MATHEMATICAL ECONOMICS: SUGGESTED SOLUTIONS TO HOMEWORK # 2

1. (ex. 5) As usual, the only demanding property to prove is the triangle inequality, the other three properties being immediate for all the three proposed metrics.

   Consider a nondecreasing function $f : A \to \mathbf{R}$, with $A \subseteq \mathbf{R}_+$ a linear subspace of $\mathbf{R}_+$, such that for any $x, y \in A$, $f(x+y) \leq f(x) + f(y)$. Then, by using the triangle inequality for the absolute value we find

$$
\begin{aligned}
f(|a-c|) &\leq f(|a-b| + |b-c|) \\
&\leq f(|a-b|) + f(|b-c|).
\end{aligned}
$$

   Thus, to show that a function is a metric, we just need to show that it induces a function $f : \mathbf{R}_+ \to \mathbf{R}$ which is nondecreasing and subadditive on $\mathbf{R}_+$.

   - $\rho(a,b) = f(|a-b|)$ for $f(x) = \sqrt{x}$, which is increasing on $\mathbf{R}_+$. As to subadditivity, notice that

$$x + y \leq x + 2\sqrt{xy} + y,$$

     which is equivalent to subadditivity (since all numbers involved are nonnegative).
   - $\sigma(a,b) = f(|a-b|)$ for $f(x) = x/1 + x$, which is increasing on $\mathbf{R}_+$. Next, notice that for any $x, y \in \mathbf{R}_+$,

$$\frac{x+y}{1+x+y} = \frac{x}{1+x+y} + \frac{y}{1+y+x} \leq \frac{x}{1+x} + \frac{y}{1+y},$$

     proving subadditivity.
   - $\tau(a,b) = f(|a-b|)$ for $f(x) = \min\{x, 1\}$, which is nondecreasing on $\mathbf{R}_+$. Finally, for any $x, y \in \mathbf{R}_+$, notice that if either $x > 1$ or $y > 1$ (or both), then $\min\{x+y, 1\} = 1$, so that

$$\min\{x+y, 1\} \leq \min\{x, 1\} + \min\{y, 1\},$$

     follows immediately. If instead $x \leq 1$ and $y \leq 1$, then the above equation boils down to $\min\{x+y, 1\} \leq x + y$, which is certainly true.

2. (Ex. 15) Suppose first that A is bounded. Then there is x_0 and $\alpha \in \mathbf{R}$ such that $d(a, x_0) \leq \alpha$ for all $a \in A$. By the triangle inequality, for any $a, b \in A$,

$$d(a, b) \leq d(a, x_0) + d(x_0, b) \leq 2\alpha.$$

   This shows that $\{d(a, b) : a, b \in A\}$ is a bounded set, so that $\text{diam}(A) < \infty$.

   Next, suppose that $\text{diam}(A) = \alpha < \infty$. Fix $x_0 \in A$. Then it follows that for any $a \in A$,

$$d(a, x_0) \leq \alpha,$$

   which shows that A is bounded.

3. (Ex. 16) Let V be a vector space, and d a metric on V that satisfies the conditions stated in the exercise. We want first to show that $\|x\| = d(x, 0)$ is a norm. Properties (i) and (ii) follow immediately from analogous properties of d. As to (iii), if $\alpha \in \mathbf{R}$, then $\|\alpha x\| = d(\alpha x, 0) = |\alpha|\, d(x, 0) = |\alpha|\, \|x\|$. Finally,

$$\|x+y\| = d(x+y, 0) = d(x, -y) \leq d(x, 0) + d(-y, 0) = \|x\| + \|y\|.$$

   To see an example of a vector space and a metric that is not induced by a norm, consider $\mathbf{R}$ with the discrete metric. Notice first of all that if $\alpha \neq \pm 1$, $d(\alpha\, x, \alpha\, y) = 1 \neq \alpha$, so that one of the properties used above fails. And indeed the function $\|x\| = d(x, 0)$ is not a norm, since $\|x\| = 1$ for all $x \in \mathbf{R} \setminus \{0\}$ (thus failing property (iii)).

