MATHEMATICAL ECONOMICS: SUGGESTED SOLUTIONS TO HOMEWORK # 5

1. Points 1, 2 and 4 are immediate. Let's look at 3. One of the implications is immediate: if C satisfies the condition that for all nonnegative $\alpha, \beta \in \mathbf{R}$, $\alpha C + \beta C = (\alpha + \beta)C$, then clearly taking $\beta = 1 - \alpha$ we see that C must be convex. In fact, in the case in which $\alpha + \beta = 1$ also the converse is true: For, suppose that C is convex and α, β are such that $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Then by convexity of C we have that $C \supseteq \alpha C + \beta C$. The converse inclusion follows immediately by taking the same element from each replica of C in the r.h.s..

   Turn now to the converse in the general case: Suppose that C is convex and α, β are nonnegative. By the argument we just made, we have that

   $$\frac{\alpha}{\alpha + \beta} C + \frac{\beta}{\alpha + \beta} C = C.$$

   But this implies that

   $$\alpha C + \beta C = (\alpha + \beta)[\frac{\alpha}{\alpha + \beta} C + \frac{\beta}{\alpha + \beta} C] = (\alpha + \beta)C.$$

   The proof of the second half of point 5 (for the closure) is actually different from that of the first point seen in class. Here you want to prove directly that given $a, b \in \text{cl}(C)$ and $\alpha \in [0, 1]$, then $\alpha a + (1 - \alpha)b \in \text{cl}(C)$. To do so, find sequences a_n and b_n in C such that $a_n \to a$ and $b_n \to b$. By convexity of C, $\alpha a_n + (1 - \alpha)b_n \in C$ for every n, and clearly $\alpha a_n + (1 - \alpha)b_n \to \alpha a + (1 - \alpha)b$, so that the limit must also be in the closure of C.

2. (The pointwise limit of a sequence of convex functions is a convex function) Suppose that $f^n \to f$ pointwise, and each f_n is convex. If f is not convex, then there are $x, y \in V$ and $\alpha \in [0, 1]$ and $\varepsilon > 0$ such that

   $$f(\alpha x + (1 - \alpha)y) > \alpha f(x) + (1 - \alpha)f(y) + \varepsilon.$$

On the other hand, each f_n is convex, so that

$$f_n(\alpha x + (1-\alpha)y) \leq \alpha f_n(x) + (1-\alpha) f_n(y).$$

Putting the two inequalities together, and denoting $z = \alpha x + (1-\alpha)y$, we get

$$(f(z) - f_n(z)) > \varepsilon + \alpha(f(x) - f_n(x)) + (1-\alpha)(f(y) - f_n(y)),$$

which by pointwise convergence leads to a contradiction (as $n \to \infty$), since ε is positive.

3. (Theorem 3.14 from class) Let $K = A - B$. Since A and B are disjoint, $0 \notin K$. By Lemma 3.13, there is a $p \in \mathbf{R}^n \setminus \{0\}$ such that $pz \geq p0 = 0$ for every $z \in K$. That is, $p(a-b) \geq 0$ for every $a \in A$ and $b \in B$.

4. (The cone $K = \{y \in \mathbf{R}^n : y = \sum_{i=1}^m \lambda_i a^i, \exists \lambda_i \geq 0, i = 1, \ldots, m\}$ is nonempty, convex and closed) It is clear that K is nonempty. Convexity is also straightforward. We prove closedness. Let y^n be a sequence in K and suppose that $y^n \to y$. For each y^n there is a corresponding vector $\lambda^n = [\lambda_1^n, \ldots, \lambda_m^n] \in \mathbf{R}^m$. Since y^n is Cauchy and

$$\|y^m - y^n\| = \|\sum_i (\lambda_i^m - \lambda_i^n) a^i\|,$$

it follows that the sequence (λ^n) must also be Cauchy and therefore converge in $\mathbf{R}^m$. Denote by λ its limit. First of all, notice that since $\lambda^n \geq 0$ for every n, then $\lambda \geq 0$. We now want to show that $z = \sum_i \lambda_i a^i \in K$ must be equal to y. Notice that

$$\|y - z\| \leq \|y - y^n\| + \|y^n - z\|,$$

and since both terms on the r.h.s. converge to 0 as $n \to \infty$, we must have that $y = z$.