# MATHEMATICAL ECONOMICS: SUGGESTED SOLUTIONS TO HOMEWORK # 7

1. An infinite horizon dynamic programming exercise: Solve letter (a) of problem 4 in the Appendix on dynamic programming by Kreps.

   **Answer:** The first important step is to make this into a dynamic programming problem. To do this, define, $S = \{-100, -99, \ldots\} \times \{E, F, I\}$, where the typical state $s = (n, \omega)$ has two coordinates, the first being an integer greater than -100, the second a state variable. The interpretation is that $s$ represents the state of being at parking spot $n$ and finding either the spot empty ($E$), full ($F$), or having in fact already parked before, so that the state of the spot is irrelevant ($I$). As we will see, $I$ plays the role of an absorbing state, which represents the eventual end of the decisions to be taken. Second, using obvious notation, define $A = \{p, c\}$. Define the feasible action correspondence as follows: $\Phi : S \to 2^A$, for every $s \in S$,

$$\Phi(n, \omega) = \begin{cases} \{p, c\} & \omega = E \\ \{c\} & \omega = F \\ \{p\} & \omega = I. \end{cases}$$

   As will be seen, the labeling of the action in the case of $\omega = I$ is irrelevant. The stochastic transition function is $f : S \times A \to S$, given by

$$f((n, \omega), a) = \begin{cases} (n+1, E) & \text{if } a = c \text{, with prob. } 1 - \alpha \\ (n+1, F) & \text{if } a = c \text{, with prob. } \alpha \\ (n+1, I) & \text{if } \omega = E, a = p \\ (n+1, I) & \text{if } \omega = I. \end{cases}$$

   Finally, the payoff function is $r : S \times A \to \mathbf{R}$ defined by

$$r((n, \omega), a) = \begin{cases} 0 & \text{if } a = c \\ 0 & \text{if } a = p,\ \omega \neq E \\ -|n| & \text{if } a = p,\ \omega = E. \end{cases}$$

The stochastic DP problem is thus to maximize $\sum_{t=0}^{\infty} r(s_t, a_t)$. If we label by $\sigma$ a strategy for this problem, we have, using the notation in the book, that

$$W(\sigma)(s) = \sum_{t=0}^{\infty} r(\sigma)(s).$$

Having thus expressed our problem as a DP problem, we start by observing an obvious feature of any optimal strategy: If $\sigma^*$ is an optimal strategy, then it must be the case that for all $s \in S$ such that $s = (n, E)$ with $n \geq 0$, $\sigma^*(h_n(s_0)) = p$ (where $s_0 = (-100, E)$, the initial state of the problem). In other words, as soon as spot -1 is passed, any optimal strategy must prescribe to grab the first empty spot. The proof is this is not hard. I will omit it for brevity, but you should think about how to do it.

Given this observation we immediately get that for any optimal strategy $\sigma^*$, for all $s = (n, \omega)$ such that $n \geq 0$,

$$V(s) = W(\sigma^*)(s) = \begin{cases} -\sum_{t=1}^{\infty}(t+n)\alpha^{t-1}(1-\alpha) & \text{if } \omega = F \\ -n & \text{if } \omega = E \\ 0 & \text{if } \omega = I, \end{cases}$$

which, using the observation that $\sum_{t=1}^{\infty} t\alpha^t = \alpha/(1-\alpha)^2$, can be rewritten as follows:

$$V(s) = W(\sigma^*)(s) = \begin{cases} -1/(1-\alpha) - n & \text{if } \omega = F \\ -n & \text{if } \omega = E \\ 0 & \text{if } \omega = I, \end{cases}$$

The observation is useful because it shows that it is licit to artificially reduce the horizon of this problem to a finite one (terminating at time T=100), using the value function $V(s)$ as the last period revenue function (stationarity is obviously lost, but we do not care for FHDP problems). It is easy to check that this FHDP problem satisfies assumptions A1-3 in chapter 11 of Sundaram (since now $r_t(s, a)$ is bounded and, trivially, continuous for all $t$). Thus to find the optimal strategy we only need to apply Bellman's principle of optimality, which says that: Strategy $\sigma^*$ is optimal *if and only if* we have for every $s \in S$,

$$W(\sigma^*)(s) = \max_{a \in \Phi(s)} \{r(s, a) + W(\sigma^*)(f(s, a))\}. \tag{1}$$