4. (Ex. 33) By the triangle inequality,

$$d(x, y) \leq d(x, x_n) + d(y, x_n).$$

   If the sequence (x_n) converges to both x and y, then $d(x, x_n) \to 0$ and $d(y, x_n) \to 0$ as $n \to \infty$, which implies that $d(x, y) = 0$, or equivalently $x = y$.

5. (Ex. 34, Ch. 3) Let's just show the most general statement (the first one follows by taking $y_n \equiv y$). By the triangle inequality (twice) we have

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y_n) \\ &\leq d(x_n, x) + d(y_n, y) + d(x, y). \end{aligned}$$

Symmetrically,

$$\begin{aligned} d(x,y) &\leq d(x_n,x)+d(x_n,y) \\ &\leq d(x_n,x)+d(y_n,y)+d(x_n,y_n). \end{aligned}$$

We thus have that for every $n \in \mathbf{N}$,

$$d(x,y)-d(x_n,x)-d(y_n,y) \leq d(x_n,y_n) \leq d(x,y)+d(x_n,x)+d(y_n,y),$$

so that by the squeezing theorem the real sequence $d(x_n,y_n)$ converges to

$$\begin{aligned} d(x,y) &= \lim_n [d(x,y)-d(x_n,x)-d(y_n,y)] \\ &= \lim_n [d(x,y)+d(x_n,x)+d(y_n,y)]. \end{aligned}$$

6. (Ex. 40) Fix $x \in \ell_1$. Then, for any given k,

$$\begin{aligned} \|x-x^{(k)}\|_1 &= \sum_{i=1}^{\infty} |x_i - x_i^{(k)}| \\ &= \sum_{i=k+1}^{\infty} |x_i| \\ &= \sum_{i=1}^{\infty} |x_i| - \sum_{i=1}^{k} |x_i|. \end{aligned}$$

Taking limits as $k \to \infty$ shows that $\|x-x^{(k)}\|_1 \to 0$.

Next, consider $x \in \ell_2$. Then,

$$\begin{aligned} \|x-x^{(k)}\|_2 &= \sqrt{\sum_{i=1}^{\infty} |x_i - x_i^{(k)}|^2} \\ &= \sqrt{\sum_{i=k+1}^{\infty} |x_i|^2} \\ &= \sqrt{\sum_{i=1}^{\infty} |x_i|^2 - \sum_{i=1}^{k} |x_i|^2}. \end{aligned}$$

Again, taking limits as $k \to \infty$ shows that $\|x - x^{(k)}\|_2 \to 0$.

Finally, consider $x = (0, 1, 0, 1, 0, 1, \ldots) \in \ell_\infty$. It is clear that for every k

$$\|x - x^{(k)}\|_\infty = \sup_i |x_i - x_i^{(k)}| = 1,$$

therefore $x^{(k)}$ never converges to x in the sup-norm.

7. (ex. 46) It is simple to see that the three proposed metrics satisfy properties (i)-(iii) of a metric. Therefore, let's concentrate on property (iv), the triangle inequality. This is trivial for d_1 and simple (but tedious) for d_2. As to d_∞, it follows from the observation that

$$\max\{\alpha + \beta, \gamma + \delta\} \leq \max\{\alpha, \gamma\} + \max\{\beta, \delta\}.$$

Next, it is straightforward to show that these metrics satisfy the property that whenever $a_n \to a$ (in d) and $x_n \to x$ (in ρ), then $(a_n, x_n) \to (a, x)$ according to the product metric, *and conversely*. (Either because the sum of two 0's is a 0, or becasue the max of two 0's is a 0.)

Finally, all these metrics are equivalent if $d_1((a_n, x_n), (a, x)) \to 0$ iff $d_2((a_n, x_n), (a, x)) \to 0$ iff $d_\infty((a_n, x_n), (a, x)) \to 0$. But these equivalences follow from what we just showed, since in all the three metrics convergence happens iff $d(a_n, a) \to 0$ and $\rho(x_n, x) \to 0$.