

Information Acquisition in Committees: Technical Addendum

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1 Proof of Proposition 3

Consider a committee of size n . We look for the optimal mechanism under the restriction that all n players acquire information.

The problem is:

$$\begin{aligned} \max_{\gamma(0), \dots, \gamma(n) \in [0,1]} & - (1 - q) P(G) + \sum_{k=0}^n v(k) \gamma(k) \\ \text{s.t.} & \sum_{k=0}^n a(k) \gamma(k) \geq c \\ & \sum_{k=0}^n b(k) \gamma(k) \geq c, \end{aligned}$$

where

$$v(k) = \binom{n}{k} f(k, n),$$

$$a(k) = \binom{n-1}{k-1} f(k, n) - \binom{n-1}{k} f(k+1, n)$$

is the coefficient of $\gamma(k)$ in $IC(i)$, and

$$b(k) = \binom{n-1}{k} f(k, n) - \binom{n-1}{k-1} f(k-1, n)$$

is the coefficient of $\gamma(k)$ in $IC(g)$. We use the convention $\binom{n-1}{-1} = \binom{n-1}{n} = 0$.

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Our optimization problem falls under the class of problems known as parametric linear programs. In particular, notice that the solution is continuous in the cost c (see, for instance, Zhang and Liu [1990]).

The goal is to show that when p is sufficiently close to one the optimal mechanism takes the form:

$$\begin{aligned} \bar{\gamma}_n(0) = \dots = \bar{\gamma}_n(\hat{k} - 1) = 0, \quad \bar{\gamma}_n(\hat{k}) = \alpha, \quad \bar{\gamma}_n(\hat{k} + 1) = \dots = \bar{\gamma}_n(k_n - 1) = 1, \\ \bar{\gamma}_n(k_n) = \dots = \bar{\gamma}_n(\bar{k} - 1) = 0, \quad \bar{\gamma}_n(\bar{k}) = \beta, \quad \bar{\gamma}_n(\bar{k} + 1) = \dots = \bar{\gamma}_n(n) = 1, \end{aligned}$$

where $\alpha, \beta \in [0, 1]$, and $0 < \hat{k} < k_n \leq \bar{k} < n$.

We assume that p is sufficiently large. Of course, $a(0) = -f(1, n) > 0$, $a(n) = f(n, n) > 0$, $b(0) = f(0, n) < 0$ and $b(n) = -f(n - 1, n) < 0$.

Notice that for $k = 1, \dots, n - 1$, we can rewrite $a(k)$ and $b(k)$ as

$$\begin{aligned} a(k) &= \binom{n-1}{k-1} \frac{1}{k} \left[(n(1-p) - k) q P(I) (1-p)^k p^{n-k-1} + (k - np) (1-q) P(G) p^k (1-p)^{n-k-1} \right], \\ b(k) &= \binom{n-1}{k-1} \frac{1}{k} \left[(k - n(1-p)) q P(I) (1-p)^{k-1} p^{n-k} + (np - k) (1-q) P(G) p^{k-1} (1-p)^{n-k} \right]. \end{aligned}$$

Clearly, $a(k) < 0$ and $b(k) > 0$ for any $k \in [n(1-p), np]$.

Since p is close to one, $n(1-p) < 1$ and $np > n - 1$ and, therefore, $a(k) < 0$ and $b(k) > 0$ for every $k = 1, \dots, n - 1$.

Throughout, we assume that n is odd and that $k_n = k_{n-1}$ (so that $IC(i)$ is the first constraint to bind when the device is Bayesian). In this case, k_n is equal to $\frac{n+1}{2}$.¹

We know that when the cost is $\hat{c} = \binom{n-1}{k_{n-1}} f(k_n, n)$, the Bayesian device satisfies the $IC(i)$ constraint with equality. For costs above \hat{c} we need to introduce distortions in order to induce all n players to acquire information. We also know from Proposition 3* in the Appendix of the paper that for c sufficiently close to \hat{c} it is optimal to distort the mechanism at $k_n = \frac{n+1}{2}$ and set $\gamma(k_n)$ smaller than one. As c increases, $\gamma(k_n)$ decreases. Notice, however, that there exists a critical value of the cost $\bar{c} > \hat{c}$ such that at \bar{c} the optimal mechanism is $\gamma(0) = \dots = \gamma(\frac{n-1}{2}) = 0$, $\gamma(\frac{n+1}{2}) \in (0, 1)$, $\gamma(\frac{n+3}{2}) = \dots = \gamma(n) = 1$, and the value of $\gamma(\frac{n+1}{2})$ is such that both constraints are satisfied with equality. To see this, note that if $\gamma(0) = \dots = \gamma(\frac{n-1}{2}) = 0$ and $\gamma(\frac{n+3}{2}) = \dots = \gamma(n) = 1$, then the LHS of the $IC(g)$ constraint is equal to $-\binom{n-1}{k_n} f(k_n; n) < 0$.