Given $\alpha < 1$, let $n^\alpha$ be the integer satisfying the following;

$$n^\alpha \equiv \inf\{n \in \{1, 2, \ldots, 100\} : \alpha^{-n} \geq 1/2\}.$$

Consider the strategy $g : S \to A$ defined as follows:

$$g(s) = \begin{cases} p & \text{if } \omega = I \\ c & \text{if } \omega = F \\ p & \text{if } \omega = E,\ n \geq -n^\alpha \\ c & \text{if } \omega = E,\ n < -n^\alpha, \end{cases}$$

We will show that $\sigma^* = [g, g, \ldots, g]$ satisfies (1). So the simple rule is to calculate (depending on the size of $\alpha$) the number $n^\alpha$, and to take the first available spot after $-n^\alpha$.

The first step in doing this is to calculate $W(\sigma^*)$. This can easily be seen to take the form

$$W(\sigma^*)(s) = \begin{cases} 0 & \text{if } \omega = I \\ -|n| & \text{if } n \geq -n^\alpha,\ \omega = E \\ -\frac{(|n|-1) - |n|\alpha + 2\alpha^{|n|}}{(1-\alpha)} & \text{if } n \geq -n^\alpha,\ \omega = F \\ -\frac{(n^\alpha - 1) - n^\alpha\alpha + 2\alpha^{n^\alpha}}{(1-\alpha)} & \text{if } n < -n^\alpha,\ \omega \neq I \end{cases}$$

The only thing that needs to be proved are the third and fourth entries. The third is easily showed by induction on $n$, and the fourth then follows immediately.

Consider $s = (n, \omega)$ (remember that now $n \leq 0$). If $\omega = F$ or $I$, then (1) holds trivially, so suppose that $\omega = E$ (implying $\Phi(s) = \{p, c\}$). We basically have two cases to discuss here, first the one for $n \leq -n^\alpha - 1$, and then that of $n \geq -n^\alpha$.

Suppose that $n \leq -n^\alpha - 1$. The two-period problem is then

$$\max_{a \in \{p,c\}} r((n, E), a) + W(\sigma^*)(f(n, E), a).$$

Choosing $c$ (as $g$ prescribes) is optimal if

$$-|n| \leq -\frac{(n^\alpha - 1) - n^\alpha\alpha + 2\alpha^{n^\alpha}}{(1-\alpha)}$$

which is immediately rewritten as follows:

$$-n \geq n^\alpha + \frac{2\alpha^{n^\alpha} - 1}{1-\alpha}$$

But the last inequality holds, in fact, because of the definition of $n^\alpha$ we have

$$-n \geq n^\alpha + 1 > n^\alpha + \frac{2\alpha^{n^\alpha} - 1}{1-\alpha},$$

since we have

$$1 > \frac{2\alpha^{n^\alpha} - 1}{1-\alpha}.$$

Coming to the (more interesting) case of $n \geq -n^\alpha$, we have that $p$ is better than $c$ if

$$-|n| \geq -(1-\alpha)(|n|-1) - \frac{\alpha}{1-\alpha}[(|n|-2) - (|n|-1)\alpha + 2\alpha^{|n|-1}],$$

which, after some rewriting, is equivalent to

$$-n + (1-\alpha)(n+1) \leq -\frac{\alpha}{1-\alpha}[(n+2) - (n+1)\alpha - 2\alpha^{-(n+1)}].$$

The latter is seen with some arithmetic to hold as long as $\alpha^{-n} \geq 1/2$, which is the case, since $-n \leq n^\alpha$. We can thus conclude that $\sigma^*$ solves the Bellman equation for every $s$ and is thus the optimal strategy.