a_1, \ldots, a_T)$$

for a finite T or $U(a) = u_0(a_0) + f(a_1, \ldots)$ for an infinite T. Then, U is a MAS utility representation with $I_0 = \{0\}$.

We conclude this section with two observations on the scope and characterization of the MAS assumption.

Remark 1 It will be seen that all our results are independent of the dimensionality of V and the interpretation that we give to its product structure (e.g., consumption plans). The driving force is the existence of a MAS utility. Precisely, all our results apply to the case in which $V = X \times Y$ and that $\succcurlyeq$ is represented by $U : X \times Y \to \mathbb{R}$ convex-ranged and such that $U(x, y) = u_X(x) + u_Y(y)$ where $u_X : X \to \mathbb{R}$ and $u_Y : Y \to \mathbb{R}$.

Remark 2 Taking advantage of Remark 1, the following is a simple characterization of MAS utilities.[9] Suppose that X and Y are non-empty sets; say that U satisfies the *rectangle condition* if for any rectangle with vertices

$$\{(x_0, x_1), (y_0, y_1), (x_0, y_1), (y_0, x_1)\} \subseteq X \times Y$$

it follows that the sum of the values at the diagonally opposite corners are equal. That is:

$$U(x_0, x_1) + U(y_0, y_1) = U(y_0, x_1) + U(x_0, y_1)$$

A utility U is MAS if and only if it satisfies the rectangle condition. To see the non-trivial implication, suppose that U satisfies the rectangle condition and fix $\bar{x} \in X$ and $\bar{y} \in Y$. Define $u_0(x_0) = U(x_0, \bar{y})$ and $u_1(x_1) = U(\bar{x}, x_1) - U(\bar{x}, \bar{y})$. It follows that

$$U(x_0, x_1) = U(\bar{x}, x_1) + U(x_0, \bar{y}) - U(\bar{x}, \bar{y}) = u_0(x_0) + u_1(x_1)$$

2.2 Utility midpoints

Let $\succcurlyeq$ be a preference with a MAS utility representation on V, with $\sim$ denoting indifference. We consider the quotient space $[V] = V/\sim$ and we denote by $A, B \in [V]$ generic indifference classes. With a slight abuse of notation, $U : [V] \to \mathbb{R}$ denotes the restriction of U to the indifference sets in $[V]$. As informally explained in the introduction, midpoints can be directly defined in terms of pairs of indifference sets, as follows:

Definition 2 For any $A, B \in [V]$, $C \in [V]$ is a *utility midpoint* of A and B, denoted $C = A \odot B$, if and only if $U(C) = \frac{1}{2}U(A) + \frac{1}{2}U(B)$.

Clearly, the definition of utility midpoint is based on the representation U. The main contribution of our paper is the following result, which shows that there is a finite preference-based procedure to elicit utility midpoints.

Theorem 1 *If $\succcurlyeq$ has a MAS representation then, for all $A, B \in [V]$ a utility midpoint $C = A \odot B$ (exists and it) can be elicited from preferences via a finite procedure.*

Notice that, because of the MAS assumption, the existence of utility midpoints is guaranteed. The novelty of the result is proving how to use behavioral data to elicit the utility midpoint of an arbitrary pair of consumption plans. More precisely, given a pair of plans $a, b \in V$ and their associated indifference sets $A, B \in [V]$, we provide an algorithm which explicitly identifies an indifference set $C \in [V]$, the elements of which have utility equal to the average of the utilities of a and b.

Our algorithm departs from the behavioral definition of midpoint introduced by Vind and Grodal (2003), where each $a \in V$ is expressed as $a = (x_0, x_1)$ with $x_0 \in X^{I_0}$ and $x_1 \in X^{I_1}$ (see also Fig. 2).[10]

[9] We thank an anonymous referee for suggesting this alternative characterization.

[10] The notation (x_0, x_1) with $x_0 \in X^{I_0}$ and $x_1 \in X^{I_1}$ is a little loose, as (x_0, x_1) does not strictly represent the concatenation of two sub-vectors x_0, x_1 but rather their combination. For example, take (a_0, a_1, a_2, a_3)

Definition 3 For any $A, B \in [V]$, $C \in [V]$ is a *preference midpoint* of A and B if and only if there exist $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$ such that $(x_0, y_1) \sim (y_0, x_1) \in C$.

A problem with this definition is that preference midpoints may not exist for arbitrary $A, B \in [V]$. For instance, consider Fig. 3. The indifference curves A', B', passing through the points (x_0, y_1) and (y_0, x_1) respectively, do not have a preference midpoint (Def. 3)—of course, they do have a utility midpoint (Def. 2).[11] In Sect. 3, we intuitively describe the procedure that allows us to elicit utility midpoints for *all* $A, B \in [V]$, which lies at the core of the proof of Theorem 1.

Remark 3 It can be seen that, if $\succcurlyeq$ is suitably unbounded, so that both u_0 and u_1 have unbounded range (above and below), preference midpoints exist for all $A, B \in [V]$ and they can then be directly employed to obtain utility midpoints. In this case, there is no need to use the algorithm discussed in Sect. 3.

Given $\succcurlyeq$, it is possible that there are multiple partitions of $\{0, \ldots, T\}$ on which the representation U of $\succcurlyeq$ is MAS. For instance, if $V = X^3$ and $U(a) = \sum_{t=0}^{2} \delta^t u(a_t)$, we could have $I_0 = \{0\}$ or $I_0' = \{0, 1\}$. The preference midpoints associated with each partition are different, as the vectors may have different dimensionality. However, the following result shows that the utility midpoints obtained in Theorem 1 are not affected by this multiplicity.

Proposition 1 *Suppose that $\succcurlyeq$ has a MAS representation with respect to the partitions I_0, I_1 and I_0', I_1' of $\{0, \ldots, T\}$, then $C \in [V]$ is a utility midpoint of $A, B \in [V]$ under I_0, I_1 if and only if it is a utility midpoint of $A, B \in [V]$ under I_0', I_1'.*

3 From preference midpoints to utility midpoints. An algorithm

In this section, we provide intuition about the finite procedure behind Theorem 1, as well as a numerical example. The detailed proof is found in Appendix C.1.

We start by observing that, whenever a preference midpoint exists, it is also a utility midpoint in any MAS utility representation. Consider the indifference curves A, B in Fig. 2. The indifference curve C satisfies the definition of preference midpoint of A and B: given the points $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$, we have $(x_0, y_1) \sim (y_0, x_1) \in C$. By the MAS assumption, the indifference is mathematically equivalent to

$$U(C) = u_0(x_0) + u_1(y_1) = u_0(y_0) + u_1(x_1) \tag{4}$$

which implies $2U(C) = u_0(x_0) + u_1(y_1) + u_0(y_0) + u_1(x_1)$, and hence

$$U(C) = \frac{1}{2}[u_0(x_0) + u_1(x_1)] + \frac{1}{2}[u_0(y_0) + u_1(y_1)] = \frac{1}{2}U(A) + \frac{1}{2}U(B)$$

and $I_0 = \{0, 2\}$ so that $I_1 = \{1, 3\}$, by writing (x_0, x_1) we mean $(x_0, x_1) = (a_0, a_1, a_2, a_3)$ (and not (a_0, a_2, a_1, a_3)).

[11] In contrast, the indifference curves A, B in Fig. 3 have a preference midpoint which, under the assumption of Theorem 1, coincides with the utility midpoint.

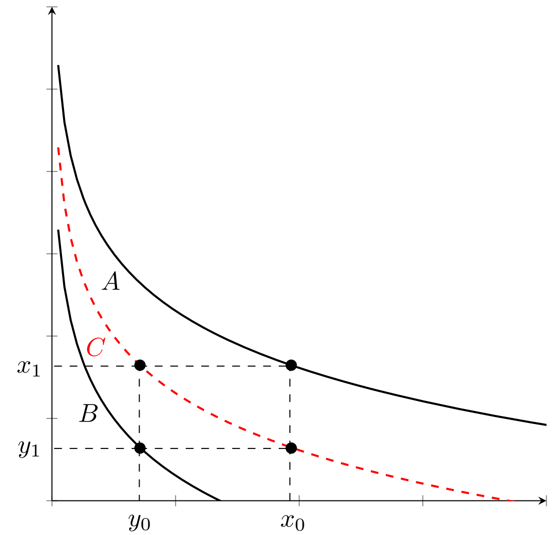

Fig. 2 The preference midpoint of A and B

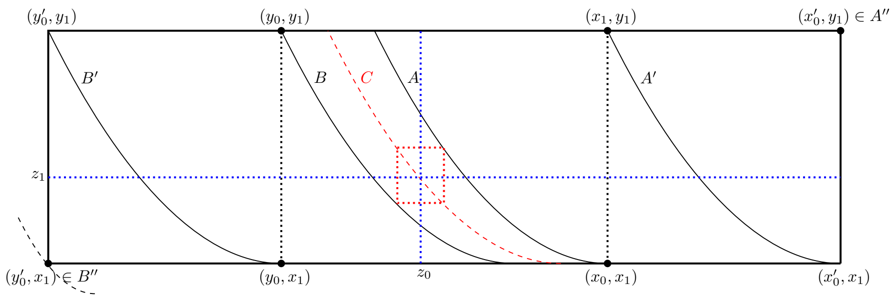

Fig. 3 Elicitation of utility midpoints

The previous argument only works for pairs $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$ for which we can establish the indifference $(x_0, y_1) \sim (y_0, x_1)$. This will not be the case for arbitrary choices of pairs of points in A and B. However, when A and B are sufficiently "close," a preference (hence utility) midpoint of A and B always exists (Lemma 2), in the sense that there will be $(x'_0, x'_1) \in A$ and $(y'_0, y'_1) \in B$ for which the indifference $(x'_0, y'_1) \sim (y'_0, x'_1)$ holds. By "closeness" of the indifference sets A and B we mean the following: there is a pair of payoffs (z_0, z_1) and points in the sets A and B which feature z_0 or z_1 as coordinates. For illustration refer to Fig. 3.

