DECISIONS AND UNCERTAINTY: SUGGESTED SOLUTIONS TO HOMEWORK # 4

1. Prove that an increasing utility function has *constant absolute risk aversion* iff . . .

   Answer: Notice that I need to assume that $K > 0$ for the statement (in Kreps) to be valid (if $K < 0$, then we find that $u(x) = ae^{-Kx} + b$ for $a > 0$). The "if" is trivial. I show the "only if": Starting from the second-order differential equation $u''(x)/u'(x) = -K$, if we let $v = u'$ and integrate, we find $v(x) = u'(x) = ce^{-Kx}$. Now we have two cases: $K = 0$ and $K > 0$. In the former case, if we integrate a second time and let $a = c$ we find

   $$u(x) = ax + b.$$

   Now, since u must be increasing, it must be $a > 0$, while b can be arbitrary. In the case in which $K > 0$, integrating and letting $a = c/K$ we find that

   $$u(x) = -ae^{-Kx} + b,$$

   where again $a > 0$, since otherwise u is not increasing, and $b \in \mathbf{R}$.

2. Solve problem 7 of Chapter 6 in Kreps.

   Answer: We know from the previous exercise that (if $K \geq 0$) our DM has CARA iff her u satisfies the condition described above. The fact that she has $K > 0$ follows from her indifference for \$ 488 and a 1/2:1/2 lottery over \$ 1000 and \$ 0. By the uniqueness properties of the utility function, it is without loss of generality to assume that then $u(x) = -e^{-Kx}$. Using the mentioned indifference, we find that K must satisfy

   $$2e^{-488K} - e^{-1000K} = 1,$$

   which can be solved numerically to find $K \approx 0.96x10^{-4}$. We can then calculate the expected utilities of the three lotteries to find that lottery (a) has expected utility

   $$(-e^{0.0096} - e^{-0.0288} - e^{-0.096})/3 \approx -0.95684,$$

while lottery (b) has expected utility

$$(-3e^{-0.0509} - 1)/4 \approx -0.96278,$$

and lottery (c) has expected utility

$$-e^{-0.037} \approx -0.96368.$$

So the DM should choose lottery (a).

3. Solve problem 13 of Chapter 6 in Kreps.

   Answer: Ignoring the questions related to bankruptcy (that is, allowing that the DM's final wealth be given by any possible wealth level), let f denote the random variable that pays 1000 with probability .4 and -500 with probability .6. Let w denote the DM's wealth when offered the gamble(s). Suppose that our DM displays the following preferences

   $$w + 1000 \succ w + 1000 + f, \quad w - 500 \succ w - 500 + f, \quad w \succ w + f,$$

   where $w + f$ is the random variable whose payoff is $w + 1000$ with probability .4 and $w - 500$ with probability .6, etc. That is, the DM does not accept to get f at any of the three possible wealth levels. Then by the independence axiom it follows that

   $$\frac{4}{10}(w+1000)+\frac{6}{10}(w-500) \succ \frac{4}{10}(w+1000+f)+\frac{6}{10}(w-500+f).$$

   But the l.h.s. is just the r.v. $w + f$, and the r.h.s. is the r.v. $w + f + f$ corresponding to getting two independent replicas of the gamble. Therefore we conclude that

   $$w \succ w + f + f.$$

   It is easy to construct a utility function that will display $w \succ w+f$ for *every* value of w, so that this DM will not accept any sum of N repetitions of f, whatever N.[1] Clearly, there may also be EU maximizing DM's who may accept to get a large number of replicas of f, but not small numbers (what should their utility function look like?).

[1] For example, consider $u(x) = -e^{-x/1000}$. For every w, the DM with such u will not take f. (Notice that the final inequality does not depend on w, in fact this u has a constant Arrow-Pratt index equal to 1/1000.)

4. You are just back from your first consulting session...

   (a) Calculate the distribution functions induced by f, g, h on the prize space $C = \{-100, -50, 0, 50, 100, 150, 200\}$.

   Answer: They are given by the following table.

	-100	-50	0	50	100	150	200
f	0.1	0.2	0.4	0	0.2	0	0.1
g	0.1	0.3	0.2	0.2	0.05	0	0.15
h	0.1	0.1	0.4	0	0.35	0	0.05

   (b) Suppose that you have elicited the utility function of the fund manager, and roughly estimated that it behaves roughly as $u(x) = \sqrt{x + 200}$. What is her optimal choice?

   Answer: h is seen to be optimal by applying expected utility. Gees! consulting work is hard, isn't it?

Suppose, instead, that you do not know what the manager's utility function is.

   (c) Are there options which your manager will *not* pick if she is greedy (the more money the better)?

   Answer: We use the table above to calculate the CDFs induced by each act, which are summarized in the following table.

	-100	-50	0	50	100	150	200
f	0.1	0.3	0.7	0.7	0.9	0.9	1
g	0.1	0.4	0.6	0.8	0.85	0.85	1
h	0.1	0.2	0.6	0.6	0.95	0.95	1

   Saying that the DM is greedy is saying that her u is nondecreasing, so we want to check for first order stochastic dominance. Using the CDFs in the table you find that the set of "admissible" choices is $\{f, g, h\}$.

   (d) What if you know that she is risk averse...

   Answer: Here we know that u is concave, but not that it is nondecreasing. Hence we want to check for riskiness in the Rothschild-Stiglitz sense. Using the CDFS above you can check that the set of admissible choices is $\{f, h\}$.

(e) Finally what if she is both greedy and risk averse?

Answer: Here you know that u is both concave and non-decreasing. Hence you have to check for second order stochastic dominance. Again, using the CDFs we conclude that the only admissible plan is h. And this without even eliciting the u function! Not bad, eh?