DECISION THEORY: SUGGESTED SOLUTIONS TO HOMEWORK # 5

1. In the Anscombe and Aumann model of Chapter 7 of Kreps, consider the following dominance axiom...

   Answer: The fact that [7.1-3, 7.14]+7.16 imply 7.17 is easily proved using the representation given in Theorem 7.17 in Kreps. The axioms are equivalent to the representation in Eq. (7.18), from which it is immediate to show that if $h_s \succeq h'_s$ (which is equivalent to $\sum_z u(z)h_s(z) \geq \sum_z u(z)h'_s(z)$) for every $s \in S$, then $h \succeq h'$.

   As for the converse implication, assume that [7.1-3, 7.14] and 7.17 hold, and suppose that for $h \in H$

   $$[h_{-s}, p] \succ [h_{-s}, q],$$

   (where I use the notation $[h_{-s}, r] \equiv (h_1, \dots, h_{s-1}, r, h_{s+1}, \dots, , h_n)$) for some $p, q \in P$. Then, using (the contrapositive of) axiom 7.17, we immediately obtain $p \succ q$. Applying axiom 7.17 again, we obtain that for every s' (not necessarily non-null)

   $$[h_{-s'}, p] \succeq [h_{-s'}, q]. \tag{1}$$

   We are only left to show that is s' is non-null then we cannot have indifference in Eq. (1). We show this by contradiction. If that were the case I claim that we would get that for any $r \in P$ we have

   $$[h_{-s'}, p] \sim [h_{-s'}, r]. \tag{2}$$

   But then s' must be null: Using the representation in (7.5) in Kreps, equation (2) implies that $u_{s'}$ must be constant. Corollary (7.15) in Kreps can then invoked to argue that s' is null. So we obtain the desired contradiction.

   I finally prove my claim that (2) must hold. If $p \succeq r \succeq q$, the result follow immediately from (1) and two applications of axiom 7.17. Suppose that $r \succ p \succ q$. Then axioms 7.1-3 imply (in the same fashion as in Lemma (5.6)(b) in Kreps) that there is a unique $\alpha \in [0,1]$ such that $p \sim \alpha r \oplus (1-\alpha)q$. Hence using 7.17 twice, 7.2 and (1) we obtain

$$\begin{aligned}[h_{-s'}, \alpha p \oplus (1-\alpha)p] &= [h_{-s'}, p] \\ &\sim [h_{-s'}, \alpha r \oplus (1-\alpha)q] \\ &\sim [[h_{-s'}, \alpha r \oplus (1-\alpha)p]\end{aligned}$$

and then (2) follows from axiom 7.2 (and 7.1). The case in which $q \succ r$ works symmetrically.

2. Solve problem 6 of Chapter 7 in Kreps.

Answer: Consider the preference relation $\succ_1$ on P defined as follows: $p \succ_1 q$ iff $(p, p^\circ) \succ (q, p^\circ)$. It is immediate to observe that H is a mixture space and that axioms (a) through (f) imply that there exists a cardinal utility u such that $p \succ_1 q$ iff $\sum u(x)\, p(x) \geq \sum u(x)\, q(x)$.

It follows from (c) that for every $p \in P$, $p^\circ \succeq_1 p \succeq_1 p_\circ$. Hence if we let $U(p) = \sum u(x)\, p(x)$, we have $U(p^\circ) \geq U(p) \geq U(p_\circ)$. Moreover (g) implies that $p^\circ \succ_1 p_\circ$. In fact, otherwise $p^\circ \sim_1 p_\circ$, which implies using (b) that $(p^\circ, p^\circ) \sim (p^\circ, p_\circ)$, so that using (d) twice we get $(p_\circ, p^\circ) \sim (p_\circ, p_\circ)$, and using (b) and (a) we then get $(p^\circ, p^\circ) \sim (p_\circ, p_\circ)$, a violation of (g). Given this, we can normalize u so to have $U(p^\circ) = 1$ and $U(p_\circ) = 0$, so that $U(p) \in [0,1]$ for every $p \in P$.

Given $a \in [0,1]$, let $s(a)$ be the binery lottery defined as follows: $s(a) = a\, p^\circ \oplus (1-a)\, p_\circ$. Fix now (p,q). By definition of $s(U(p))$, we get that

$$(p, p^\circ) \sim (s(U(p)), p^\circ),$$

so that by (d) twice we get $(p,q) \sim (s(U(p)), q)$. Analogously, $(p^\circ, q) \sim (p^\circ, s(U(q)))$ which with the above and (a) implies

$$(p,q) \sim (s(U(p)), q) \sim (s(U(p)), s(U(q))).$$

Consider the ordering $\succeq^*$ on $[0,1]^2$ defined by

$$(a,b) \succeq^* (a',b') \Longleftrightarrow (s(a), s(b)) \succeq (s(a'), s(b')).$$

It is by now clear that

$$(p,q) \succeq (p',q') \Longleftrightarrow (U(p), U(q)) \succeq^* (U(p'), U(q')).$$

We have to show that $\succeq$ is a weak order on $[0,1]^2$. This requires showing that for every $a \in [0,1]$ there is a $p \in P$ such that $a = U(p)$. But this is obvious: Take $p = a\, p^\circ \oplus (1-a)\, p_\circ$. The fact that $\succeq^*$ is a weak order now follows from the fact that $\succeq$ is a weak order as well.