The indifference sets A and B in the center of the rectangle are "close" (see the payoffs z_0 and z_1 and the intersections of the blue-dotted lines with the sets A and B). In contrast, the indifference sets A' and B' are not "close," as it is impossible to find an analogous pair (z_0, z_1) (precisely there is no z_0 that works).

When pairs of indifference classes are not "close," utility midpoints can still be elicited by exploiting a consequence of the MAS representation, called the Diagonal property (Lemma 3): If $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$ and C is the utility midpoint of A and B, then C is also the utility midpoint of the classes A' and B' which respectively contain the vectors (x_0, y_1) and (y_0, x_1) derived from (x_0, x_1) and (y_0, y_1) by switching the second coordinate. Referring again to Fig. 3, the Diagonal property then implies that the indifference set C is also the utility midpoint of A' and B'. Another application of the Diagonal property and another coordinate switch allow us to conclude that C is also the utility midpoint of A'' and B''.

Reversing this procedure yields the algorithm to elicit the utility midpoint of two arbitrary indifference sets $A'', B'' \in [V]$. Suppose without loss of generality that $A'' \succ B''$. First, we find vectors belonging to A'' and B'' that are Pareto ranked, such as $(y_0', x_1) \in B''$ and $(x_0', y_1) \in A''$. This allows us to restrict attention to the closed and bounded (in terms of utility values) rectangle defined by the vertices (y_0', x_1), (y_0', y_1), (x_0', x_1), (x_0', y_1) as in Fig. 3. Next, we use coordinate switches to construct a *finite* sequence of pairs of indifference sets $\{(A_n, B_n)\} \in [V] \times [V]$ such that, eventually, A_n and B_n are "close," and hence have a utility midpoint C (see Lemma 4). The Diagonal property then ensures that C is the utility midpoint of A_n and B_n for all n, and hence of the initial A'' and B''. The fact that the sequence is finite follows from the observation that the rectangle defined above is compact (in terms of utilities), and that at every step, the utility difference $U(A_n) - U(B_n)$ is reduced by a constant amount.

For concreteness, consider $X = \mathbb{R} \times \mathbb{R}$ with $U(x_0, x_1) = x_0 + x_1$, and assume $U(A'') = 10$ and $U(B'') = 0$. We can find two Pareto-ranked vectors $(y_0', x_1) \in B''$ and $(x_0', y_1) \in A''$; for example, $(y_0', x_1) = (0, 0)$ and $(x_0', y_1) = (8, 2)$. Using a coordinate switch, we obtain $(0, 2) \in B'$ and $(8, 0) \in A'$. Moving along the indifference curves of $(0, 2)$ and $(8, 0)$, we find the vectors $(2, 0) \in B'$ and $(6, 2) \in A'$, which are again Pareto-ranked. A second coordinate switch yields the vectors $(2, 2) \in B$ and $(6, 0) \in A$. These two vectors are "close" since there are vectors $(3.5, 0.5) \in B$ and $(4.5, 1.5) \in A$ such that $(3.5, 1.5) \sim (4.5, 0.5)$. Therefore, the procedure terminates in $n = 2$ steps, identifying the preference midpoint of A'' and B'' to be C with $U(C) = 5$.

In general, in the proof of Theorem 1 we prove that the number of steps required to find a midpoint of A and B, starting from two Pareto-ranked vectors $A \ni (x_0', y_1) \succ (y_0', x_1) \in B$, is the smallest integer n such that $n \geq \frac{1}{2}\frac{u_0(x_0')-u_0(y_0')}{u_1(y_1)-u_1(x_1)}$ (with the example above, $n \geq \frac{1}{2}\frac{8-0}{2-0} = 2$, as we showed).

4 Applications to choice under uncertainty

This section contains some applications of utility midpoints. Having defined the operator $\odot$, we use it to obtain a subjective mixture operation in a standard setting for decision making under uncertainty. Precisely, the decision maker has a binary relation $\succcurlyeq$ defined on the set $\mathcal{F}$ of all simple Σ-measurable functions $f : S \to V$, where S is a state space endowed with an algebra of events Σ and the set of consequences V is a finite or infinite product space X^{T+1}.

We first provide a fully subjective axiomatic characterization of the Monotone, Bernoullian and Archimedean (MBA) preferences (Cerreia-Vioglio et al. 2011), and then of the Second-Order Subjective Expected Utility (SOSEU) model.

4.1 Monotone, Bernoullian and Archimedean preferences

We assume the following basic axioms:

Axiom 1 (Preference Order—P) $\succcurlyeq$ *is a complete, nontrivial and transitive binary relation on* $\mathcal{F}$.

Axiom 2 (Monotonicity—M) *If* $f(s) \succcurlyeq g(s)$ *for all* $s \in S$*, then* $f \succcurlyeq g$.

Axiom 3 (Minimally Additive Separability on Consequences—MASC) *The restriction of* $\succcurlyeq$ *to* V *has a MAS representation* $U : V \to \mathbb{R}$.

By Theorem 1, the MAS assumption allows us to elicit utility midpoints for every pair of consequences $a, b \in V$. Given $f : S \to V$ and $\succcurlyeq$, the act f induces the correspondence $F : S \rightrightarrows V$, $F(s) = \{a \in V : a \sim f(s)\} \in [V]$. We denote by $\mathbb{F}$ the family of correspondences that are generated by $\mathcal{F}$. It follows from axioms P and M that the binary relation $\succcurlyeq$ can be extended to $\mathbb{F}$.[12] We can then use the operator $\odot$ to define the *quotient-act mixture* $1/2F \oplus 1/2G$ for any $F, G \in \mathbb{F}$ as follows: for any $s \in S$,

$$\left(\frac{1}{2}F \oplus \frac{1}{2}G\right)(s) \equiv F(s) \odot G(s). \tag{5}$$

Thus, suitably restricting the MAS representation U of $\succcurlyeq$ to $[V]$, we obtain

$$U\left[\left(\frac{1}{2}F \oplus \frac{1}{2}G\right)(s)\right] = \frac{1}{2}U(F(s)) + \frac{1}{2}U(G(s))$$

We then consider iterated quotient-act mixtures such as $\frac{1}{2}F \oplus \left(\frac{1}{2}F \oplus \frac{1}{2}G\right)$, which corresponds to a $\frac{3}{4} : \frac{1}{4}$ mixture of F and G. By standard continuity arguments (see Appendix C in Ghirardato et al. 2002) iterated quotient-act mixtures can then be used to define $\alpha F \oplus (1-\alpha)G$ for any $\alpha \in [0, 1]$, so that for any $s \in S$,

$$U[(\alpha F \oplus (1-\alpha)G)(s)] = \alpha U(F(s)) + (1-\alpha)U(G(s)) \tag{6}$$

These mixtures allow us to state the next axiom, which is familiar from traditional Anscombe-Aumann treatments:

Axiom 4 (Full Continuity—FC) *For all* $F, G, H \in \mathbb{F}$*, the sets*

$$\{\alpha \in [0, 1] : \alpha F \oplus (1-\alpha)G \succcurlyeq H\} \quad and \quad \{\alpha \in [0, 1] : H \succcurlyeq \alpha F \oplus (1-\alpha)G\}$$

[12] That is, given $f, f' \in \mathcal{F}$ such that $f \mapsto F$ and $f' \mapsto F$, it follows from axiom P and M and the definition of F that $f \sim f'$.

are closed.

Notice that it follows from axioms P, M, MASC and FC that for every $f \in \mathcal{F}$ there is $A_f \in [V]$ such that $A_f \sim F$. That is, every act has a certainty equivalent in quotient space (notice that, for any $a \in A_f, a \sim f$). Adapting the argument in Cerreia-Vioglio et al. (2011), we show that the axioms stated so far are necessary and sufficient to provide $\succcurlyeq$ with an MBA representation:[13]

Proposition 2 *Axioms P, M, MASC, and FC hold if and only if there exist a non-constant MAS representation $U : V \to \mathbb{R}$, and a monotonic, continuous and normalized functional $I : B_0(\Sigma, U(V)) \to \mathbb{R}$ such that*

$$f \succcurlyeq g \iff I(U(f)) \geq I(U(g))$$

Notice that we do not need to assume Risk Independence, since our U is, by construction, affine with respect to the mixture operator $\oplus$ (see Eq. 6).

The class of MBA preferences includes as special cases most of the models of ambiguity-sensitive preferences: the Maxmin EU model of Gilboa and Schmeidler (1989), the Variational Preferences model of Maccheroni et al. (2006), the Confidence Preferences model of Chateauneuf and Faro (2009), the Smooth Ambiguity model of Klibanoff et al. (2005), the Vector EU model of Siniscalchi (2009), and the Uncertainty Averse Preferences model of Cerreia-Vioglio et al. (2011).

