MATHEMATICAL ECONOMICS: SUGGESTED SOLUTIONS TO HOMEWORK # 3

1. (Ex. 17, Ch. 4) First, we show that A is open iff $A^\circ = A$. Recalling the definition of A° we immediately get that $A^\circ \subseteq A$ for any A. If A is itself open, then clearly $A \subseteq A^\circ$, proving the equality. Conversely, if $A = A^\circ$, then A is open.

   As to the proof that A is closed iff $A = \bar{A}$, the "only if" is again trivial. The proof of the "if" is analogous to the one above.

2. (Exs. 33 and 34, Ch. 4) Given $A \subseteq M$ and $x \in M$, suppose that x is a limit point of A. We want to show that for every $\varepsilon > 0$, the set $B_\varepsilon(x)$ contains infinitely many points of A (indeeed, $A \setminus \{x\}$). This is shown by contradiction. Suppose that for some ε, $B_\varepsilon(x)$ only contains finitely many points of $A \setminus \{x\}$. Then there must be some $y \neq x$ which is in $B_\varepsilon(x) \cap A$ and is closest to x. Let $\delta = d(x, y)$. Then $B_\delta(x) \cap (A \setminus \{x\}) = \emptyset$, which is a contradiction.

   Given the result we just proved, we can show that there is a sequence $(x_n) \subseteq A \setminus \{x\}$ such that $x_n \to x$. For every $n \in \mathbf{N}$, choose $x_n \in A \setminus \{x\}$ so that $x_n \in B_{1/n}(x)$. This selection is feasible since by what we just proved every such open ball contains infinitely many elements. (It does require the axiom of choice, though.) By construction it yields a sequence that converges to x.

3. (Ex. 46, Ch. 4) The fact that D (the statement "A is dense") implies (a) follows immediately from Corollary 4.11 in the book. The implication (a)⇒(b) is obvious, as is the implication (b)⇒(c).

   Saying that A^c has nonempty interior amounts to saying that there is $x \in A^c$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A^c$; equivalently, there is an open ball $B_\varepsilon(x)$ such that $B_\varepsilon(x) \cap A = \emptyset$. This proves (c)⇒(d) (by proving its contrapositive).

   To prove that (d)⇒D, begin by observing that for every $A \subseteq M$, $(\bar{A})^c = (A^c)^\circ$. In fact, $x \in (\bar{A})^c$ if there is $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \emptyset$. The latter set equality is equivalent to saying that $B_\varepsilon(x) \subseteq A^c$, so that for some $\varepsilon > 0$, $B_\varepsilon(x) \subseteq A^c$, which is equivalent to saying that $x \in (A^c)^\circ$.

Given the equality $(\bar{A})^c = (A^c)^\circ$, if A^c has empty interior then $\emptyset = (A^c)^\circ = (\bar{A})^c$, which is equivalent to $M = \bar{A}$.