DECISION THEORY: SUGGESTED SOLUTIONS TO HOMEWORK # 1

1. In class I claimed that a cardinal utility is necessary for some decision criteria to make sense...

   Answer: This is a routine exercise. For instance, take the example I gave in class to illustrate the Maximin risk rule:

   $$\begin{bmatrix} 0 & 100 \\ 1 & 1 \end{bmatrix},$$

   where the choice that the criterion prescribes is that corresponding to the first row of payoffs. Suppose that we took another utility function that represents the same ordering of the final payoffs, to get the following matrix:

   $$\begin{bmatrix} 0 & 15 \\ 10 & 10 \end{bmatrix}.$$

   If we now calculate the risk matrix, we find

   $$\begin{bmatrix} -10 & 0 \\ 0 & -5 \end{bmatrix},$$

   so that the "maximin" choice is now the one corresponding to the bottom row. Hence the transformation of the utilities has changed the presciption of the criterion. Similar examples can be constructed for the $\alpha \min + (1-\alpha) \max$ criterion of Hurwicz, and for the "equal probabilities" criterion.

2. Solve problems 1 and 2 of Chapter 1 in Kreps.

   Answers: We first have to prove that axioms 1.1-1.5 and 1.9 are inconsistent on P, the set of all simple lotteries on $[0, 100]$. As Kreps suggests, we assume that, as is actually the case, theorem 1.6 is true (it's a corollary of the von Neumann-Morgenstern theorem we'll see later). Of course this could also be proved directly, but that is *much* more painful. So we know that axioms 1.1-1.5 are equivalent to the existence of a strictly increasing function $u : [0, 100] \to \mathbf{R}$ such that every lottery $p \in P$ is evaluated via the expected utility functional

$$E_p(u) = \sum_{r\in[0,100]} u(r)p(r).$$

Given the utility function u obtained by the theorem, let (without loss of generality) $u(0) = 0$ and $u(100) = 1$. We know that u is stricyly increasing. that does not mean however that it is continuous, only that it can have at most countably many discontinuity points. Let $r \in (0, 100)$ be a continuity point of u. Then for every $\epsilon > 0$ we can find a point $r' < r$ such that $|u(r)-u(r')| < \epsilon$. Consider now the two simple lotteries p and p' defined as follows: $p(r') = 1/2,\ p(100) = 1/2$ and $p'(r) = 1$. Then, for ϵ small enough,

$$\begin{aligned} E_p(u) &= (1/2)u(r') + (1/2)u(100) \\ &= (1 + u(r'))/2 \\ &> (1 + u(r))/2 \\ &> u(r). \end{aligned}$$

So, whatever u is, it is always possible to construct two simple lotteries belonging to P such that axiom 1.9 fails. Since the axiom was stated implicitly with an existential quantifier ("for every p and p'"), we have thus shown that the axioms are inconsistent. Notice that the argument works because there can only be countably many discontinuity points of u. Also observe that consistency can only be proved with respect to a given domain of the preference relation (P in this case): it is always possible to construct a domain (smaller than P) on which all axioma can be satisfied, so that they are consistent.

Proving that axioms 1.1 through 1.5 are consistent is immediate. The preference relation induced by taking the expected utility of every lottery with respect to some strictly increasing u (for instance the identity) is immediately seen to satisfy all axioms.

Analogously it is very simple to show that axioms 1.1-1.5 imply axiom 1.10 by using the expected utility representation (again it would be much more painful to show it directly, and it can be done) and observing that

$$E_{ap+(1-a)p'}(u) = aE_p(u) + (1 - a)E_{p'}(u).$$

3. Prove the following statement from class: Suppose that $\sim$ is an equivalence relation...

   Answer: I prove first that if $\sim$ is an equivalence, then $X/\sim$ is a partition. For any $x \in X$, let $[x] = \{y \in X : y \sim x\}$. By reflexivity, it follows that $x \in [x]$, which immediately implies that $\cup_{A \in X/\sim} A = X$. Suppose now that $A, B \in X/\sim$, $x \in A \cap B$. I show that $A = B$. There are $z, z' \in X$ such that $A = \{y \sim z\}$ and $B = \{y \sim z'\}$. For any $y \in A$, $y \sim z$, and since $x \sim z$ and $x \sim z'$, by transitivity $y \sim z'$ so that $y \in B$. Similarly, if $y \in B$, then $y \sim z$, so that $y \in A$.

   As to the "converse". Suppose that $\sim$ is not symmetric. Then there are $x, y \in X$ such that $x \sim y$ and $y \not\sim x$. This is immediately seen to violate the assumptions, for $x \in [y]$ and $x \in [x]$, but $[y] \neq [x]$. Suppose that there are $x, y, z \in X$ such that $x \sim y$, $y \sim z$ and $x \not\sim z$. Then $x \in [y]$ and $x \notin [z]$, so that, since $X/\sim$ is a partition, $[y] \cap [z] = \emptyset$. This is a contradiction, since $y \in [y] \cap [z]$.

   A counterexample to the plain converse is the following: $X = \{x, y\}$, $\sim = \{(x, y), (y, x)\}$, so that $\{z \sim x\} = \{y\}$ and $\{z \sim x\} = \{x\}$, a partition even if $\sim$ is not reflexive).