4.2 Second-order subjective expected utility

As noted in Strzalecki (2011), in a pure Savage-style setting SOSEU is observationally equivalent to SEU, as we lack a method to cardinally identify the Bernoulli utility U. Two tools have been employed to solve this identification problem. The first is the assumption of the availability of an objective randomization device (Grant et al. 2009; Cerreia-Vioglio et al. 2012), thus providing the set of consequences with an objective mixture-space structure. The second is the assumption of the existence of multiple sources of uncertainty (Nau 2006; Ergin and Gul 2009), such that the decision maker has SEU preferences with respect to at least one source.

We now show that when consequences are consumption plans and preferences satisfy the MASC axiom, our notion of utility midpoint provides a third tool. Building on the axiomatization of MBA preferences given above and assuming that S is finite,[14] we characterize the following version of SOSEU:

Definition 4 A binary relation $\succcurlyeq$ has a *SOSEU* representation* if there exist a non-constant MAS representation $U : V \to \mathbb{R}$, a continuous and strictly increasing ϕ :

[13] As customary, we denote by $B_0(\Sigma, \Gamma)$ the set of simple Σ-measurable functions on S with values in the interval $\Gamma \subseteq \mathbb{R}$ and we say that $I : B_0(\Sigma, \Gamma) \to \mathbb{R}$ is: *monotonic*, if $I(\phi) \geq I(\psi)$ when $\phi \geq \psi$; *continuous*, if it is sup-norm continuous; *normalized*, if $I(\gamma 1_S) = \gamma$ for all $\gamma \in \Gamma$. In Proposition 2, we omit the uniqueness statement, which is analogous to that in Cerreia-Vioglio et al. (2011).

[14] Finite S is an assumption which is common to most previous axiomatizations of SOSEU (e.g. Nau 2006; Grant et al. 2009).

$U(V) \to \mathbb{R}$ and a probability distribution p on S, such that $\succcurlyeq$ is represented by $J : \mathcal{F} \to \mathbb{R}$ defined by:

$$J(f) = \sum_{s \in S} \phi\left(U(f(s))\right) p(s)$$

We use SOSEU* in place of SOSEU to underscore two main differences between our representation and the traditional model. First, our U necessarily satisfies the MAS assumption. Second, the separate identification of ϕ and U builds just on the vector structure of the set of consequences, and it is not necessarily tied to the availability of an additional source of uncertainty (see also Remark 8).

In what follows, we provide an axiomatic characterization of the SOSEU* representation, as well as some interesting special cases. In particular, we obtain the *Exponential SOSEU** representation, which is defined for $\theta \in (-\infty, 0) \cup (0, \infty]$ as follows:

$$\phi_\theta(U) = \begin{cases} -\exp\left(-\frac{U}{\theta}\right) & \theta \in (-\infty, 0) \cup (0, \infty) \\ U & \theta = \infty \end{cases}$$

We also characterize the popular *Multiplier Preferences* of Hansen and Sargent (2001) (which corresponds to an Exponential SOSEU* with $\theta \in (0, \infty]$). Finally, we discuss the *Power SOSEU** representation, which is defined for $\theta \in (0, \infty)$ as follows:

$$\phi_\theta(U) = U^\theta$$

and the particular case of *Concave Power SOSEU**, corresponding to $\theta \in (0, 1]$.

Remark 4 The functional form of SOSEU* is related to the Generalized Expected Discounted Utility model of DeJarnette et al. (2020) which, in a setting of choice under risk, is designed to capture aversion towards *time lotteries*—lotteries in which the prize is fixed but the award date is random. As they observe, such aversion has significant experimental support, but it is at odds with geometric discounted expected utility. In our setting, assume that $X = [w, b] \subset \mathbb{R}$ and that $U(a) = \sum_{t \geq 0} \delta^t u(x_t)$ with $u(0) = 0$. A time lottery corresponds to a binary act $a_{t-k} E a_{t+k}$, where $a_s = (0, 0, \ldots, x, \ldots, 0, 0)$ is a consumption plan that pays x at time s and 0 otherwise. If $p(E) = \frac{1}{2}$, a SOSEU* with a ϕ which is a concave transformation of ln (see Proposition 2 in DeJarnette et al. 2020) will prefer the constant act a_t to the act $a_{t-k} E a_{t+k}$, thus displaying "risk aversion" towards time acts.

4.2.1 An axiomatic characterization of SOSEU*

To characterize SOSEU*, we need three additional axioms. First of all, we have Savage's Sure-Thing Principle (P2), restated below.[15]

[15] For $f, g \in \mathcal{F}$ and $E \in \Sigma$, $f_E g$ denotes the act which follows f on E and g on E^c. In particular, given $a \in V$ and $f \in \mathcal{F}$, $a_s f$ denotes the act which pays a in state s and follows f otherwise.

Axiom 5 (P2) *For all $E \in \Sigma$ and acts $f, g, h, h' \in \mathcal{F}$, if $f_E h \succcurlyeq g_E h$ then $f_E h' \succcurlyeq g_E h'$.*

Next, we impose an additional monotonicity property in which we define a state $s \in S$ to be *null* if for all $f \in \mathcal{F}$ and $a, b \in V$, $a_s f \sim b_s f$.

Axiom 6 (Statewise Monotonicity—M^+) *For all non-null $s \in S$ and all $a, b, c \in V$, if $a \succ b$ then $a_s c \succ b_s c$.*

The axioms introduced so far allow us to obtain a "state-dependent" version of SOSEU*:

Lemma 1 *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, M^+, MASC, FC, and P2 if and only if there exists a non-constant MAS representation $U : V \to \mathbb{R}$ and continuous functions $\phi_s : U(V) \to \mathbb{R}$, strictly increasing (resp. constant) for any non-null state s (resp. for any null state s), such that $\succcurlyeq$ is represented by:*

$$J(f) = \sum_{s \in S} \phi_s(U(f(s)))$$

If $\{\phi'_s\}_{s \in S}$ also represent $\succcurlyeq$, there are $\alpha > 0$ and $\beta_s \in \mathbb{R}$ such that $\phi_s = \alpha \phi'_s + \beta_s$.

Grant et al. (2009, Theorem 1) provide a similar representation result. The main difference is that they require Risk Independence, which presumes the presence of objective mixtures (see Footnote 1). Instead, we only rely on vector-valued consequences and the MAS assumption.

In the full-fledged SOSEU* representation, the "second-order utility ϕ" is independent of s. We obtain this result by leveraging on the results in the previous sections. Specifically, we derive a mixture space structure conditional on each state s, and we require consistency of mixtures across states.

In order to construct the mixture space structure in state s, we begin by observing that the functional $J(f) = \sum_{s \in S} \phi_s(U(f(s)))$ is an additively separable representation of $\succcurlyeq$, hence J is a MAS representation on $\mathcal{F}$. Therefore, there exists a (utility) midpoint operator $\hat{\oplus}$ on $[\mathcal{F}]$, the quotient of $\mathcal{F}$ with respect to $\succcurlyeq$, identified as follows: for any pair of indifference classes $[f], [g] \in [\mathcal{F}]$, we let

$$[f] \hat{\oplus} [g] = [h] \in [\mathcal{F}] \quad \Longleftrightarrow \quad J([h]) = \frac{1}{2} J([f]) + \frac{1}{2} J([g])$$

where, analogously to what we did for U, we denote by $J([f])$ the restriction of J to $[\mathcal{F}]$. The argument of Theorem 1 can be adapted to the present setting to show that $\hat{\oplus}$ can be elicited from preferences. Given $\hat{\oplus}$, we can define, for each $s \in S$, an operator $\oplus_s$ on the space $[V]$ as follows:

Definition 5 For any $A, B, C \in [V]$, $C = A \oplus_s B$ if and only if there are $a \in A, b \in B, c \in C$ and $h \in \mathcal{F}$ such that $[a_s h] \hat{\oplus} [b_s h] = [c_s h]$.

That is, $C = A \oplus_s B$ if, conditional on state s, for some $h \in \mathcal{F}$,[16] c can be substituted to either a or b so that the act $c_s h$ corresponds to a utility midpoint of the acts $a_s h$ and $b_s h$. By construction, $C = A \oplus_s B$ if and only if C is a midpoint of A and B according to the state-dependent utility function $\phi_s \circ U$. To see this, notice that $C = A \oplus_s B$ if and only if

$$\frac{1}{2} J(a_s h) + \frac{1}{2} J(b_s h) = J(c_s h)$$

which is equivalent to

$$\frac{1}{2}\left(\phi_s(U(a)) + \sum_{s' \in S \backslash s} \phi_{s'}(U(h(s)))\right) + \frac{1}{2}\left(\phi_s(U(b)) + \sum_{s' \in S \backslash s} \phi_{s'}(U(h(s)))\right)$$
$$= \phi_s(U(c)) + \sum_{s' \in S \backslash s} \phi_{s'}(U(h(s)))$$

which, in turn, is equivalent to $\phi_s(U(c)) = \frac{1}{2}\phi_s(U(a)) + \frac{1}{2}\phi_s(U(b))$.

