## Decisions and Uncertainty: Suggested Solutions to HOMEWORK \# 5

1. Imagine a world in which the result of any lottery depends only on one event $A$ happening or not...

## Answers:

(a) The $45^{\circ}$ line is the set of all the degenerate acts which give some prize with certainty. It's called the certainty line.
(b) The required indifference for $f=\left(f_{1}, f_{2}\right)$ is given by

$$
I(f)=\left\{g=\left(g_{1}, g_{2}\right) \in \mathbf{R}^{2}: p u\left(g_{1}\right)+(1-p) u\left(g_{2}\right)=\bar{u}\right\}
$$

where $\bar{u}=p u\left(f_{1}\right)+(1-p) u\left(f_{2}\right)$. Since the DM is riskneutral, we can assume that her utility function has the affine form $u(x)=a x+b$ where $a \geq 0$ and $b \in \mathbf{R}$. Now if $a=0$ then the utility is constant (this is the trap), so that the indifference set is really all $\mathbf{R}^{2}$. Suppose now that $a>0$. Then $g \in I(f)$ if

$$
a\left(p g_{1}+(1-p) g_{2}\right)+b=a\left(p f_{1}+(1-p) f_{2}\right)+b \equiv a(\bar{f})+b
$$

which immediately tells you that the indifference locus is a line with equation

$$
g_{2}=\frac{\bar{f}}{1-p}-\frac{p}{1-p} g_{1} .
$$

The line is depicted in figure 1. The point $g^{\prime}$ at which $I(f)$ cuts the certainty line is equal to $g^{\prime}=(\bar{f}, \bar{f})$. Clearly its slope at $g^{\prime}$ is equal to $-(p /(1-p))$, the likelihood ratio. An indifference locus corresponding to higher utility, in the case $a>0$, is another line with the same slope $-(p /(1-p))$ to the right of $I(f)$, see figure 1 . If $a=0$ there is no indifference set corresponding to higher utility.
(c) If $u$ is concave but not (strictly) increasing then indifference sets could look like "thick" curves (see also point (e) below). So suppose that $u$ is increasing and concave (hence


Figure 1:
continuous). Pick the point $g^{\prime}=(\bar{f}, \bar{f})$ on the certainty line in figure 1. Its indifference locus if $u$ is affine is given by the straight line in the picture with slope $-(p /(1-p))$, as we saw in the previous point. What if $u$ is concave, or in words the DM is risk averse? Suppose that you picked two points $f$ and $g$ (different from $g^{\prime}$ ) on the indifference locus $I\left(g^{\prime}\right)$, and consider a mixture with weight $\alpha$ of $f$ and $g$, that is consider the act $h=\left(\alpha f_{1}+(1-\alpha) g_{1}, \alpha f_{2}+(1-\alpha) g_{2}\right)$. This is represented graphically as a point in the line connecting $f$ with $g$. I want to show that $h$ is, if $u$ is locally strictly concave, strictly preferred to $g^{\prime}$. But this is immediate since

$$
\begin{aligned}
u(h) & =p u\left(\alpha f_{1}+(1-\alpha) g_{1}\right)+(1-p) u\left(\alpha f_{2}+(1-\alpha) g_{2}\right) \\
& \geq p\left(\alpha u\left(f_{1}\right)+(1-\alpha) u\left(g_{1}\right)\right)+(1-p)\left(\alpha u\left(f_{2}\right)+(1-\alpha) u\left(g_{2}\right)\right) \\
& =\alpha\left(p u\left(f_{1}\right)+(1-p) u\left(f_{2}\right)\right)+(1-\alpha)\left(p u\left(g_{1}\right)+(1-p) u\left(g_{2}\right)\right) \\
& =u(\bar{f})
\end{aligned}
$$