Suppose now that also (h) holds. We have that $p \succeq_1 q$ implies that $(p,q) \sim (q,q)$. Given $(p,q), (p',q')$, suppose w.l.o.g. that $p \succeq_1 q$ and $p' \succeq_1 q'$. Then $(p,q) \succeq (p',q')$ iff $(q,q) \succeq (q',q')$. We now show that $U(q) \geq U(q')$ iff $(q,q) \succeq (q',q')$. First assume that $(q,q) \succeq (q',q')$. From (c) it follows that $(p^\circ, p^\circ) \succeq (p^\circ, q)$, so that (h) implies $(p^\circ, q) \sim (q,q)$. Similarly, $(p^\circ, q') \sim (q',q')$, so that using (a) we get $(q,q) \succeq (q',q')$ implies $q \succeq_1 q'$, which in turn implies $U(q) \geq U(q')$. Conversely, assume that $U(q) \geq U(q')$, so that $q \succeq_1 q'$. Applying (d) twice we get $(q,q) \succeq (q,q')$ and $(q',q) \succeq (q',q')$ so that by (b) and (a) we get $(q,q) \succeq (q',q')$.

Summing up, we can conclude as requested that

$$(p,q) \succeq (p',q') \Longleftrightarrow \min[U(p), U(q)] \geq \min[U(p'), U(q')].$$

The preference relation $\succ$ clearly satisfies axiom 7.1 and 7.14. Axiom 7.3 is verified by looking at its mathematical representation above. Axioms 7.2 and 7.16 fail. To see 7.2, take $p, q \in P$ such that $U(p) > U(q)$: we have $(p,p) \succ (p,q)$ but if we mix with (q,p) we get

$$(\frac{1}{2}p + \frac{1}{2}q, p) \sim (\frac{1}{2}p + \frac{1}{2}q, \frac{1}{2}p + \frac{1}{2}q).$$

As for 7.16 choose $p, q, r, r' \in P$ such that $U(p) > U(r) > U(q) \geq U(r')$. Then $(p,r) \succ (q,r)$ and $(r',p) \sim (r',q)$, in violation of the axiom with $h = (r', r)$ (notice that state 2 is clearly non-null).

3. Solve problem 5 in Chapter 8 in Kreps.

   Answer: This is a routine proof. The only point that requires some conceptual work is point (d). To do that, you first show that $A \sim^* B$ and $C \sim^* D$, with B and D disjoint, then $B \cup D \succeq^* A \cup C$ (you have to use point (b) to show this). It is now easy to use this result twice to get to the conclusion.

4. (WARNING: ONLY FOR THE MASOCHIST!!!) Solve problem 8 in Chapter 8 of Kreps.

   Answer: The only interesting things to prove here are that there are no probabilities whatsoever which represent $\succ^*$ in the two cases, and that the relation in example 1 is tight, while the one in example 2 is fine. The first claim can be proved using the standard argument that there is no utility representing the lexicographic ordering on R^2. In fact in both cases we can represent a set a as a pair of *numbers* (in example 1 the pair is given by $(\ell(a_1), \ell(a_2))$, in example 2 the pair is $(\ell(a_1) + \ell(a_2), \ell(a_1)))$, and then the relation $\succ^*$ is just the lexicographic ordering. I now show that $\succ^*$ in example 1 is tight. Suppose that $a \succ^* b$. Then either $\ell(a_1) > \ell(b_1)$ or $\ell(a_1) = \ell(b_1)$ and $\ell(a_2) > \ell(b_2)$. Suppose that $\ell(a_1) > \ell(b_1)$. Then let $c = c_1$ where $c_1 \cap b_1 = \emptyset$ (clearly $b_1 \neq [0, 1]$ for otherwise $\ell(a_1) > 1$, a contradiction) and $0 < \ell(c_1) < \ell(a_1) - \ell(b_1)$. Then $a \succ^* b \cup c \succ^* b$, as required. In the case in which $\ell(a_1) = \ell(b_1)$, choose $c = c_2$ where $c_2 \cap b_2 = \emptyset$ and $0 < \ell(c_2) < \ell(a_2) - \ell(b_2)$ and again $a \succ^* b \cup c \succ^* b$. The proof of the fineness of $\succ^*$ in example 2 is straightforward.