Remark 5 It is important to understand that the operators $\odot$ and $\oplus_s$, while both defined on $[V]$, are not necessarily isomorphic. For an example, consider the case $V = \mathbb{R}_{++} \times \mathbb{R}_{++}$ and $U(x, y) = x + y$. Given two vectors $a = (x, x)$ and $b = (y, y)$, their midpoint C (such that $C = A \odot B$) is such that, for every $c \in C$, $U(c) = x + y$. However, if for example $\phi_s = \ln$ then $c' \in A \oplus_s B$ is such that $U(c') = e^{\frac{1}{2}\ln(2x) + \frac{1}{2}\ln(2y)} \neq U(c)$. Notice that this is still true even if $\phi_s = \ln$ for all $s \in S$; that is, if the $\oplus_s$ operator is state-independent.

Remark 6 The mixture operator $\hat{\oplus}$ can be used to state the following version of the independence axiom: for all $f, g, h \in \mathcal{F}$, $[f] \succcurlyeq [g]$ implies $[f]\hat{\oplus}[h] \succcurlyeq [g]\hat{\oplus}[h]$. It is simple to prove that SOSEU satisfies such axiom, since J is affine with respect to $\hat{\oplus}$: $J([f]\hat{\oplus}[g]) = \frac{1}{2}J([f]) + \frac{1}{2}J([g])$. On the other hand, imposing the independence axiom with respect to the objective mixture $+$ of Anscombe-Aumann, implies that the preference satisfies SEU. This shows that the mixture structure given by $\hat{\oplus}$ is "weaker" than the objective mixture $+$ of Anscombe-Aumann.

Since U is cardinally unique by the MASC axiom, the mixture $\oplus_s$ allows us to cardinally identify the state-dependent function ϕ_s (without cardinally identifying U, it would only be possible to cardinally identify $\phi_s \circ U$).[17] It also enables us to state the desired restriction on preferences:

Axiom 7 (State-wise Cardinal Symmetry—SCS) *For all non-null $s', s'' \in S$ and for all $a, b \in V$, if $c' \in A \oplus_{s'} B$ and $c'' \in A \oplus_{s''} B$, then $c'_s d \sim c''_s d$ for some $d \in V$ and all $s \in S$.*

[16] Under axiom P2 this is true for any $h \in \mathcal{F}$.

[17] To clarify further, without the MASC axiom—hence without cardinally identifying U—the uniqueness part of Lemma 1 would say that if $\{(\phi_s \circ U)'\}_{s \in S}$ also represent $\succcurlyeq$, there are $\alpha > 0$, $\beta_s \in \mathbb{R}$ such that $\phi_s \circ U = a(\phi_s \circ U)' + \beta_s$.

The role of this axiom is quite transparent. The act $c'_s d$ is equal to d in all states different from $s \in S$ and equal to c' in state s. The indifference with $c''_s d$ means that the decision maker is indifferent (by Monotonicity) between the constants c' and c''. Therefore, since c', c'' are the state-dependent utility midpoints of a, b in state s' and s'' respectively, $c'_s d \sim c''_s d$ implies that such midpoints are *state-independent*. It follows that, in the representation of Lemma 1, $\phi_{s'}$ and $\phi_{s''}$ are *cardinally equivalent* for any $s', s'' \in S$.

We can finally state the main result of this section:

Theorem 2 *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, M$^+$, MASC, FC, P2 and SCS if and only if $\succcurlyeq$ has a SOSEU* representation. If (U', ϕ', p') also represents $\succcurlyeq$, there are $\alpha, \kappa > 0$, $\beta, \zeta \in \mathbb{R}$ such that $p = p'$, $U' = \alpha U + \beta$ and $\phi'(\alpha r + \beta) = \kappa\phi(r) + \zeta$ for all $r \in U(V)$.*

Remark 7 Following the discussion in Remark 5, it can be seen that imposing $A \odot B = A \oplus_s B$ for all $A, B \in [V]$, on top of the axioms of Theorem 2, implies that $\phi = \phi_s = \alpha \mathrm{id} + \beta$, for some $\alpha > 0$ and $\beta \in \mathbb{R}$. That is, the preference is SEU. Thus, we see that SEU intuitively corresponds to the case in which the utility midpoint of A and B under certainty $A \odot B$, is indifferent to the utility midpoint under uncertainty $A \oplus_s B$ for every $s \in S$ (thus implying Axiom SCS).

4.2.2 Some special cases

With respect to the existing literature, Theorem 2 has two advantages: first, it does not require the existence of an objective randomization device or multiple sources of uncertainty. Second, it does not constrain the ambiguity attitude of the decision maker, since ϕ is a general monotone function.[18] Additional properties of ϕ can be obtained by imposing additional restrictions on preferences over act mixtures. To begin, the concavity of ϕ can be obtained by adding the following axiom:

Axiom 8 (Ambiguity Hedging—AH) *If $F \sim G$ then $\alpha F \oplus (1-\alpha)G \succcurlyeq F$ for all $\alpha \in [0, 1]$.*

Corollary 1 *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, M$^+$, MASC, FC, P2, SCS and AH if and only if $\succcurlyeq$ has a SOSEU* representation with concave ϕ.*

This result is analogous to Theorem 3 in Grant et al. (2009), who assume Uncertainty Aversion (i.e., a version of AH with objective act mixtures) and a condition called "Translation Invariance at Certainty" (TIC).[19]

Next, we characterize Exponential SOSEU* by a subjective version of the Weak Certainty Independence axiom of Maccheroni et al. (2006). With a standard abuse of

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18 For example, Grant et al. (2009) only obtain a SOSEU representation with concave ϕ.

19 Translation Invariance at Certainty holds if: $\succcurlyeq$ is locally EU at a with respect to p^a and locally EU at b with respect to p^b, then $p^a = p^b$. Where, $\succcurlyeq$ is locally EU at $a \in A$ w.r.t. a probability p on S if, for all $f \in \mathcal{F}$ and $\mathbb{E}_{p^a}[U(f)] > U(a)$, there exists an $\bar{\alpha} \in (0, 1]$ such that, for all $\alpha \in (0, \bar{\alpha}]$, $\alpha f \oplus (1-\alpha)a \succ a$ and $U(a) > E_{p^a}[U(f)]$ implies there exists $\bar{\alpha} \in (0, 1]$ such that, for all $\alpha \in (0, \bar{\alpha}]$, $a \succ \alpha f \oplus (1-\alpha)a$. TIC is weaker than SCS, since it does not imply the state-independence of ϕ_s without ancillary assumptions.

notation, we identify $A \in [V]$ with the constant correspondence that delivers A for every $s \in S$.

Axiom 9 (Weak Certainty Independence—WCI) *For all $F, G \in \mathbb{F}$ and $A, B \in [V]$, and $\alpha \in (0, 1)$,*

$$\alpha F \oplus (1-\alpha)A \succcurlyeq \alpha G \oplus (1-\alpha)A \quad \Rightarrow \quad \alpha F \oplus (1-\alpha)B \succcurlyeq \alpha G \oplus (1-\alpha)B$$

Corollary 2 *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, M^+, MASC, FC, P2, SCS and WCI if and only if $\succcurlyeq$ has an Exponential SOSEU* representation.*

The argument is similar to that given by Strzalecki (2011, Theorem 1) in his characterization of Multiplier Preferences. However, our SCS and WCI do not characterize Multiplier Preferences. The reason for this is that our derivation of SOSEU* does not entail restrictions on the curvature of ϕ. To obtain the characterization of Multiplier Preferences, we need to add axiom AH:

Corollary 3 *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, M^+, MASC, FC, P2, SCS, WCI and AH if and only if $\succcurlyeq$ has a Multiplier Preference representation.*

Our final axiom is based on the work of Chateauneuf and Faro (2009), and requires independence only with respect to mixtures with the "worst" payoff:

Axiom 10 (Worst Independence—WI) *There exists $Z_* \in [V]$ such that $F \succcurlyeq Z_*$ for all $F \in \mathcal{F}$ and, for all $F, G \in \mathcal{F}$ and all $\alpha \in (0, 1)$,*

$$F \sim G \quad \Longrightarrow \quad \alpha F \oplus (1-\alpha)Z_* \sim \alpha G \oplus (1-\alpha)Z_*.$$

By Axiom WI, the function ϕ is positively homogeneous. Therefore, the only SOSEU* model that satisfies WI is the Power SOSEU*:

Corollary 4 *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, M^+, MASC, FC, P2, SCS, and WI if and only if $\succcurlyeq$ has a Power SOSEU* representation.*

This result is similar to Proposition 6 of Gumen and Savochkin (2012), with the difference that we do not require risk independence. The last corollary, which proof is immediate, shows that Concave Power SOSEU* (i.e. $\theta \in (0, 1]$) is characterized by adding axiom AH to those of Power SOSEU*:

Corollary 5 *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, M^+, MASC, FC, P2, SCS, WI and AH if and only if $\succcurlyeq$ has a Concave Power SOSEU* representation.*

Figure 4 summarizes the above results with a graphical illustration of the relations between axioms and representations within the SOSEU* model.

We conclude with two observations on the relation of our results with previous characterizations of SOSEU.

