DECISIONS AND UNCERTAINTY: SUGGESTED SOLUTIONS TO HOMEWORK # 6

1. Solve problem 5 in Chapter 8 in Kreps.

   Answer:

   I will use capital letters for events, rather than lowercase. Also, I will prove items (e) and (f) *before* item (d).

   The proof of (a) is rather straightforward (use the fact that $C \subseteq B$ to argue that $C \setminus B \neq \emptyset$, hence $C \setminus B \succeq^* \emptyset$ by axiom QP2, finally use axiom QP3).

   As to item (b), suppose that $A \sim^* B$ and $A \cap B \neq \emptyset$. Since $B = (B \setminus C) \cup (B \cap C)$ and $B \cap (C \setminus B) = \emptyset$, axiom QP3 implies

   $$[(B \setminus C) \cup (B \cap C)] \cup (C \setminus B) \sim^* A \cup (C \setminus B)$$

   which is equivalent to $B \cup C \sim^* A \cup (C \setminus B)$. By item (a), $A \cup C \succeq^* A \cup (C \setminus B)$, hence (using axiom QP1), $A \cup C \succeq^* B \cup C$.

   The proof of item (c) works like the proof of (b), with $\succ^*$ replacing $\sim^*$ throughout.

   As to item (d), assume $A \sim^* B, C \sim^* D, A \cap C = \emptyset$. Since $(D \setminus A) \cap A = \emptyset$, (b) implies that $B \cup (D \setminus A) \preceq^* A \cup (D \setminus A) = A \cup D$. Since $(A \setminus D) \cap D = \emptyset$, (b) and $C \sim^* D$ imply $A \cup D = (A \setminus D) \cup D \preceq^* (A \setminus D) \cup C$. So by axiom QP1, $B \cup (D \setminus A) \preceq^* C \cup (A \setminus D)$. Finally, since $(A \cap D) \cap (C \cup (A \setminus D)) = \emptyset$, items (b) and (c) imply

   $$[B \cup (D \setminus A)] \cup (A \cap D) \preceq^* [C \cup (A \setminus D)] \cup (A \cap D)$$

   or equivalently $B \cup D \preceq^* A \cup C$.

   For item (f), follow the previous argument, replacing $C \sim^* D$ with $C \succ^* D$ and using (c) instead of (b) to obtain strict inequalities throughout.

   Finally, for item (d), use (e) twice to obtain both $B \cup D \preceq^* A \cup C$ and $B \cup D \succeq^* A \cup C$, which under axiom QP1 yields the result.

2. (WARNING: ONLY FOR THE MASOCHIST!!!) Solve problem 8 in Chapter 8 of Kreps.

Answer: The only interesting things to prove here are that there are no probabilities whatsoever which represent $\succ^*$ in the two cases, and that the relation in example 1 is tight, while the one in example 2 is fine. The first claim can be proved using the standard argument that there is no utility representing the lexicographic ordering on $\mathbb{R}^2$. In fact in both cases we can represent a set a as a pair of *numbers* (in example 1 the pair is given by $(\ell(a_1), \ell(a_2))$, in example 2 the pair is $(\ell(a_1) + \ell(a_2), \ell(a_1)))$, and then the relation $\succ^*$ is just the lexicographic ordering. I now show that $\succ^*$ in example 1 is tight. Suppose that $a \succ^* b$. Then either $\ell(a_1) > \ell(b_1)$ or $\ell(a_1) = \ell(b_1)$ and $\ell(a_2) > \ell(b_2)$. Suppose that $\ell(a_1) > \ell(b_1)$. Then let $c = c_1$ where $c_1 \cap b_1 = \emptyset$ (clearly $b_1 \neq [0, 1]$ for otherwise $\ell(a_1) > 1$, a contradiction) and $0 < \ell(c_1) < \ell(a_1) - \ell(b_1)$. Then $a \succ^* b \cup c \succ^* b$, as required. In the case in which $\ell(a_1) = \ell(b_1)$, choose $c = c_2$ where $c_2 \cap b_2 = \emptyset$ and $0 < \ell(c_2) < \ell(a_2) - \ell(b_2)$ and again $a \succ^* b \cup c \succ^* b$. The proof of the fineness of $\succ^*$ in example 2 is straightforward.

