

DECISIONS AND UNCERTAINTY: SUGGESTED SOLUTIONS TO
HOMEWORK # 1

1. In class I claimed that a cardinal utility is necessary for some decision criteria to make sense...

Answer: This is a routine exercise. For instance, take the example I gave in class to illustrate the Maximin risk rule:

$$\begin{bmatrix} 0 & 100 \\ 1 & 1 \end{bmatrix},$$

where the choice that the criterion prescribes is that corresponding to the first row of payoffs. Suppose that we took another utility function that represents the same ordering of the final payoffs, to get the following matrix:

$$\begin{bmatrix} 0 & 15 \\ 10 & 10 \end{bmatrix}.$$

If we now calculate the risk matrix, we find

$$\begin{bmatrix} -10 & 0 \\ 0 & -5 \end{bmatrix},$$

so that the “maximin” choice is now the one corresponding to the bottom row. Hence the transformation of the utilities has changed the prescription of the criterion. Similar examples can be constructed for the $\alpha \min + (1 - \alpha) \max$ criterion of Hurwicz, and for the “equal probabilities” criterion.

2. Solve problem 1 of Chapter 1 in Kreps.

Answers: We first have to prove that axioms 1.1-1.5 and 1.9 are inconsistent on P , the set of all simple lotteries on $[0, 100]$. As Kreps suggests, we assume that, as is actually the case, theorem 1.6 is true (it’s a corollary of the von Neumann-Morgenstern theorem we’ll see later). Of course this could also be proved directly, but that is *much* more painful. So we know that axioms 1.1-1.5 are equivalent to the existence of a strictly increasing function $u : [0, 100] \rightarrow \mathbf{R}$ such that every lottery $p \in P$ is

evaluated via the expected utility functional

$$E_p(u) = \sum_{r \in [0,100]} u(r)p(r).$$

Given the utility function u obtained by the theorem, let (without loss of generality) $u(0) = 0$ and $u(100) = 1$. We know that u is strictly increasing, that does not mean however that it is continuous, only that it can have at most countably many discontinuity points. Let $r \in (0, 100)$ be a continuity point of u . Then for every $\epsilon > 0$ we can find a point $r' < r$ such that $|u(r) - u(r')| < \epsilon$. Consider now the two simple lotteries p and p' defined as follows: $p(r') = 1/2$, $p(100) = 1/2$ and $p'(r) = 1$. Then, for ϵ small enough,

$$\begin{aligned} E_p(u) &= (1/2)u(r') + (1/2)u(100) \\ &= (1 + u(r'))/2 \\ &> u(r). \end{aligned}$$

So, whatever u is, it is always possible to construct two simple lotteries belonging to P such that axiom 1.9 fails. Since the axiom was stated implicitly with an existential quantifier (“for every p and p' ”), we have thus shown that the axioms are inconsistent. Notice that the argument works because there can only be countably many discontinuity points of u . Also observe that consistency can only be proved with respect to a given domain of the preference relation (P in this case): it is always possible to construct a domain (smaller than P) on which all axioms can be satisfied, so that they are consistent.

Proving that axioms 1.1 through 1.5 are consistent is immediate. The preference relation induced by taking the expected utility of every lottery with respect to some strictly increasing u (for instance the identity) is immediately seen to satisfy all axioms.

3. Solve exercise 2 of Chapter 2 in Kreps.

Answer: The proof is right only if the relation has the additional property that for every $x \in X$ there is $y \in X$ such that xRy . But this will not necessarily be the case, so that the step of the proof which says “take some $y \in X$ such that xRy ” might

fail. For instance consider $X = [0, 1]$ and the relation defined by $R = [1/4, 3/4] \times [1/4, 3/4]$.¹ It is easy to check that R is symmetric (easy to see graphically: R is symmetric around the diagonal of the square) and transitive (this is also easy to see graphically: how?), but it is not reflexive, because $\neg xRx$ for $x < 1/4$ and $x > 3/4$.

4. Prove the following statement from class: Suppose that \sim is an equivalence relation...

Answer: I prove first that if \sim is an equivalence, then X/\sim is a partition. For any $x \in X$, let $[x] = \{y \in X : y \sim x\}$. By reflexivity, it follows that $x \in [x]$, which immediately implies that $\cup_{A \in X/\sim} A = X$. Suppose now that $A, B \in X/\sim$, $x \in A \cap B$. I show that $A = B$. There are $z, z' \in X$ such that $A = \{y \sim z\}$ and $B = \{y \sim z'\}$. For any $y \in A$, $y \sim z$, and since $x \sim z$ and $x \sim z'$, by transitivity $y \sim z'$ so that $y \in B$. Similarly, if $y \in B$, then $y \sim z$, so that $y \in A$.

As to the “converse”. Suppose that \sim is not symmetric. Then there are $x, y \in X$ such that $x \sim y$ and $y \not\sim x$. This is immediately seen to violate the assumptions, for $x \in [y]$ and $x \in [x]$, but $[y] \neq [x]$. Suppose that there are $x, y, z \in X$ such that $x \sim y$, $y \sim z$ and $x \not\sim z$. Then $x \in [y]$ and $x \notin [z]$, so that, since X/\sim is a partition, $[y] \cap [z] = \emptyset$. This is a contradiction, since $y \in [y] \cap [z]$.

A counterexample to the plain converse is the following: $X = \{x, y\}$, $\sim = \{(x, y), (y, x)\}$, so that $\{z \sim x\} = \{y\}$ and $\{z \sim x\} = \{x\}$, a partition even if \sim is not reflexive).

¹ Little puzzle: how can we write R as a property that x has in relation to y . That is, find how to express R in words (and formulas) so as to obtain that x is in the relation R to y if and only if $(x, y) \in R$.