DECISION THEORY: SUGGESTED SOLUTIONS TO HOMEWORK # 3

1. Show that axioms (NM0) NM1, NM3 and the following property do *not* imply NM2...

   Answer: The hint really gives this away. The idea is how to define $\succeq$ on $\mathcal{P}_S$ so that NM1 and NM3 are satisifed, but NM2 is not. For any $r \in \mathcal{P}_S$, let $r \sim q$. Then NM1 is obviously satisfied. NM3 holds trivially because there are only two indifference classes (so that the assumption that there are p, q, r such that $p \succ q \succ r$ is never true). Similarly it's immediate to check that the property stated in the exercise holds. What about NM2? We have that $p \succ q$, but for any α and any r it is clearly the case that $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$ (why?). So $\succeq$ shows that NM1, NM3 and the property above do not imply NM2.

2. Solve problem 1 of Chapter 5 in Kreps.

   Answer: The axiom which is violated is NM3, the Archimedean axiom, whereas all the others hold. To see that NM3 fails, consider the lotteries $p = [1/3, 1/3, 1/3]$, $q = [1/4, 5/12, 1/3]$, $r = [0, 0, 1]$. We have that $p \succ q \succ r$, but it is easy to convince oneself that there is no $\alpha \in (0, 1)$ such that $\alpha p + (1-\alpha)r) \succ q$. It is immediate to verify that $\succeq$ satisfies axiom NM1 (the lexic order is a strict order, after all). As for axiom NM2, suppose that $p \succ q$. That means that either $p_3 < q_3$ or $p_3 = q_3$ and $p_2 < q_2$. Thus, either $\alpha p_3 + (1-\alpha)r_3 < \alpha q_3 + (1-\alpha)r_3$, or $\alpha p_3 + (1-\alpha)r_3 = \alpha q_3 + (1-\alpha)r_3$ and $\alpha p_2 + (1-\alpha)r_2 < \alpha q_2 + (1-\alpha)r_2$. This proves that
   $$\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r.$$

3. Solve problem 4 in chapter 5 of Kreps.

   Answer: Consider the ordering induced in the set of outcomes X. Since X is finite, there are $\underline{x}, \overline{x} \in X$ such that, for any $x \in X$, $\delta_{\overline{x}} \succeq \delta_x \succeq \delta_{\underline{x}}$. We want to show that this is also the case for any (nondegenerate) lottery $p \in \mathcal{P}_S$. One way to do this is by proving the following

Lemma 1 *Suppose that $p_0, p_1, \ldots, p_n$ are simple lotteries, and $\{\alpha\}_{i=1}^n \subseteq [0,1]$ are weights such that $\sum_{i=1}^n \alpha_i = 1$. Then if $p_0 \succeq p_i$ for $i = 1, \ldots, n$ we have that $p_0 \succeq \sum_{i=1}^n \alpha_i p_i$. If instead $p_i \succeq p_0$ for $i = 1, \ldots, n$, we have that $\sum_{i=1}^n \alpha_i p_i \succeq p_0$.*

Proof: We prove the lemma by induction on n. If $n = 1$ there is nothing to prove. So let $n > 1$ and suppose that the statement is true for $n - 1$. Assume that $p_i \succeq p_0$ for all i. Using reduction of compound lotteries

$$\sum_{i=1}^{n} \alpha_i p_i = (1 - \alpha_n) \sum_{i=1}^{n-1} (\alpha_i / 1 - \alpha_n) p_i + \alpha_n p_n.$$

By the induction hypothesis $\sum_{i=1}^{n-1} (\alpha_i/1 - \alpha_n) p_i \succeq p_0$. Applying axiom 5.2 (the independence axiom) we obtain

$$(1 - \alpha_n) \sum_{i=1}^{n-1} (\alpha_i / 1 - \alpha_n) p_i + \alpha_n p_n \succeq (1 - \alpha_n) p_0 + \alpha_n p_n,$$

and applying 5.2 once again we obtain

$$(1 - \alpha_n) p_0 + \alpha_n p_n \succeq (1 - \alpha_n) p_0 \alpha_n p_0 = p_0.$$

By axiom 5.1 (weak order) we thus get $\sum_{i=1}^n \alpha_i p_i \succeq p_0$. The proof of the other statement is symmetric. ∎

Now, let p be the lottery which yields δ_{x_i} with probability p_i, for $i = 1, \ldots, n$. By definition $\delta_{\overline{x}} \succeq \delta_{x_i} \succeq \delta_{\underline{x}}$. Applying the lemma above we obtain $\delta_{\overline{x}} \succeq p \succeq \delta_{\underline{x}}$.

4. Solve problem 5 in Chapter 5 of Kreps.

   Answer: This is quite straightforward. It is immediate to verify that the function F defined from u satisfies (5.13), so that part (d) of Theorem 5.11 holds. Using (d) to show that part (a), (b) and (c) are satisfied is immediate.