We now show that as the cost increases above \bar{c} it is optimal to continue decreasing the value of $\gamma(\frac{n+1}{2})$ and to start increasing the value of $\gamma(\frac{n-1}{2})$. More generally, we prove the following.

¹The cases in which n is even and/or $k_n = k_{n-1} + 1$ can be analyzed in a similar way.

Claim 1 Assume that we are at a point $c > \bar{c}$ where the optimal mechanism is

$$\begin{aligned} \bar{\gamma}_n(0) = \dots = \bar{\gamma}_n(\hat{k}) = 0, & \quad \bar{\gamma}_n(\hat{k} + 1) = \dots = \bar{\gamma}_n(k_n - 1) = 1, \\ \bar{\gamma}_n(k_n) = \dots = \bar{\gamma}_n(\bar{k} - 1) = 0, & \quad \bar{\gamma}_n(\bar{k}) = \beta, \quad \bar{\gamma}_n(\bar{k} + 1) = \dots = \bar{\gamma}_n(n) = 1, \end{aligned} \tag{1}$$

$\beta \in (0, 1)$, and $0 < \hat{k} < k_n \leq \bar{k} < n$. Suppose that the cost increases. Then it is optimal to continue decreasing $\bar{\gamma}_n(\bar{k})$ and to start increasing $\bar{\gamma}_n(\hat{k})$.

In what follows, we provide a proof for Claim 1. A symmetric claim also holds:

Claim 2 Assume that we are at a cost $c > \bar{c}$ where the optimal mechanism is

$$\begin{aligned} \bar{\gamma}_n(0) = \dots = \bar{\gamma}_n(\hat{k} - 1) = 0, & \quad \bar{\gamma}_n(\hat{k}) = \alpha, \quad \bar{\gamma}_n(\hat{k} + 1) = \dots = \bar{\gamma}_n(k_n - 1) = 1, \\ \bar{\gamma}_n(k_n) = \dots = \bar{\gamma}_n(\bar{k} - 1) = 0, & \quad \bar{\gamma}_n(\bar{k}) = \bar{\gamma}_n(\bar{k} + 1) = \dots = \bar{\gamma}_n(n) = 1, \end{aligned}$$

where $\alpha \in (0, 1)$, and $0 < \hat{k} < k_n \leq \bar{k} < n$. Suppose that the cost increases. Then it is optimal to continue increasing $\bar{\gamma}_n(\hat{k})$ and to start decreasing $\bar{\gamma}_n(\bar{k})$.

The proof of Claim 2 is identical to that of Claim 1 and is thus omitted. The combination of these two claims (together with Remark 3 below) provide the proof of Proposition 3.²

Proof of Claim 1

Note that the optimal device is the solution to a linear programming problem with two constraints, $IC(i)$ and $IC(g)$, and the additional constraints that every $\gamma(k)$ belongs to $[0, 1]$. It follows that there will be at most two values of k at which $\gamma(k)$ is different from 0 or 1 (see, e.g., Luenberger [1965], Chapter 3). Clearly, the optimal mechanism is continuous in c . Thus, if we start from the device (1) and increase c by a small amount, the optimal mechanism is such that the value of $\bar{\gamma}_n(\bar{k})$ is close to β . Therefore, if we start from (1) and increase c , one change must pertain to $\bar{\gamma}_n(\bar{k})$.

In principle, there are different ways to satisfy the constraints when c increases:

1. Decrease the value of $\gamma(\bar{k})$ and increase the value of $\gamma(k)$ for some $k = 1, \dots, \hat{k}$;
2. Decrease the value of $\gamma(\bar{k})$ and increase the value of $\gamma(k)$ for some $k = k_n (= \frac{n+1}{2}), \dots, \bar{k} - 1$;
3. Increase the value of $\gamma(\bar{k})$ and decrease the value of $\gamma(k)$ for some $k = \bar{k} + 1, \dots, n - 1$;

²Note that in generic environments the optimal distortionary device entails randomization for at least one profile of reports. Our proof does, however, extend to non-generic cases in which for some cost levels, the optimal distortionary device entails no randomization.