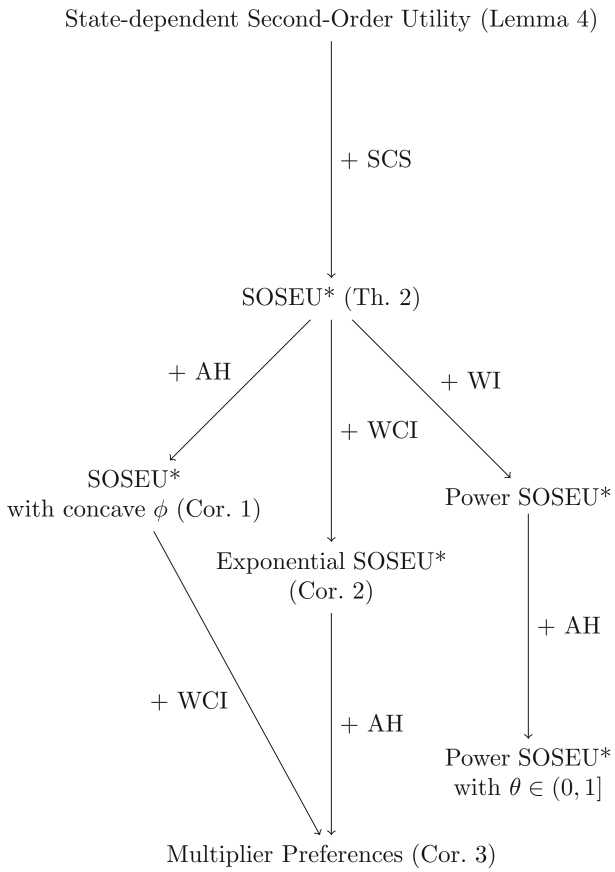

Fig. 4 Axioms and SOSEU*

Remark 8 Our characterization of SOSEU can be compared to Nau (2006) and Ergin and Gul (2009), who identify ambiguity attitude by using multiple sources of uncertainty in lieu of an objective randomization device. In their setting, the states of the world are pairs $(s_1, s_2) \in S_1 \times S_2$ and acts are functions from the states to consequences $f : S_1 \times S_2 \to X$. By assuming $|S_2| < \infty$ one can interpret one such act as a "compound act": a map from S_1 to a homogeneous product space with finitely many coordinates $V = X^{S_2}$. A SEU evaluation for acts that depend only on the source of uncertainty S_2 is a MAS representation of the restriction of $\succcurlyeq$ to X^{S_2}: $U(f(s_1)) = \sum_{s_2 \in S_2} u(f(s_1, s_2))q(s_2)$. Then, assuming moreover that $|S_1| < \infty$, the axioms of Theorem 2 can be adapted to obtain the following representation of $\succcurlyeq$:

$$J(f) = \int_{S_1} \phi \left(\sum_{s_2 \in S_2} u(f(s_1, s_2))q(s_2) \right) dp(s_1)$$

Our result actually requires a weaker condition than SEU over $V = X^{S_2}$. Indeed, it would be sufficient to have a MAS representation (not necessarily EU) of $\succcurlyeq$ on $V = X^{S_2}$: $U(f(s_1)) = \sum_{s_2 \in S_2} u_{s_2}(f(s_1, s_2))$. Such U allows us to define $\odot$ and $\oplus_s$ and to reproduce the proof of Theorem 2 to obtain:

$$J(f) = \int_{S_1} \phi \left(\sum_{s_2 \in S_2} u_{s_2}(f(s_1, s_2)) \right) dp(s_1)$$

Remark 9 Another axiomatization of SOSEU without restrictions on the function ϕ is that provided by Cerreia-Vioglio et al. (2012). They assume a rich state space as in Savage (1954) and the existence of an objective randomization device. Our technique can be applied to substitute the assumption of the presence of an objective randomization with the assumption that the set of consequences is a (finite or infinite) product space $V = X^{T+1}$. The advantage of this extension is that we do not need to assume Risk Independence, as the utility function is automatically affine with respect to $\oplus$. We can then prove:

Corollary 6 *A binary relation $\succcurlyeq$ satisfies axioms P1-P6 of Savage, MASC and FC if and only if there exist a non-constant MAS representation $U : V \to \mathbb{R}$, a continuous increasing $\phi : U(V) \to \mathbb{R}$, and a non-atomic and finitely additive probability p on S such that $\succcurlyeq$ is represented by $J : \mathcal{F} \to \mathbb{R}$ defined by:*

$$J(f) = \int_S (\phi \circ U)(f) dp.$$

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A Axioms for a MAS representation

The following definitions and results are taken from Krantz et al. (1971) Section 6.2 (pp. 250–261) to which we refer the reader for the definitions of (strictly bounded) *standard sequence* and *essential component*. Suppose that X_0, X_1 are nonempty sets and $\succcurlyeq$ is a binary relation on $X_0 \times X_1$. Consider the following axioms:

Axiom 11 (Weak Ordering) *$\succcurlyeq$ is a weak order.*

Axiom 12 (Independence) *For* $x_0, y_0 \in X_0$, $(x_0, w_1) \succcurlyeq (y_0, w_1)$ *for some* $w_1 \in X_1$ *implies* $(x_0, z_1) \succcurlyeq (y_0, z_1)$ *for all* $z_1 \in X_1$*; and, for* $x_1, y_1 \in X_1$, $(w_0, x_1) \succcurlyeq (w_0, y_1)$ *for some* $w_0 \in X_0$ *implies* $(z_0, x_1) \succcurlyeq (z_0, y_1)$ *for all* $z_0 \in X_0$.

Axiom 13 (Thomsen) *For every* $x_0, y_0, z_0 \in X_0$ *and* $x_1, y_1, z_1 \in X_1$*, if* $(x_0, z_1) \sim (z_0, y_1)$ *and* $(z_0, x_1) \sim (y_0, z_1)$*, then* $(x_0, x_1) \sim (y_0, y_1)$.

Axiom 14 (Restricted Solvability) *Whenever there exist* $x_0, x', x'' \in X_0$ *and* $x_1, y_1 \in X_1$ *for which* $(x', y_1) \succcurlyeq (x_0, x_1) \succcurlyeq (x'', y_1)$*, then there exists* $z_0 \in X_0$ *such that* $(z_0, y_1) \sim (x_0, x_1)$*. A similar condition holds for* X_1.

Axiom 15 (Archimedean) *Every strictly bounded standard sequence is finite.*

Axiom 16 (Essentiality) *Each component is essential.*

The following result is Theorem 2 p. 257 in Krantz et al. (1971):

Theorem 3 *Suppose that a binary relation* $\succcurlyeq$ *on* $X_1 \times X_2$ *satisfies axioms Weak Ordering, Independence, Thomsen, Restricted Solvability, Archimedean and Essentiality then there exist functions* $u_i : X_i \to \mathbb{R}$ $i = 0, 1$ *such that for all* $(x_0, x_1), (y_0, y_1) \in X_0 \times X_1$,

$$(x_0, x_1) \succcurlyeq (y_0, y_1) \quad \Longleftrightarrow \quad u_0(x_0) + u_1(x_1) \geq u_0(y_0) + u_1(y_1).$$

If u'_0, u'_1 *are two functions with the same property, then there are* $\alpha > 0$ *and* $\beta_0, \beta_1 \in \mathbb{R}$ *such that* $u'_0 = \alpha u_0 + \beta_0$ *and* $u'_1 = \alpha u_1 + \beta_1$.

Under suitable topological assumptions on the sets X_0 and X_1, the Archimedean and Restricted Solvability axioms can be dispensed with (see Th. 20 in Vind and Grodal 2003).

B A simple sufficient condition for existence

As we observed earlier, while utility midpoints exist for any pair A and B of indifference sets when $\succcurlyeq$ has a MAS representation, this is not the case for preference midpoints. That is, given two arbitrary vectors $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$, a preference midpoint of (x_0, x_1) and (y_0, y_1) may not exist. Here, we introduce a sufficient condition for the existence of preference midpoints, which is a particular case of the informal notion of "closeness" we introduced in Sect. 3 (formally defined in Definition 7). An advantage of this definition is that it is entirely formulated in terms of vectors, rather than indifference sets. Thus, it is in principle easier to verify and implement.

Definition 6 $\succcurlyeq$ satisfies the *Triangle condition* at $(y_0, y_1) \in V$ and $(x_0, x_1) \in V$ if and only if there exist $z \in X^{I_0}$ and $z_1 \in X^{I_1}$ such that $(y_0, z_1) \sim (z_0, y_1) \sim (x_0, x_1)$.

Figure 5 graphically illustrates the condition, making it also clear that the indifference sets A and B are "close."

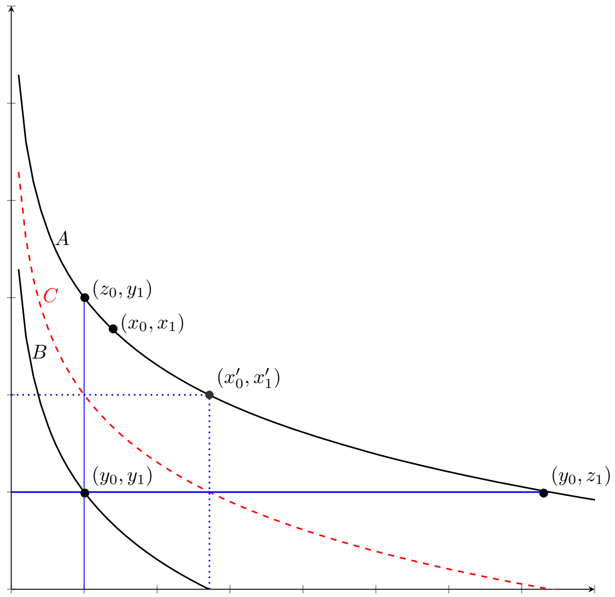

Fig. 5 Triangle condition at (y_0, y_1) and (x_0, x_1)

Proposition 3 *Suppose that* $\succcurlyeq$ *has a MAS representation. For arbitrary* (y_0, y_1) *and* (x_0, x_1)*, if the Triangle condition holds at* (y_0, y_1) *and* (x_0, x_1)*, then there exists* $(x'_0, x'_1) \sim (x_0, x_1)$ *such that* $(y_0, x'_1) \sim (x'_0, y_1)$.