with strict inequality if $u$ is strictly concave over the relevant interval. Thus the indifference curve will be convex
(see the dashed line in figure 1). As for the slope of $I\left(g^{\prime}\right)$ at $g^{\prime}$, a simple application of the implicit function theorem (I am assuming that $\bar{f}$ is a point of differentiability of $u$ now) shows that the slope is equal to $-(p /(1-p))$.
(d) Looking at figure 1 , take $f$ as your nondegenerate act. Then its indifference locus is $I\left(g^{\prime}\right)$ and its certainty equivalent is given by $c(f)=\bar{f}$. It is unique (and it exists) because we took $u$ to be increasing as well as concave.
(e) If $u$ is not continuous then the indifference locus does not have to intersect the certainty line (so that $c(f)$ might be empty). In fact then there might be some pairs of payoffs $\left(f_{1}, f_{2}\right)$ such that there is no $\bar{f}$ for which $u(\bar{f})=p u\left(f_{1}\right)+$ $(1-p) u\left(f_{2}\right)$. If $u$ is not nondecreasing then the indifference loci are not necessarily giving higher utility as we move rightwards. And if $u$ is not increasing then there might be "thick" indifference sets (so that $c(f)$ might be an interval). I also mentioned earlier how some specific answers change if $u$ is not increasing.
(f) Consider once again the point $g^{\prime}$. What is the set of points $f=\left(f_{1}, f_{2}\right)$ which are indifferent to $g^{\prime}$ such that $f_{2} \geq f_{1}$, labelled $I\left(g^{\prime} \mid f_{2} \geq f_{1}\right)$ ? Looking at the formula for the utility of a RDEU DM (and using the assumption that $u(x)=x$ ) we have that then

$$
I\left(g^{\prime} \mid f_{2} \geq f_{1}\right)=\left\{g \in \mathbf{R}^{2}: \varphi(p) g_{1}+(1-\varphi(p)) g_{2}=\bar{f}\right\} .
$$

So the the indifference locus above the certainty line looks like a straight line with slope $-(\varphi(p) /(1-\varphi(p))$. Instead
$I\left(g^{\prime} \mid f_{1} \geq f_{2}\right)=\left\{g \in \mathbf{R}^{2}:(1-\varphi(1-p)) g_{1}+\varphi(1-p) g_{2}=\bar{f}\right\}$,
which is a straight line with slope $-(1-\varphi(1-p)) / \varphi(1-p)$. Now given the assumptions on $\varphi$ it is easy to verify that, unless $p=1 / 2$ we will have

$$
\frac{\varphi(p)}{1-\varphi(p)}>\frac{1-\varphi(1-p)}{\varphi(1-p)}
$$

so the slope above the certainty line will be steeper than the slope below the certainty line. A typical indifference curve is depicted in figure 2.


Figure 2:
(g) A key observation is that the model behaves exactly like expected utility (with a risk-neutral DM) on each side of the certainty line. So for every act there is one and only one (remember that $u(x)=x) \bar{f}$ such that $\bar{f}=(\bar{f}, \bar{f})$ is indifferent to it. The risk premium $\pi(f)$ is given by the difference between $E(f)$ and $c(f)$. The former can be represented as the point where the line with slope $-(p /(1-p))$ passing through $f$ intersects the certainty line, and $c(f)$ can be represented by $\bar{f}$, see figure 2 . The point is that even if she has a linear utility function, a RDEU with strictly concave $u$ will have a positive risk premium.
(h) Each branch of the indifference curve will become convex (see the dashed line in figure 2). The reasoning is exactly the same as for the EU model. Hence the risk premium will increase (again, see figure 2).
(i) Given the point $f$, a lottery of the type $f+t \epsilon$ is for instance a point on the upper branch (with slope $-p /(1-p)$ ) departing from $f$, as in figure 3. Letting $t \rightarrow 0$ amounts in


Figure 3:
the picture to taking a sequence of points on the branch moving towards $f$. Clearly $\pi(t) \rightarrow 0$. Fix some $t$ (corresponding to the point $f+t \epsilon$ in the figure) and consider the difference between the risk premium of a EU DM and a RDEU DM. The EU DM has an indifference curve which has slope $-p /(1-p)$ when it reaches the certainty line, whereas (again assuming $p \neq 1 / 2$ ) the indifference curve for the RDEU DM (dashed in the figure) is to the left of the other one (and the risk premium higher) and it intersects the certainty line with a steeper slope. If you imagine now letting $t$ go to zero you can see that the risk premium for the EU DM will go to zero faster than the one for the RDEU DM. In fact it is possible to show that the former goes to zero as $t^{2}$ and the latter as $t$. This is the reason why it is said that RDEU preferences have first order and EU preferences second order risk aversion.
2. In the Anscombe and Aumann model of Chapter 7 of Kreps,
consider the following dominance axiom...
Answer: The fact that [7.1-3, 7.14]+7.16 imply 7.17 is easily proved using the representation given in Theorem 7.17 in Kreps. The axioms are equivalent to the representation in Eq. (7.18), from which it is immediate to show that if $h_{s} \succeq h_{s}^{\prime}$ (which is equivalent to $\left.\sum_{z} u(z) h_{s}(z) \geq \sum_{z} u(z) h_{s}^{\prime}(z)\right)$ for every $s \in S$, then $h \succeq h^{\prime}$.
As for the converse implication, assume that [7.1-3, 7.14] and 7.17 hold, and suppose that for $h \in H$

