MATHEMATICAL ECONOMICS: SUGGESTED SOLUTIONS TO HOMEWORK # 1

1. (Ex. 3) This is straightforward.
2. (Ex. 14) This is also straightforward.
3. (Ex. 24) Suppose first that (a_n) is a bounded sequence. Notice that then

$$R_n = \sup\{-a_n, -a_{n+1}, \ldots\} = -\inf\{a_n, a_{n+1}, \ldots\} = -t_n.$$

Moreover, (R_n) and (t_n) are also bounded, so

$$\begin{aligned} \limsup_n(-a_n) &= \inf_n R_n \\ &= \inf_n -(t_n) \\ &= -\sup_n(t_n) \\ &= -\liminf_n a_n. \end{aligned}$$

Next, consider the case in which (a_n) is unbounded.

Given a sequence (b_n), if $\limsup b_n = -\infty$, that means that $T_n = \sup\{b_n, b_{n+1}, \ldots\}$ is unbounded below and diverges to $-\infty$. Indeed, then the sequence (b_n) itself must be diverging to $-\infty$ (thus, we end up solving exercise 25 along with 24, aren't we lucky?). For, suppose not. Then there is a $\beta \in \mathbf{R}$ such that for every N, we can find $n \geq N$ for which $b_n \geq \beta$ (check that this is really what it means to negate that $\lim b_n = -\infty$). This implies, however, that $T_N \geq \beta$ for every N; i.e., T_n is bounded below, contradicting the fact that $T_n \to -\infty$.

Thus, if $\limsup -a_n = -\infty$, then $(-a_n) \to -\infty$, so that $a_n \to +\infty$. This implies (why?) that $\liminf a_n = +\infty$, so that

$$-\liminf a_n = -\infty = \limsup -a_n.$$

Next, we show that, given a sequence (b_n), if $\limsup b_n = +\infty$, there must be a subsequence of (b_n) that diverges to $+\infty$ (thus concluding the proof of Exercise 25). Indeed, if

$$\inf_n(\sup\{b_n, b_{n+1}, \ldots\}) = +\infty$$

then $\sup\{b_n, b_{n+1}, \ldots\} \equiv +\infty$; that is, every tail of (b_n) is unbounded above. This implies that there is a subsequence of (b_n) that diverges properly. For, notice that given $k \in \mathbf{N}$ there must be an element $b_{n_k} \geq k$. This can be repeated with $k+1$, taking care to consider only the tail of the sequence $(b_{n_k+1}, b_{n_k+2}, \ldots)$, and so on. (A careful proof of this is done by induction, but it is clear how this works.)[1]

Thus, if $\limsup -a_n = +\infty$, then there is a subsequence $(-a_{n_k}) \to +\infty$. This implies that $(a_{n_k}) \to -\infty$, so that for every n,

$$t_n = \inf\{a_n, a_{n+1}, \ldots\} = -\infty.$$

It follows that $\liminf a_n = \sup_n t_n = -\infty$, so that, once again,

$$-\liminf a_n = +\infty = \limsup -a_n.$$

4. (Ex. 27) Let (a_n) be a sequence. Notice to start that if $\limsup a_n = \pm\infty$ then we have seen in the previous exercise how to find a subsequence that converges to $\limsup a_n$.

   So, suppose that $\limsup a_n = M \in \mathbf{R}$. As we know, the previous equality implies that for every $\varepsilon > 0$, $a_n > M - \varepsilon$ for infinitely many n and $a_n < M + \varepsilon$ for all but finitely many n. Notice, first of all, that because of this we can find n_1 which is large enough so that $|a_{n_1} - M| < 1$. Similarly, for every $k > 1$ we can find an index $n_k > n_{k-1}$ such that $|a_{n_k} - M| < 1/k$. We thus obtain inductively a subsequence (a_{n_k}) of (a_n) such that $(a_{n_k}) \to M$.

   The result about $\liminf a_n$ follows from the result proved above (ex. 24) that
   $$\liminf a_n = -\limsup(-a_n),$$
   as the sequence $(-a_{n_k})$ that we obtain by the above reasoning (applied to $(-a_n)$) converges to $-M = \liminf a_n$.

5. (Ex. 33) The first statement is easy to prove: Suppose the original sequence (x_n) converges to some limit x. Take any sequence

[1] Notice that (b_n) itself does not have to diverge to $+\infty$. Consider the sequence $(1, 2, 1, 4, 1, 6, ...)$ which coincides with n on even indices and is identically 1 on odd indices. Such sequence satisfies $\limsup b_n = +\infty$, but b_n does not diverge to $+\infty$ (it doesn't have a limit).

n_k of natural numbers and consider the corresponding subsequence of the original sequence. For any $\epsilon > 0$ there exists an integer N such that for all $n \geq N$

$$|x_n - x| < \epsilon$$

But then we also have the same inequality for the subsequence as long as $n_k \geq N$. Therefore any subsequence *itself* must converge to the same limit x.

The second statement is almost as easy. Suppose (x_n) is a sequence such that every subsequence extracted from it has a further subsequence that converges to the same limit x. Suppose that (x_n) *does not* converge to x. Then for any k, there exist n_k for which $|x_{n_k} - x| > 1/k$. But then the subsequence (x_{n_k}) cannot have a further subsequence that converges to x, violating my assumption.

6. (Ex. 45) This is immediate.