MATHEMATICAL ECONOMICS: SUGGESTED SOLUTIONS TO HOMEWORK # 6

1. A finite horizon dynamic programming exercise: Solve problem 2 in chapter 11 (p.278) of Sundaram.

   Answer:

   (a) First we formulate this as a FHDP problem. We take $S = [0, \bar{y}]$, where $\bar{y} = f^T(y)$, which is what can be maximally produced in periods 0 to T-1 if nothing is ever put to market. Thus a state is the quantity y of fish in the fishery (what? you didn't know that fish is infinitely divisible?). The action space is the quantity to harvest, so $A = [0, \bar{y}]$ as well. The period t reward is given by $r_t(y, x) = \pi(x)$. This shows that r_t is independent of time and of state. The transition function F is given by $F_t(y, x) = f(y - x)$, which is also independent of time. Finally the feasible action correspondence is given by $\Phi_t(y) = [0, y]$, which is also time independent.

   (b) These are obtained if the functions and correspondences of the FHDP problem described above satisfy A1-3 of chapter 11 of Sundaram. Compact-valuedness and continuity of Φ are obvious, and do not require any assumption. In order to satisfy A1, we require π to be continuous on A (remember that r does not depend on S, so if this assumption is satisfied r is continuous on $S \times A$), boundedness follows from the compactness of $S \times A$. Finally A2 holds if f is continuous on $\mathbf{R}$ (notice that, since $y - x$ is a continuous function on $\mathbf{R}_+ \times \mathbf{R}_+$, this is enough to insure that F is continuous on $S \times A$).

   (c) Suppose now that $\pi(x) = \ln x$ and $f(x) = x^\alpha$, for $\alpha \in (0, 1]$. These satisfy the conditions described above, so that an optimal strategy exists. To find such strategy, we start, by backwards induction, from the last period problem. Since π is increasing, it is obvious that the optimal strategy $g_T(y) = y$ for every $y \in S$. This implies that $V_T(y) = \ln y$. Consider now the problem at $T-1$. Now we have to solve

$$\max_{x\in[0,y]} \ln x + V_T((y-x)^\alpha) = \ln x\,(y-x)^\alpha.$$

Since this is strictly convex, first-order conditions are necessary and sufficient for a maximum, so we get that

$$g_{T-1}(y) = \frac{y}{1+\alpha} \quad \text{and} \quad V_{T-1}(y) = (1+\alpha)\ln y + K,$$

where K is a sum of terms which does not depend on y (that, as we shall see presently, we do not really have to care about). This seems to suggest that for period t we have

$$g_t(y) = \frac{y}{1+\alpha+\alpha^2+\cdots+\alpha^{T-t}}, \tag{1}$$

and

$$V_t(y) = (1+\alpha+\alpha^2+\cdots+\alpha^{T-t})\ln y + K(\alpha,t), \tag{2}$$

where once again the terms $K(\alpha,t)$ is a sum which does not depend on y (but only on t and α). We now verify this guess (that works by definition for T-1) by induction. Suppose that g_τ and V_τ have resp. the form (1) and (2) for every $\tau \in \{t+1,\ldots,T\}$. We want to show that then they have it for t. Given y, the firm is

$$\max_{x\in[o,y]} \ln x + V_{t+1}((y-x)^\alpha) = \max_{x\in[0,y]} \ln x + (1+\cdots+\alpha^{T-t-1})\ \ln(y-x)^\alpha.$$

Taking first-order conditions, we find that the optimal $\hat{x}$ satisfies

$$\hat{x} = \frac{y}{1+\alpha+\alpha^2+\cdots+\alpha^{T-t}},$$

so that $g_t(y)$ does indeed satisfy (1). Plugging this into the objective function immediately shows that $V_t(y)$ also has the form (2), which concludes our induction step, and shows that $\sigma = [g_0,\ldots,g_T]$ is an optimal strategy for the FHDP problem.

2. Solve exercise 3 of Chapter 12 (p.309) in Sundaram.

   Answer: We have that $f(x) = ax + b$, hence

   $$|f(x) - f(y)| = |a(x - y)| = |a||x - y|.$$

   So, for f to be a contraction, we need $b \in \mathbf{R}$ and $a \in [-1, 1]$.

3. Solve exercise 21 of Chapter 12 (p.311) in Sundaram.

   Answers:

   (a) f is not continuous at $x = (1, 1/2)$. To see this, let $x_n = (1, 1/2 - 1/n)$, for $n = 1, 2, \ldots$, and notice that $x_n \to x$. However, $f(x_n) = 0$ for all n, so that $f(x_n) \to 0 \neq f(x) = 1$.

   (b) If $\delta = 0$, then the optimal choice for $s = 0$ is $a = 0$ and for $s = 1$ is $a = 0$. This gives $V(0) = 0$ and $V(1) = 1$.

   (c) If $s_0 = 0$, then the system is stuck at 0 forever, so that the optimal choice is $a = 0$ regardless of the discount factor. What if $s_0 = 1$? If the DM chooses any $a < 1/2$, then $s_1 = 0$, so that the future optimal payoff will hence be 0 (see the previous case). Conditional on this range of values for a, the optimal choice is then given by $a = 0$, which provides a total stream of payoffs of 1. If the DM chooses $a \geq 1/2$, then $s_1 = 1$. Assuming that any strategy that satisfies Bellman's equation is optimal also in this (discontinuous) case (it is, why?), we thus conjecture that the optimal strategy prescribes $a = 1/2$ any time that $s = 1$, which provides $V(1) = 1/[2(1 - \delta)]$. In order for Bellman's equation to be satisfied we need

   $$1 \leq \frac{1}{2(1 - \delta)},$$

   that is, $\delta \geq 1/2$. So, if $\delta \geq 1/2$, the optimal strategy is $\pi^*(0) = 0$ and $\pi^*(1) = 1/2$. If, instead, $\delta < 1/2$, the optimal strategy is $\pi^*(0) = 0$ and $\pi^*(1) = 0$.

   (d) For $\delta = 1$, the solution 'should' be the same as that for $\delta > 1/2$ above.

   (e) If f has this shape, f is again discontinuous (proof?), but in this case the optimal strategy is missing because of an 'openness' problem. When $s_0 = 0$, the optimal thing to do is still $a = 0$, but when $s_0 = 1$, the DM would like to choose as close to 1/2 as possible, without getting there. As game theorists would say, there is thus no optimal strategy, but there are ϵ-optimal strategies (strategies that deliver an expected utility which is less than ϵ away from optimality, for every $\epsilon > 0$). The morale is that, when continuity of f fails, one could still find an optimal strategy (like in (c)), or one could not.