$$
\left[h_{-s}, p\right] \succ\left[h_{-s}, q\right],
$$

(where I use the notation $\left[h_{-s}, r\right] \equiv\left(h_{1}, \ldots, h_{s-1}, r, h_{s+1}, \ldots, h_{n}\right)$ ) for some $p, q \in P$. Then, using (the contrapositive of) axiom 7.17 , we immediately obtain $p \succ q$. Applying axiom 7.17 again, we obtain that for every $s^{\prime}$ (not necessarily non-null)

$$
\begin{equation*}
\left[h_{-s^{\prime}}, p\right] \succeq\left[h_{-s^{\prime}}, q\right] . \tag{1}
\end{equation*}
$$

We are only left to show that is $s^{\prime}$ is non-null then we cannot have indifference in Eq. (1). We show this by contradiction. If that were the case I claim that we would get that for any $r \in P$ we have

$$
\begin{equation*}
\left[h_{-s^{\prime}}, p\right] \sim\left[h_{-s^{\prime}}, r\right] . \tag{2}
\end{equation*}
$$

But then $s^{\prime}$ must be null: Using the representation in (7.5) in Kreps, equation (2) implies that $u_{s^{\prime}}$ must be constant. Corollary (7.15) in Kreps can then invoked to argue that $s^{\prime}$ is null. So we obtain the desired contradiction.
I finally prove my claim that (2) must hold. If $p \succeq r \succeq q$, the result follow immediately from (1) and two applications of axiom 7.17. Suppose that $r \succ p \succ q$. Then axioms 7.1-3 imply (in the same fashion as in Lemma (5.6)(b) in Kreps) that there is a unique $\alpha \in[0,1]$ such that $p \sim \alpha r \oplus(1-\alpha) q$. Hence using 7.17 twice, 7.2 and (1) we obtain

$$
\begin{aligned}
{\left[h_{-s^{\prime}}, \alpha p \oplus(1-\alpha) p\right] } & =\left[h_{-s^{\prime}}, p\right] \\
& \sim\left[h_{-s^{\prime}}, \alpha r \oplus(1-\alpha) q\right] \\
& \sim\left[\left[h_{-s^{\prime}}, \alpha r \oplus(1-\alpha) p\right]\right.
\end{aligned}
$$

and then (2) follows from axiom 7.2 (and 7.1). The case in which $q \succ r$ works symmetrically.
3. Solve problem 6 of Chapter 7 in Kreps.

Answer: Consider the preference relation $\succ_{1}$ on $P$ defined as follows: $p \succ_{1} q$ iff $\left(p, p^{\circ}\right) \succ\left(q, p^{\circ}\right)$. It is immediate to observe that $H$ is a mixture space and that axioms (a) through (f) imply that there exists a cardinal utility $u$ such that $p \succ_{1} q$ iff $\sum u(x) p(x) \geq \sum u(x) q(x)$.
It follows from (c) that for every $p \in P, p^{\circ} \succeq_{1} p \succeq_{1} p_{\circ}$. Hence if we let $U(p)=\sum u(x) p(x)$, we have $U\left(p^{\circ}\right) \geq U(p) \geq U\left(p_{\circ}\right)$. Moreover (g) implies that $p^{\circ} \succ_{1} p_{\circ}$. In fact, otherwise $p^{\circ} \sim_{1} p_{\circ}$, which implies using (b) that $\left(p^{\circ}, p^{\circ}\right) \sim\left(p^{\circ}, p_{\circ}\right)$, so that using (d) twice we get $\left(p_{\circ}, p^{\circ}\right) \sim\left(p_{\circ}, p_{\circ}\right)$, and using (b) and (a) we then get $\left(p^{\circ}, p^{\circ}\right) \sim\left(p_{\circ}, p_{\circ}\right)$, a violation of (g). Given this, we can normalize $u$ so to have $U\left(p^{\circ}\right)=1$ and $U\left(p_{\circ}\right)=0$, so that $U(p) \in[0,1]$ for every $p \in P$.
Given $a \in[0,1]$, let $s(a)$ be the binery lottery defined as follows: $s(a)=a p^{\circ} \oplus(1-a) p_{0}$. Fix now $(p, q)$. By definition of $s(U(p))$, we get that