3. Solve exercise 2 of Chapter 9 in Kreps.

   Answer: Given what is already done in Kreps' text, this is straightforward.

4. Solve exercise 4 of Chapter 9 in Kreps.

   Answer: This is fairly straightforward. Suppose that $A \cap B = \emptyset$ and that $f \succ_A g$ and $f \succ_B g$. By axioms P1 and P2 and the definition of $\succeq_{A \cup B}$, $f \succ_{A \cup B} g$ if and only if

$$f' \equiv [f, A \cup B; h, (A \cup B)^c] \succ [f, A \cup B; h, (A \cup B)^c] \equiv g'.$$

   Let $f_1 = [g, A; f, B; h, (A \cup B)^c]$. It follows again from P1-2 and the definition of $\succeq_A$ that $f' \succ f_1$. By the same token, $f_1 \succ [g, A; g, B; h, (A \cup B)^c] = g'$. It follows from P1 that $f' \succ g'$, which is what we wanted to show.

   If $A \cap B \neq \emptyset$ then the result may actually fail. Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and that, with obvious notation $f = [0, 100, 0]$ and $g = [40, 40, 40]$. Then if $P = [1/3, 1/3, 1/3]$ and $u(x) = x$, the DM will find $f \succ_{\{\omega_1, \omega_2\}} g$ since

$$100P(\{\omega_2\} | \{\omega_1, \omega_2\}) = 50 > 40,$$

and she will find $f \succ_{\{\omega_2,\omega_3\}} g$, since

$$100P(\{\omega_2\}|\{\omega_2,\omega_3\}) = 50 > 40.$$

On the other hand $g \succ_\Omega f$, since

$$100P(\{\omega_2\}) = 100/3 < 40.$$

5. Consider the simpler case of the Savage theorem (9.16 in Kreps) when all acts are simple (i.e., they have a finite range).

   Answers:

   (a) This is straightforward (if a bit tedious). Notice that only a proof of necessity was requested, not sufficiency.

   (b) No, if we change the property of p to non-atomicity the axioms are no more necessary. It is immediate to observe that axioms P1-5 and P7 are necessary, so the problem is with axiom P6. In fact if axiom P6 did hold then it would follow that p has a convex range. So we need to find an example of a probability charge which is non-atomic but doesn't have a convex range. Unfortunately the example is a bit technical, so read it at your own risk! Suppose that $S = [0, 1]$ and that $\mathcal{A}$ is the Borel σ-algebra on S. Let λ be the Lebesgue measure on $(S, \mathcal{A})$,[1] and let τ be a 0-1 valued charge on $(S, \mathcal{A})$ which is such that for every $B \in \{A \in \mathcal{A} : \lambda(A) = 0\}$, $\tau(B) = 0$ (such objects do exist, and they only take values 0 and 1, hence their name). Finally let $\mu = (\lambda + 2\tau)/3$. Then one can see that μ is non-atomic, but not convex-ranged. In fact if $\mu(A) > 0$ then $\lambda(A) > 0$ so that we can find $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$ (notice that if $\tau(A) = 0$ then $\tau(B) = 0$). However, μ is not convex-ranged, for given any A such that $\mu(A) > 0$, it is not possible to find $B \subseteq A$ such that $\mu(B) = 1/2$ (any point in the (1/3, 2/3) interval, for that matter).

   (c) No, in the finite case axiom 9.9 (Savage's P7) is implied by the others. The proof of this you have already given, since

[1] Though I will not bother to do it, it is possible to find a finitely additive extension of λ to all the subsets of $(0, 1)$. As it is well-known, this is *not* the case if we want λ to satisfy σ-additivity.

you have shown in (a) that all axioms are necessary in the finite case, including axiom 9.9. Now, we have seen in class that the other axioms are sufficient for the result, hence we are home. There is also a direct proof but I will spare you.

(d) The evasive answer is that if u is not bounded then the axioms cannot be necessary. The simple reason is that Fishburn and Savage proved that axioms P1-7 imply that u is bounded (see Fishburn, thm. 14.5). Hence if u is unbounded one of the axioms must be failing. For instance you can construct a counterexample which violates P2 (see Fishburn, ex. 14.17).