Mathematical Economics

Final (full) Examination

May 13, 2024

Please, answer at least 5 of the following 6 questions. ***Time allowed****: two hours and 45 minutes. PLEASE PLEASE, make an effort to write in a legible and organized fashion.*

Question 1. Metric spaces and normed vector spaces. Definitions, properties and examples. In particular, discuss the example of the space $C([a, b])$.

Question 2. The Weierstrass theorem in its two variants, with appropriate definitions and examples.

Question 3. Pointwise and uniform convergence of functions. Definitions and examples. Is there a connection between uniform convergence and convergence in a specific norm? Results about properties of uniform limits and their consequences for spaces of functions.

Question 4. Upper and lower hemi-continuity. Definitions and examples. Relations with other properties (closed- , compact- and convex-valued, closed- and convex-graph). The maximum theorem.

Finally, consider the correspondence $\varphi(x) : [0, 3] \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} [1, 2] & x \in [0, 1) \\ \{1\} & x \in [1, 2) \\ [0, 1] & x \in [2, 3] \end{cases}$$

Identify all the properties it satisfies.

Question 5. Introduce a stationary discounted dynamic programming (SDDP) problem and define an optimal strategy for such problem. After recalling the important property that the value function V must satisfy, present, in as much detail as possible (including its mathematical background), the result which shows that an optimal strategy must induce a revenue function ($w(\sigma)(s)$) that satisfies such property. State clearly which assumptions on the SDDP you are making.

Question 6. Let $S = \{0.1\}$, $A = \mathbb{R}_+$ and $\Phi(s) = A$ for all $s \in S$. Let the state transition function be given by

$$f(s, a) = \begin{cases} 0 & \text{if } s = 0 \text{ or } a < 1/3 \\ 1 & \text{if } s = 1 \text{ and } a \geq 1/3 \end{cases}$$

Let $r : S \times A \to \mathbb{R}$ be given by

$$r(s, a) = \begin{cases} -a & \text{if } s = 0 \\ 1 - a & \text{if } s = 1 \end{cases}$$

Is f continuous? Assuming $\delta \in (2/3, 1)$ show (by constructing it!) that, nonetheless, there is an optimal strategy. (Notice that Bellman's equation is still valid, and explain why referring to the answer to the previous exercise).