4. Increase the value of $\gamma(\bar{k})$ and decrease the value of $\gamma(k)$ for some $k = \hat{k} + 1, \dots, k_n - 1 (= \frac{n-1}{2})$;
5. Increase the value of $\gamma(\bar{k})$ and increase the value of $\gamma(0)$;
6. Decrease the value of $\gamma(\bar{k})$ and decrease the value of $\gamma(n)$.

In all cases, the optimal thing to do is to satisfy both constraints with equality. Recall that we start at a point where both constraints are binding and the mechanism is not Bayesian. If we end up with a mechanism under which one constraint is not binding, the mechanism cannot be optimal.³

Below we prove the following facts:

- A** In case 1, the optimal distortion is to use \hat{k} , the largest k available.
- B** Any change in which we increase $\gamma(k')$ and decrease $\gamma(k'')$, where $k' = k_n, \dots, n - 2$ and $k'' = k' + 1, \dots, n - 1$ has a negative effect on the designer's expected utility (the objective function). Furthermore, this change is worse than any change in which we decrease $\gamma(k')$ and increase $\gamma(k)$, where $k = 1, \dots, \hat{k}$.
- C** Case 4 is not feasible.
- D** Case 5 is not feasible.
- E** Case 6 is not feasible.

Note that the distortions mentioned in Fact A certainly generate a decrease in the expected value of the designer's objective function. Fact B implies that case 3 cannot be optimal directly. In fact, it implies that distortions of the type specified in case 3 generate lower expected values to the designer than distortions of the type specified in case 1. In particular, the former yield a decrease in the designer's expected value as well. Fact B also implies that case 2 cannot be optimal. Indeed, suppose we end up with a device in which $\gamma(\bar{k}) \in (0, 1)$ and $\gamma(k) \in (0, 1)$ for some $k = k_n, \dots, \bar{k} - 1$. Then consider the following deviation. Decrease the value of $\gamma(k)$ and increase the value of $\gamma(\bar{k})$ so that the LHS of both constraints decreases by the same (small) amount δ . It follows from the first part of Fact B that this change will *increase* the value of the objective function by some amount $\Delta > 0$.⁴ Now, decrease the value of

³The proof of this fact depends on which case -1 through 6- we are considering. In each case, it is straightforward to identify a deviation that does not violate either constraint and improves the utility. For the sake of brevity, we do not include the relevant calculations.

⁴We know from Fact B that if we increase $\gamma(k')$ and decrease $\gamma(k'')$, where $k' = k_n, \dots, n - 2$ and $k'' = k' + 1, \dots, n - 1$, then the expected utility decreases. Notice that $\bar{k} \leq n - 1$. Therefore, if we decrease the value of $\gamma(k)$ for some $k = k_n, \dots, \bar{k} - 1$ and increase the value of $\gamma(\bar{k})$ (i.e., we take a "mirror image" of the type of changes described in Fact B), then the expected utility must increase.

$\gamma(k)$ and increase the value of $\gamma(\tilde{k})$, for some $\tilde{k} = 1, \dots, \hat{k}$, so that the LHS of both constraints increases by δ given above. This will *decrease* the value of the objective function by $\Delta' > 0$. The second part of Fact B implies that $\Delta > \Delta'$ and so the combination of the two changes is feasible and strictly beneficial.

Proof of Fact A

The goal of this section is as follows. Fix $k' = k_n (= \frac{n+1}{2}), \dots, n-1$ and $k = 1, \dots, \frac{n+1}{2} - 2 (= \frac{n-3}{2})$. Suppose that we decrease $\gamma(k')$ by $\eta > 0$ and increase the value of $\gamma(k)$ by $\varepsilon > 0$ to increase the LHS of both constraints by the same (small) number $\delta > 0$ (we will show that this is possible). Let $Z(k)$ denote the change of the value of the objective function. We show that $Z(k) < Z(k+1) < 0$.