Proof First consider the case $u_0(y_0) + u_1(y_1) = k' = U(B) < U(A) = k$. Define $f : X^{I_0} \to \mathbb{R}$ as follows $f(x) = u_0(y_0) + k - 2u_0(x) - u_1(y_1)$. By the Triangle condition, take $x = y_0$, then $f(y_0) = u_0(y_0) + k - 2u_0(y_0) - u_1(y_1) = k - (u_0(y_0) + u_1(y_1)) = k - k' > 0$. Again by the Triangle condition, take $x = z_0$, then $f(z_0) = u_0(y_0) + u_0(z_0) + u_1(y_1) - 2u_0(z_0) - u_1(y_1) = u_0(y_0) - u_0(z_0) < 0$, because $k' = u_0(y_0) + u_1(y_1) < u_0(z_0) + u_0(y_0) = k$. By continuity of u_0, there exists $f(x^*) = 0$. The case $u_0(y_0) + u_0(y_0) = U(B) > U(A)$ is symmetric. $\square$

Notice that the indifference $(y_0, x'_1) \sim (x'_0, y_1)$, as in Eq. (4), guarantees that (y'_0, x_1) is a preference midpoint of (x_0, x_1) and (y_0, y_1).[20]

C Proofs

C.1 Proof of Theorem 1

The proof of the theorem builds on several lemmas.

We begin by showing that a midpoint of $A, B \in [V]$ exists when A and B satisfy the following preference condition:

[20] More precisely, $(x'_0, y_1) \sim (y_0, x'_1) \in C$ and C is a preference midpoint of $A \ni (x_0, x_1)$ and $B \ni (y_0, y_1)$.

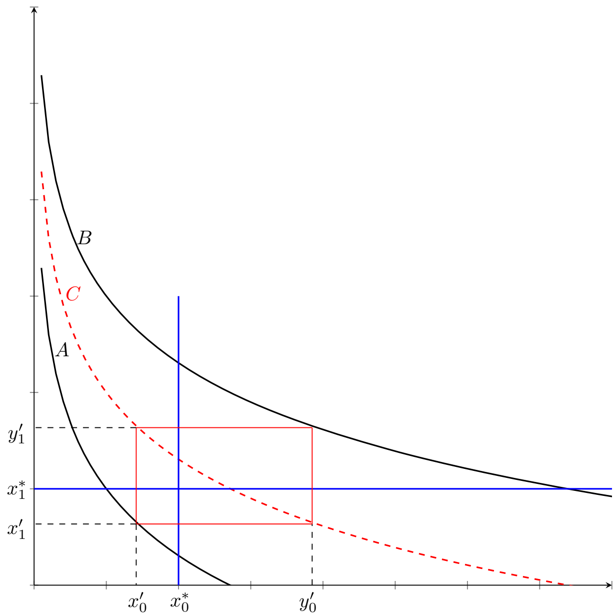

Fig. 6 Crossing property and midpoints

Definition 7 A MAS utility satisfies *Crossing* at $A, B \in [V]$ if and only if there are $x_0^* \in X^{I_0}, x_1^* \in X^{I_1}$ such that $(x_0^*, x_1) \in A$ and $(x_0^*, y_1) \in B$ for some $x_1, y_1 \in X^{I_1}$ and $(w_0, x_1^*) \in A$ and $(z_0, x_1^*) \in B$ for some $w_0, z_0 \in X^{I_0}$ with $u_0(z_0) \geq u_0(x_0^*) \geq u_0(w_0)$ and $u_1(y_1) \geq u_1(x_1^*) \geq u_1(x_1)$.

Figure 6 provides a graphical representation of the Crossing property.

Lemma 2 *Given $A, B \in [V]$, if $\succcurlyeq$ has a MAS representation and Crossing holds at A, B, then there exist $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$ such that $(x_0, y_1) \sim (y_0, x_1)$ (hence, there exists $C \in [V]$ such that $C = A \odot B$).*

Proof In the space of utilities, given $U(A) = k, U(B) = k'$, assume w.l.o.g that $U(B) > U(A)$. By Crossing, there are $(x_0^*, x_1) \in A$ and $(x_0^*, y_1) \in B$ for some $x_1, y_1 \in X^{I_1}$ and $(w_0, x_1^*) \in A$ and $(z_0, x_1^* \in B$ for some $w_0, z_0 \in X^{I_0}$, such that $u_0(z_0) \geq u_0(x_0^*) \geq u_0(w_0)$ and $u_1(y_1) \geq u_1(x_1^*) \geq u_1(x_1)$. Define $a' = (x_0', x_1')$ by $u_0(x_0') = 0.5u_0(x_0^*) + 0.5u_0(w_0)$ and $u_1(x_1') = 0.5u_1(x_1^*) + 0.5u_1(x_1)$. Notice that $U(a') = U(A)$. Similarly, define $b' = (y_0', y_1')$ by $u_0(y_0') = 0.5u_0(x_0^*) + 0.5u_0(z_0)$ and $u_1(y_1') = 0.5u_1(x_1^*) + 0.5u_1(y_1)$. Notice that $U(b') = U(B)$. Both a' and b' are well-defined because of Crossing at A, B and continuity of the preference. Then, it follows that $(x_0', y_1') \sim (y_0', x_1')$, indeed:

$$\begin{aligned} u_0(x_0') + u_1(y_1') &= 0.5u_0(x_0^*) + 0.5u_0(w_0) + 0.5u_1(x_1^*) + 0.5u_1(y_1) \\ &= 0.5(u_0(x_0^*) + u_1(y_1)) + 0.5(u_1(w_1) + u_0(x_0^*)) \\ &= 0.5U(B) + 0.5U(A) \end{aligned}$$

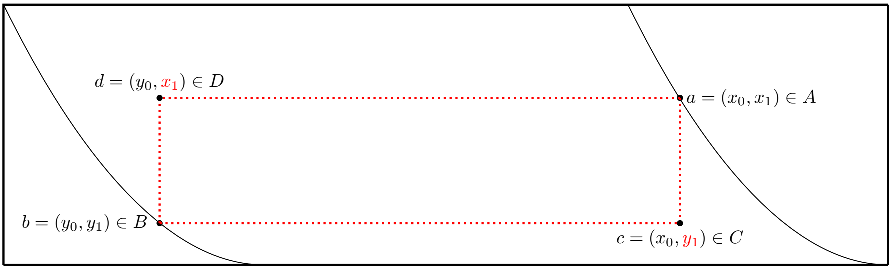

Fig. 7 Smoothing swap of a and b

and

$$\begin{aligned} u_0(y_0') + u_1(x_1') &= 0.5u_0(x_0^*) + 0.5u_0(z_0) + 0.5u_1(x_1^*) + 0.5u_1(x_1) \\ &= 0.5(u_0(x_0^*) + u_1(x_1)) + 0.5(u_0(z_0) + u_1(x_1^*)) \\ &= 0.5U(A) + 0.5U(B) \end{aligned}$$

□

If Crossing does not hold at A and B, we need to take additional steps in order to identify the midpoint of A and B. The following definition outlines such additional steps:

Definition 8 An *elementary step* is:

(SS) *A smoothing swap of* $a = (x_0, x_1)$ *and* $b = (y_0, y_1)$: The pair $c = (x_0, y_1)$ and $d = (y_0, x_1)$ is substituted to a and b respectively (see Fig. 7).
(II) *An indifference substitution of* $a = (x_0, x_1)$: The vector $b = (x_0', x_1') \sim a$ is substituted to a.

Suppose that $a = (x_0, x_1) \in A$ Pareto dominates $b = (y_0, y_1) \in B$. A smoothing swap of a and b generates two points $c = (x_0, y_1) \in C$ and $d = (y_0, x_1) \in D$ that are "interior," in terms of preferences, to a and b; i.e., $a \succcurlyeq \{c, d\} \succcurlyeq b$. Moreover, the MAS assumption implies the Diagonal property (see Lemma 15 p. 75 in Vind and Grodal 2003):

Lemma 3 *If* $\succcurlyeq$ *has a MAS representation,* $(x_0, x_1) \in A$ *and* $(y_0, y_1) \in B$ *and* $C = A \odot B$*, then* $C = A' \odot B'$ *where* $(x_0, y_1) \in A'$ *and* $(y_0, x_1) \in B'$.

Given $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$, if they have a utility midpoint, their midpoint is the same as that of $(x_0, y_1) \in A'$ and $(y_0, x_1) \in B'$. Therefore, if Crossing holds at a pair C, D generated by a smoothing swap of $a \in A$ and $b \in B$, then Lemma 2 implies the existence of a utility midpoint E of C and D and the Diagonal property guarantees that E is also the utility midpoint of A and B.

