DECISION THEORY: SUGGESTED SOLUTIONS TO HOMEWORK # 2

1. Solve problem 5 of chapter 3 in Kreps...

   Answer: Here are two counterexamples to the statement in the exercise:

   Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as follows:

   $$f(x) \equiv \begin{cases} x & x < 1 \\ 1 & 1 \leq x \leq 2 \\ x - 1 & x > 2 \end{cases}$$

   We want to construct u and v such that $v = f \circ u$. For instance consider $X = [0, 1] \cup (2, +\infty)$ and let $u(x) = x$ for all $x \in X$. Then $v(x) = f(u(x)) = f(x)$ for all $x \in X$. It can be seen immediately that there is no (strictly) *increasing* g such that $v = g \circ u$. In fact such g must satisfy the following: $g(1) = 1$ and $1 < g(2 + \epsilon) = 1 + \epsilon$ for every $\epsilon > 0$, which promptly gives a contradiction (take the limit as ϵ goes to zero). Another possible counterexample, which does not use a disconnected X set, is the following. Let $X = \mathbf{R}$ and consider the two functions $v(x) = x$ and

   $$u(x) \equiv \begin{cases} x & x \leq 1 \\ x + 1 & x > 1 \end{cases}$$

   Then it is easy to see that $v = f \circ u$, and by the same reasoning as above, there is no strictly increasing g such that $v = g \circ u$. (Interesting question: Does this work if we exchange the labels of u and v? Why was this clear from theorem 3.6?)

   Finally, a counterexample to the theorem in Kreps. On $X = (0, 1)$, consider $u(x) = x$ and $v(x) = 1/(1 - x)$. Clearly, u and v represent the same ordering. On Range$(u) = (0, 1)$ the function $\varphi = 1/(1 - x)$ clearly does the job (of being such that $v = \varphi \circ u$). However, it is impossible to extend φ to $\mathbf{R}$ so that φ is *nondecreasing* outside $(0, 1)$.

2. Let $\succeq$ be a relation on $X \subseteq \mathbf{R}^n$. Suppose that $\succeq$ is represented by a continuous utility function $u : X \to \mathbf{R}$. Show that then $\succeq$ is a continuous weak order.

Answer: Suppose that $\succeq$ is represented by a continuous $u : X \to \mathbf{R}$. It is immediate to check that $\succeq$ must then be a weak order. To check that it must be continuous, remember that, since u is continuous, we have that if $\{x_n\}$ is a sequence of points such that $x_n \to x$ for some $x \in X$, then $u(x_n) \to u(x)$. Suppose that $x \succ y$, so that $u(x) > u(y)$. From the definition of limit, for any $\epsilon > 0$ there is N such that for all $n \geq N$, $|u(x_n) - u(x)| < \epsilon$. One just needs to take ϵ less than the difference $|u(x) - u(y)|$ to do the job of showing that $x_n \succ y$ for $n \geq N$. The case of $y \succ x$ is treated symmetrically. This concludes the proof that $\succeq$ is a continuous weak order.

3. Something that I forgot to do in class...

   Answer: For any nonempty and compact B and $x \in B$, let $b(x, B) = \{y \in B : y \succeq x\}$; i.e., $b(x, B)$ is the set of the points in B which are weakly preferred to x.

   We first show that if we consider any finite collection $x_1, x_2, \ldots, x_n \in B$ (for any n), we must have that $x^i \in \cap_{j=1}^n b(x_j, B)$ for some $i = 1, \ldots, n$. This is proved by induction on n. The statement is obvious for $n = 1$ (since $\succeq$ is reflexive). Suppose it holds for every $n-1$-tuple of points of B, and consider $x_1, x_2, \ldots, x_n$. It follows from the induction hypothesis that there exists x_i such that $x_i \succeq x_j$ for every $j = 1, \ldots, n-1$. If $x_i \succeq x_n$, then $x_i \in \cap_{j=1}^n b(x_j, B)$. If instead $x_n \succ x_i$, then by transitivity $x_n \succ x_j$ for all $j = 1, \ldots, n$, proving that $x_n \in \cap_{j=1}^n b(x_j, B)$. Either way, $\cap_{j=1}^n b(x_j, B) \neq \emptyset$.

   It now follows from the "finite-intersection property" characterization of compactness (which says that a set B is compact iff the following is true: given a family of closed subsets of B such that every finite subfamily has a nonempty intersection, the family itself has a nonempty intersection) that since B is compact and every set $b(\cdot, B)$ is closed (by the continuity axiom), the family of sets $\{b(x, B) : x \in B\}$ has a nonempty intersection. That is, there is a $\hat{x} \in B$ such that $\hat{x} \in b(x, B)$ for every $x \in B$. Clearly, such $x \in C^*(B, \succeq)$, proving that the latter set is nonempty.