MATHEMATICAL ECONOMICS: SUGGESTED SOLUTIONS TO HOMEWORK # 4

1. (Ex. 1, Ch. 5) Most of these questions are straightforward, so I will not test your patience by repeating their proof here [:-)]. Here is an example that $f(A \cap B) \subset f(A) \cap f(B)$. Consider the projection of $\mathbf{R}^2$ into $\mathbf{R}$ given by $f(x, y) = x$. Then any two constant hyperplanes of the form $(\cdot, y)$ and $(\cdot, y')$ map to the same set (all of $\mathbf{R}$), even though they have empty intersection if $y \neq y'$.

2. (Ex. 17, Ch. 5) Since D is dense in M, for every $x \in M$, there is a sequence $(x_n) \in D$ such that $x_n \to x$. Given such a sequence, we have that $f(x_n) = g(x_n)$ for every n, so that since f and g are continuous, $f(x) = g(x)$ (remember that sequences in $\mathbf{R}$ can have at most one limit!).

   Suppose now that f is onto. Given any $y \in N$, there is some $x \in M$ such that $y = f(x)$. Find a sequence $(x_n) \subseteq D$ that converges to x, and notice that $(f(x_n))$ is a sequence in $f(D)$ that converges to y. Therefore $\overline{f(D)} = N$.

3. (Ex. 25, Ch. 5) Take $\delta = \epsilon/K$.

4. (Ex. 53, Ch. 5) Just consider the function $f(z) = |z - x|$. It is continuous, and clearly $|f(x_n) - f(x)| = |x_n - x|$, so that $x_n \to x$ if $f(x_n) \to f(x)$.

5. (Ex. 57, Ch. 5) Let f be a bijective function. We first show that $(i) \Rightarrow (ii)$. If $G \subseteq M$ is closed, then G^c is open, so that $f(G^c)$ is open. But notice that, since f is bijective,

$$
\begin{aligned}
(f(G))^c = N \setminus f(G) &= \{y \in N : y \neq f(x), \forall x \in G\} \\
&= \{y \in N : y = f(x), \exists x \in M \setminus G\} = f(G^c).
\end{aligned}
$$

   (Subtle question: where do we use the fact that f is injective in this argument?) Thus $f(G)$ is closed. The proof that $(ii) \Rightarrow (i)$ follows along the same lines. Finally, the proof that $(i) \Leftrightarrow (iii)$ is immediate from Ex. 1 above and the fact that f is bijective.

6. (Ex. 6, Ch. 8) This is (an easy version of) a well-known result, called *Tychonoff's theorem*. We want to prove that if A and B are both compact, then $A \times B$, thus one (the hard one, in fact) of the implications of the theorem. We want to use the sequential characterization of compactness given in Theorem 8.2 in the book. That is, given an arbitrary sequence $(a_n, b_n) \in A \times B$, we want to show that it has a subsequence that converges to a point $(a, b) \in A \times B$. Notice first of all that, since A is compact, then (a_n) has a subsequence that converges. That is, there exist $a \in A$ and an increasing sequence of natural numbers (n_k) such that $a_{n_k} \to a$ (as $k \to \infty$). Consider now the subsequence (b_{n_k}). Since B is compact, it must itself have a converging subsequence. That is, there is $b \in B$ and an increasing subsequence n_{k_h} of n_k such that $b_{n_{k_h}} \to b$ (as $h \to \infty$). It follows that the corresponding subsequence $(a_{n_{k_h}}, b_{n_{k_h}}) \to (a, b)$ (as $h \to \infty$), proving that $A \times B$ is compact.

7. (Ex. 23, Ch. 8) We just need to show that f^{-1} is continuous. For instance, we can show that if $A \subseteq M$ is closed, then $f(A)$ must be closed. Since A is a closed subset of a compact set, it is compact (Cor. 8.3 in the book). Thus, $f(A)$ is compact because f is continuous (Thm. 8.4), and being compact it must be closed (Cor. 8.3 again).

8. (Ex. 30, Ch. 8) Suppose that condition (a) holds, and let $\mathcal{F}$ be a collection of closed sets such that finite intersections are all nonempty. Then $\cap_{F \in \mathcal{F}} F = \cap_{G \in \mathcal{G}} G^c = [\cup_{G \in \mathcal{G}} G]^c$ where $\mathcal{G} = \{G \subseteq M : G = F^c, \exists F \in \mathcal{F}\}$. Contradicting (b), suppose that $\cap_{F \in \mathcal{F}} F = \emptyset$. Then $\cup_{G \in \mathcal{G}} G \supseteq M$, so that by condition (a) there are a finite collection $G_1, G_2, \ldots, G_n \in \mathcal{G}$ such that $\cup_{i=1}^n G_i \supseteq M$. Taking complements, this means that, if we let $F_i = (G_i)^c$,

$$\cap_{i=1}^n F_i = \emptyset.$$

This yields a contradiction. The proof that (b) implies (a) is symmetric.