Lemma 4 *If,* $A \succ B$ *and for some* $(x_0, x_1) \in A$ *and* $(y_0, y_1) \in B$*,* $u_0(x_0) > u_0(y_0)$ *and* $u_1(x_1) > u_1(y_1)$ *then, there exists* $C = A \odot B$ *which can be determined in* n

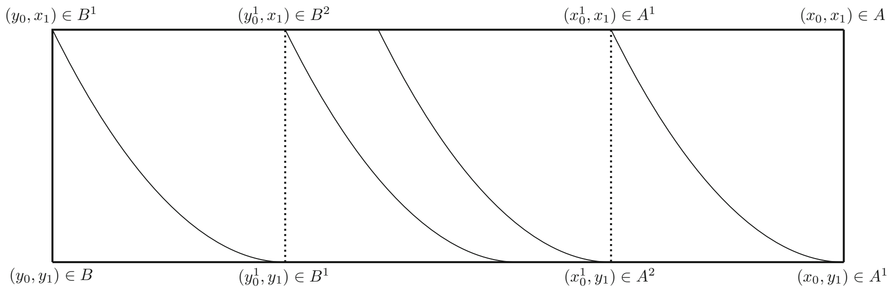

Fig. 8 Elementary steps

elementary steps. The number n is the smallest integer such that

$$n \geq \frac{1}{2} \max \left\{ \frac{u_0(x_0) - u_0(y_0)}{u_1(x_1) - u_1(y_1)}, \frac{u_1(x_1) - u_1(y_1)}{u_0(x_0) - u_0(y_0)} \right\} \tag{7}$$

Proof If Crossing holds at A, B, by Lemma 2 there exists a midpoint. If not, consider $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$ and refer to Fig. 8. Assume first that $(x_0, y_1) \succ (y_0, x_1)$ (or equivalently $u_0(x_0) - u_0(y_0) > u_1(x_1) - u_1(y_1)$). Now apply a smoothing swap. The pairs (x_0, y_1) and (y_0, x_1) are such that $(x_0, x_1) \succ (x_0, y_1) \in A^1$ and $(y_0, y_1) \prec (y_0, x_1) \in B^1$. If Crossing holds at A^1, B^1 then, by Lemma 2 and by the Diagonal property there exists a midpoint of A and B. If not, the condition $u_0(x_0) - u_0(y_0) > u_1(x_1) - u_1(y_1)$ implies $A^1 \succ B^1$. Now, find $y_0^1 \in X^{I_0}$ such that $(y_0^1, y_1) \in B^1$. It exists by continuity and the fact that $(x_0, y_1) \in A^1 \succ B^1 \succ (y_0, y_1) \in B$. Similarly, find $x_0^1 \in X^{I_0}$ such that $(x_0^1, x_1) \in A^1$. Since Crossing does not hold at A^1, B^1, then $u_0(x_0^1) > u_0(y_0^1)$, so (y_0^1, y_1) and (x_0^1, x_1) are strictly Pareto-ranked. Apply a smoothing swap to (y_0^1, y_1) and (x_0^1, x_1), to find $(x_0^1, y_1) \in B^2$ and $(y_0^1, x_1) \in A^2$. If Crossing holds at A^2, B^2 then by Lemma 2 and the Diagonal property there exists a midpoint of A^1 and B^1, by another application of the Diagonal property, there exists a midpoint of A and B. If Crossing does not hold at A^2 and B^2, repeat the argument to find $A^3, A^4, \ldots$ and $B^3, B^4, \ldots$ as before. The following claim shows that Crossing holds for suitably large (but finite) n:

Claim 1 *Crossing holds at* A^n, B^n *for* $n \geq \frac{1}{2} \frac{u_0(x_0) - u_0(y_0)}{u_1(x_1) - u_1(y_1)}$.

Proof of Claim 1 To prove this claim, first observe that $u_0(y_0^1) - u_0(y_0) = u_0(y_0^1) + u_1(y_1) - u_0(y_0) - u_1(y_1) = U(B^1) - U(B) = u_0(y_0) + u_1(x_1) - u_0(y_0) - u_1(y_1) = u_1(x_1) - u_1(y_1)$ and $u_0(x_0) - u_0(x_0^1) = u_0(x_0) + u_1(x_1) - u_0(x_0^1) - u_1(x_1) = U(A) - U(A^1) = u_0(x_0) + u_1(x_1) - u_0(x_0) - u_1(y_1) = u_1(x_1) - u_1(y_1)$.

Crossing holds at A^n and B^n for given n if $u_0(y_0^n) \geq u_0(x_0^n)$ with $(y_0^n, y_1) \in B^n$ and $(x_0^n, x_1) \in A^n$, choosing $x_0^* \in X^{I_0}$ such that $u_0(y_0^n) \geq u_0(x_0^*) \geq u_0(x_0^n)$ (and choosing $x_1^* \in X^{I_1}$ so that $(x_0^*, x_1^* \in B^n)$ that exists by continuity). Iterating the previous calculations, $u_0(y_0^n) = u_0(y_0) + n(u_1(x_1) - u_1(y_1))$ and $u_0(x_0^n) = u_0(x_0) - n(u_1(x_1) - u(y_1))$. Therefore, $u_0(y_0^n) \geq u_0(x_0^n)$ will hold as soon as n

satisfies $u_0(y_0) + n(u_1(x_1) - u_1(y_1)) \geq u_0(x_0) - n(u_1(x_1) - u(y_1))$; that is

$$n \geq \frac{1}{2} \frac{u_0(x_0) - u_0(y_0)}{u_1(x_1) - u_1(y_1)} \tag{8}$$

□

To conclude the proof of the lemma, we just observe that, in the case in which $(x_0, y_1) \prec (y_0, x_1)$ (equivalently $u_0(x_0) - u_0(y_0) < u_1(x_1) - u_1(y_1)$), we can adapt the argument above to derive the symmetric inequality

$$n \geq \frac{1}{2} \frac{u_1(x_1) - u_1(y_1)}{u_0(x_0) - u_0(y_0)}$$

Combining the two cases, we get inequality (7). □

Proof of Theorem 1 Given arbitrary $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$, if $\succcurlyeq$ satisfies Crossing at A and B, a midpoint of A and B exists by Lemma 2. Suppose $\succcurlyeq$ does not satisfy Crossing at A, B, and assume w.l.o.g. that $A \succ B$. Then for arbitrary $(x_0, x_1) \in A$ and $(y_0, y_1) \in B$, there are three possible cases:

1. $u_0(x_0) > u_0(y_0)$ and $u_1(y_1) > u_1(x_1)$
2. $u_0(y_0) > u_0(x_0)$ and $u_1(x_1) > u_1(y_1)$
3. $u_0(x_0) \geq u_0(y_0)$ and $u_1(x_1) \geq u_1(y_1)$

For Case 1, we apply a smoothing swap to find $c = (x_0, y_1) \in C$ and $d = (y_0, x_1) \in D$ for some $C, D \in [V]$. $c \succ d$ and they are strictly Pareto-ranked. Hence, we can apply Lemma 4 (the condition of Lemma 4 is satisfied because $A \succ B$). Therefore, there exists a midpoint E of C, D. By the Diagonal property (Lemma 3), $E = A \odot B$.

Case 2 can be treated as Case 1, up to relabeling of the axes.

Case 3 has three subcases:

a. $u_0(x_0) > u_0(y_0)$ and $u_1(x_1) = u_1(y_1)$
b. $u_0(x_0) = u_0(y_0)$ and $u_1(x_1) > u_1(y_1)$
c. $u_0(x_0) > u_0(y_0)$ and $u_1(x_1) > u_1(y_1)$

For case a, since $u_1(X^{I_1})$ is an interval in $\mathbb{R}$, if $u_1(y_1) \in u_1(X^{I_1})^\circ$ or $u_1(y_1) = \max_{w_1 \in X^{I_1}} u_1(w_1)$, we can always find $\epsilon > 0$ such that $u_0(x_0) - u_0(y_0) > \epsilon$, $u_1(y_1) - \epsilon = u_1(z_1)$, and $(y_0, y_1) \sim (z_0, z_1)$ for some $z_0 \in X^{I_0}$. By construction, $u_0(x_0) > u_0(z_0)$, because

$$\begin{aligned} u_0(x_0) &> u_0(y_0) + \epsilon \\ &= u_0(y_0) + u_1(y_1) - u_1(z_1) \\ &= u_0(z_0) + u_1(z_1) - u_1(z_1) \\ &= u_0(z_0) \end{aligned}$$

therefore, (z_0, z_1) and (x_0, x_1) are strictly Pareto-ranked. We can thus apply Lemma 4 to prove existence of the midpoint of A and B.