$$
\left(p, p^{\circ}\right) \sim\left(s(U(p)), p^{\circ}\right)
$$

so that by (d) twice we get $(p, q) \sim(s(U(p)), q)$. Analogously, $\left(p^{\circ}, q\right) \sim\left(p^{\circ}, s(U(q))\right)$ which with the above and (a) implies

$$
(p, q) \sim(s(U(p)), q) \sim(s(U(p)), s(U(q)))
$$

Consider the ordering $\succeq^{*}$ on $[0,1]^{2}$ defined by

$$
(a, b) \succeq^{*}\left(a^{\prime}, b^{\prime}\right) \Longleftrightarrow(s(a), s(b)) \succeq\left(s\left(a^{\prime}\right), s\left(b^{\prime}\right)\right)
$$

It is by now clear that

$$
(p, q) \succeq\left(p^{\prime}, q^{\prime}\right) \Longleftrightarrow(U(p), U(q)) \succeq^{*}\left(U\left(p^{\prime}\right), U\left(q^{\prime}\right)\right)
$$

We have to show that $\succeq$ is a weak order on $[0,1]^{2}$. This requires showing that for every $a \in[0,1]$ there is a $p \in P$ such that $a=U(p)$. But this is obvious: Take $p=a p^{\circ} \oplus(1-a) p_{\circ}$. The fact that $\succeq^{*}$ is a weak order now follows from the fact that $\succeq$ is a weak order as well.

Suppose now that also (h) holds. We have that $p \succeq_{1} q$ implies that $(p, q) \sim(q, q)$. Given $(p, q),\left(p^{\prime}, q^{\prime}\right)$, suppose w.l.o.g. that $p \succeq_{1} q$ and $p^{\prime} \succeq_{1} q^{\prime}$. Then $(p, q) \succeq\left(p^{\prime}, q^{\prime}\right)$ iff $(q, q) \succeq\left(q^{\prime}, q^{\prime}\right)$. We now show that $U(q) \geq U\left(q^{\prime}\right)$ iff $(q, q) \succeq\left(q^{\prime}, q^{\prime}\right)$. First assume that $(q, q) \succeq\left(q^{\prime}, q^{\prime}\right)$. From (c) it follows that $\left(p^{\circ}, p^{\circ}\right) \succeq\left(p^{\circ}, q\right)$, so that (h) implies $\left(p^{\circ}, q\right) \sim(q, q)$. Similarly, $\left(p^{\circ}, q^{\prime}\right) \sim\left(q^{\prime}, q^{\prime}\right)$, so that using (a) we get $(q, q) \succeq\left(q^{\prime}, q^{\prime}\right)$ implies $q \succeq_{1} q^{\prime}$, which in turn implies $U(q) \geq U\left(q^{\prime}\right)$. Conversely, assume that $U(q) \geq U\left(q^{\prime}\right)$, so that $q \succeq_{1} q^{\prime}$. Applying (d) twice we get $(q, q) \succeq\left(q, q^{\prime}\right)$ and $\left(q^{\prime}, q\right) \succeq\left(q^{\prime}, q^{\prime}\right)$ so that by (b) and (a) we get $(q, q) \succeq\left(q^{\prime}, q^{\prime}\right)$.
Summing up, we can conclude as requested that

$$
(p, q) \succeq\left(p^{\prime}, q^{\prime}\right) \Longleftrightarrow \min [U(p), U(q)] \geq \min \left[U\left(p^{\prime}\right), U\left(q^{\prime}\right)\right]
$$

The preference relation $\succ$ clearly satisfies axiom 7.1 and 7.14. Axiom 7.3 is verified by looking at its mathematical representation above. Axioms 7.2 and 7.16 fail. To see 7.2 , take $p, q \in P$ such that $U(p)>U(q)$ : we have $(p, p) \succ(p, q)$ but if we mix with $(q, p)$ we get

$$
\left(\frac{1}{2} p+\frac{1}{2} q, p\right) \sim\left(\frac{1}{2} p+\frac{1}{2} q, \frac{1}{2} p+\frac{1}{2} q\right) .
$$

As for 7.16 choose $p, q, r, r^{\prime} \in P$ such that $U(p)>U(r)>U(q) \geq$ $U\left(r^{\prime}\right)$. Then $(p, r) \succ(q, r)$ and $\left(r^{\prime}, p\right) \sim\left(r^{\prime}, q\right)$, in violation of the axiom with $h=\left(r^{\prime}, r\right)$ (notice that state 2 is clearly non-null).