If $u_1(y_1) = \min_{w_1 \in X^{I_1}} u_1(w_1)$, take $\epsilon > 0$ with $u_0(x_0) - u_0(y_0) > \epsilon$, define $u_1(x_1) + \epsilon = u_1(y_1) + \epsilon = u_1(z_1)$, and find $z_0 \in X^{I_0}$ such that $(z_0, z_1) \in a$. By construction, $u_0(z_0) > u_0(y_0)$, because

$$\begin{aligned} u_0(y_0) &< u_0(x_0) - \epsilon \\ &= u_0(x_0) + u_0(y_0) - u_0(z_0) \\ &= u_0(z_0) + u_1(z_1) - u_1(z_1) \\ &= u_0(z_0) \end{aligned}$$

Therefore, (z_0, z_1) strictly Pareto dominates (y_0, y_1), and Lemma 4 can by applied.
Case b is symmetric to case a. For Case c we can directly apply Lemma 4. □

Proof of Proposition 1 Suppose that I_0, I_1 and I'_0, I'_1 are two partitions of $\{0, \ldots, T\}$ associated with a MAS utility U. That is, $U(d) = u_0(d_{I_0}) + u_1(d_{I_1})$ and $U(d) = u'_0(d_{I'_0}) + u_1(d_{I'_1})$ for any $d \in V$. Suppose that $C \in [V]$ is a midpoint of $A, B \in [V]$, then for any $a \in A$, $b \in B$ and $c \in C$,

$$\begin{aligned} U(c) = \frac{1}{2}U(a) + \frac{1}{2}U(b) &= \frac{1}{2}\left(u_0(a_{I_0}) + u_1(a_{I_1})\right) + \frac{1}{2}\left(u_0(b_{I_0}) + u_1(b_{I_1})\right) \\ &= \frac{1}{2}\left(u'_0(a_{I'_0}) + u'_1(a_{I'_1})\right) + \frac{1}{2}\left(u'_0(b_{I'_0}) + u'_1(b_{I'_1})\right) = \frac{1}{2}U(a) + \frac{1}{2}U(b). \end{aligned}$$

So, C is a utility midpoint of A and B regardless of the partition. □

C.2 Proofs for Section 4

Proof of Proposition 2 Necessity is straightforward. For sufficiency, if $\succcurlyeq$ satisfies MASC and P, its restriction to V has a non-constant MAS representation U. By Theorem 1, we can define a mixture operator $\oplus$, so that axiom FC can be applied. By P and FC, for each $F \in \mathbb{F}$, there exists an $A \in [V]$ such that $A \sim F$, so that for any $a \in A$, $f \sim a$, where $f \mapsto F$. Then, we can define $I(U(f)) = U(a)$ for some $a \in A$. The functional $I : B_0(\Sigma, U(V)) \to \mathbb{R}$ is monotone, continuous and normalized (see Prop. 1 in Cerreia-Vioglio et al. 2011). □

Proof of Lemma 1 Necessity is straightforward. For sufficiency, if $\succcurlyeq$ satisfies P, M, MASC and FC, by Proposition 2, we can define $I(U(f)) = U(a_f)$. The functional I induces a preference $\succcurlyeq_U$ on $U(V)^S$, defined as $U \circ f \succcurlyeq_U U \circ g$ if and only if $I(U(f)) \geq I(U(g))$. By P2 and Theorem 3 of Debreu (1959), the preference $\succcurlyeq_U$ has an additively separable representation: there are continuous, increasing functions $\phi_s : U(V) \to \mathbb{R}$, three of them non-constant, such that $I(U(f)) \geq I(U(g))$ if and only if $\sum_{s \in S} \phi_s(U(f(s))) \geq \sum_{s \in S} \phi_s(U(g(s)))$. The result follows from defining $J(f) = \sum_{s \in S} \phi_s(U(f(s)))$. Assume that, for non-null $s \in S$, ϕ_s is not strictly increasing. Then, there are $U(a) \neq U(b)$ such that $\phi_s(U(a)) = \phi_s(U(b))$. W.l.o.g. suppose that $U(a) > U(b)$, since U represents the restriction of $\succcurlyeq$ on V, it follows that $a \succ b$. By M^+, $a_s d \succ b_s d$, but $\phi_s(U(a)) + \sum_{s' \in S \setminus s} \phi_{s'}(U(d)) = \phi_s(U(b)) + \sum_{s' \in S \setminus s} \phi_{s'}(U(d))$, a contradiction. Hence, ϕ_s is strictly increasing. The uniqueness part of the representation follows from Debreu (1959). □

Proof of Theorem 2 By Lemma 1, axioms P, M, M^+, MASC, FC and P2 implies the existence of continuous and increasing functions (three of them strictly increasing) $\phi_s : U(V) \to \mathbb{R}$ such that $J(f) = \sum_{s \in S} \phi_s(U(f(s)))$ represents $\succcurlyeq$. By axiom SCS, for all $s \in S$ and for non-null $s', s'' \in S$ (there are at least three of them) $\phi_s(U(c')) = \phi_s(U(c''))$. If $s \in S$ is null the equality is always true. Take a non-null $s \in S$, then ϕ_s is strictly increasing, hence $U(c') = U(c'')$. Therefore, $U(c'') = \phi_{s''}^{-1}\left(\frac{1}{2}\phi_{s''}(U(a)) + \frac{1}{2}\phi_{s''}(U(b))\right)$ if and only if $U(c'') = \phi_{s'}^{-1}\left(\frac{1}{2}\phi_{s'}(U(a)) + \frac{1}{2}\phi_{s'}(U(b))\right)$. Therefore $\phi_{s'}(U(c')) = \phi_{s'}\left(\phi_{s''}^{-1}\left(\frac{1}{2}\phi_{s''}(U(a)) + \frac{1}{2}\phi_{s''}(U(b))\right)\right)$, hence, by defining $\psi = \phi_{s'} \circ \phi_{s''}^{-1}$ we have:

$$
\begin{aligned}
\psi\left(\frac{1}{2}\phi_{s''}(U(a)) + \frac{1}{2}\phi_{s''}(U(b))\right) &= \psi \circ \phi_{s''}\left(U(c'')\right) \\
&= \phi_{s'}\left(U(c''\right) \\
&= \phi_{s'}\left(U(c')\right) \\
&= \frac{1}{2}\phi_{s'}(U(a)) + \frac{1}{2}\phi_{s'}(U(b)) \\
&= \frac{1}{2}\psi\left(\phi_{s''}(U(a))\right) + \frac{1}{2}\psi\left(\phi_{s''}(U(b))\right)
\end{aligned}
$$

Therefore, ψ is affine, i.e. there is $\alpha_s > 0$, $\beta_{s'} \in \mathbb{R}$ with $\phi_{s'} \circ U = \alpha_{s'}\phi_{s''} \circ U + \beta_{s'}$.

By assumption $\alpha_{s'} > 0$ for at least 3 states. Therefore, $J(f) = \sum_{s \in S} \alpha_s \phi(u(f(s))) + \beta_s$ where $\phi \triangleq \phi_s$. Renormalizing by $K = \left(\sum_{s \in S} \alpha_s\right) > 0$ gives a SOSEU* with $p(s) = \frac{a_s}{K}$. □

Proof of Corollary 2 The argument is similar to the one of Theorem 1 in Strzalecki (2011): WCI implies translation invariance of the functional $I : B_0(\Sigma, U(V)) \to \mathbb{R}$ defined by $I(U \circ f) = \phi^{-1}\left(\sum_{s \in S} p(s)\phi\left(U(f(s))\right)\right)$ (i.e. $I(\xi + k) = I(\xi) + k$ for all $\xi \in B_0(\Sigma, U(V))$ and $k \in \mathbb{R}$). In turn, translation invariance forces the function ϕ to satisfy a generalized Pexider's functional equations whose solution is the exponential function $\phi(r) = \gamma e^{\alpha r} + \beta$ for $\gamma, \alpha \neq 0$ and arbitrary β (see Aczél 1966, Cor. 1, p. 150). By non-triviality and monotonicity $\gamma > 0$. □

Proof of Corollary 4 The argument is similar to the one in Proposition 6 in Gumen and Savochkin (2012): Worst Independence implies positive homogeneity of the functional

$I : B_0(\Sigma, U(V)) \to \mathbb{R}$ defined by $I(U \circ f) = \phi^{-1}\left(\sum_{s\in S} p(s)\phi(U(f(s)))\right)$ (i.e. $I(\gamma\xi) = \gamma I(\xi)$ for all $\xi \in B_0(\Sigma, U(V))$ and all $\gamma \geq 0$). In turn, the normalization $U(Z_*) = 0$, positive homogeneity and the uniqueness of the SOSEU* representation force the function ϕ to satisfy the multiplicative functional equation $\phi(\gamma t) = \alpha(\gamma)\phi(t)$ for all $\gamma, t \geq 0$. By Theorem 4 (p. 144) in Aczél (1966), there are $\beta, \theta \in \mathbb{R}$ such that $\phi(t) = \beta t^\theta$.[21] By Monotonicity and non-triviality $\beta > 0$ and $\theta > 0$. □

Proof of Corollary 6 The proof is straightforward. By Savage's result, there exist $W : V \to \mathbb{R}$ and a non-atomic and finitely additive p over S such that $J(f) = \int_S W(f)dp$ represents $\succcurlyeq$. Therefore $\succcurlyeq$ satisfies Axiom M. By P, M and FC, for each $f \in \mathcal{F}$ there exists $a_f \in V$ such that $f \sim a_f$. Let $U : V \to \mathbb{R}$ be the MAS representation on V, which exists by MASC. By defining $I(U \circ f) = U(a_f)$, we obtain $f \succcurlyeq g$ if and only $I(U \circ f) \geq I(U \circ g)$. I is well-defined by M. Since both U and W represents $\succcurlyeq$ on V, there exists a monotone function ϕ such that $W = \phi \circ U$. The proof that ϕ is continuous is identical to that of Proposition 3 in Cerreia-Vioglio et al. (2012). □

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21 According to this result, the solutions of the functional equation $f(xy) = g(x)h(y)$ for positive x, y, with function f continuous in a point, are $f(z) = a\beta z^\theta$, $g(z) = az^\theta$ and $h(z) = \beta z^\theta$ when $z > 0$. Since in the case of SOSEU* $f(xy) = g(x)f(y)$, $a = 1$.

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