Consider k . To find ε and η , we need to solve

$$\begin{aligned} a(k)\varepsilon - a(k')\eta &= \delta, \\ b(k)\varepsilon - b(k')\eta &= \delta. \end{aligned}$$

The solution to this system is

$$\begin{aligned} \varepsilon &= \frac{a(k')-b(k')}{b(k)a(k')-b(k')a(k)}\delta, \\ \eta &= \frac{a(k)}{a(k')} \frac{a(k')-b(k')}{b(k)a(k')-b(k')a(k)}\delta - \frac{1}{a(k')}\delta. \end{aligned}$$

Notice that $a(k') - b(k') < 0$ and $a(k') < 0$. Thus, to show that $\varepsilon > 0$ and $\eta > 0$, it is necessary and sufficient that

$$b(k)a(k') - b(k')a(k) < 0.$$

To simplify the notation we define:

$$\begin{aligned} a_1(k) &= \binom{n-1}{k-1} \frac{1}{k} (n(1-p) - k) qP(I) p^{n-k-1} \\ a_2(k) &= \binom{n-1}{k-1} \frac{1}{k} (k - np) (1-q) P(G) p^k \end{aligned}$$

so that

$$a(k) = a_1(k) (1-p)^k + a_2(k) (1-p)^{n-k-1}.$$

Similarly, define

$$\begin{aligned} b_1(k) &= \binom{n-1}{k-1} \frac{1}{k} (k - n(1-p)) qP(I) p^{n-k} \\ b_2(k) &= \binom{n-1}{k-1} \frac{1}{k} (np - k) (1-q) P(G) p^{k-1} \end{aligned}$$

so that

$$b(k) = b_1(k) (1-p)^{k-1} + b_2(k) (1-p)^{n-k}$$

Notice that

$$\begin{aligned} b_1(k) a_1(k') &= b_1(k') a_1(k) \\ b_2(k) a_2(k') &= b_2(k') a_2(k) \end{aligned}$$

and so

$$\begin{aligned} &b(k) a(k') - b(k') a(k) = \\ &b_1(k) a_2(k') (1-p)^{n-k'+k-2} + b_2(k) a_1(k') (1-p)^{n-k+k'} \\ &- b_1(k') a_2(k) (1-p)^{n-k+k'-2} - b_2(k') a_1(k) (1-p)^{n-k'+k}. \end{aligned}$$

Note that the smallest power of the term $(1-p)$ in the expression above is $n-k'+k-2$. Therefore, for p close to 1 the sign of $b(k) a(k') - b(k') a(k)$ coincides with the sign of $b_1(k) a_2(k')$, which is negative.

The total effect $Z(k)$ on the utility is then

$$\begin{aligned} Z(k) &= v(k) \varepsilon - v(k') \eta = \\ &\left(v(k) - v(k') \frac{a(k)}{a(k')} \right) \frac{a(k') - b(k')}{b(k) a(k') - b(k') a(k)} \delta + \frac{v(k')}{a(k')} \delta, \end{aligned}$$

which is negative. In a similar way, for $k+1$ we get

$$Z(k+1) = \left(v(k+1) - v(k') \frac{a(k+1)}{a(k')} \right) \frac{a(k') - b(k')}{b(k+1) a(k') - b(k') a(k+1)} \delta + \frac{v(k')}{a(k')} \delta.$$

Recall that we need to show that $Z(k+1) > Z(k)$. We subtract $\frac{v(k')}{a(k')} \delta$ from $Z(k)$ and $Z(k+1)$. We then multiply both terms by the positive quantity (recall $a(k') < 0$ and $b(k') > 0$)

$$\frac{a(k')}{a(k') - b(k')} \frac{1}{\delta}.$$

We need to show

$$\frac{v(k+1) a(k') - v(k') a(k+1)}{b(k+1) a(k') - b(k') a(k+1)} > \frac{v(k) a(k') - v(k') a(k)}{b(k) a(k') - b(k') a(k)}.$$

We multiply both sides by

$$[b(k+1) a(k') - b(k') a(k+1)] [b(k) a(k') - b(k') a(k)] > 0$$

and obtain

$$\begin{aligned} &[v(k+1) a(k') - v(k') a(k+1)] [b(k) a(k') - b(k') a(k)] > \\ &[v(k) a(k') - v(k') a(k)] [b(k+1) a(k') - b(k') a(k+1)]. \end{aligned} \tag{2}$$

Each side of the inequality contains several terms. However, as p approaches 1, it suffices to consider the terms with the smallest power of $(1 - p)$ to determine whether the inequality is satisfied or not.

We now write

$$v(k) = v_1(k)(1-p)^k + v_2(k)(1-p)^{n-k},$$

where we define

$$\begin{aligned} v_1(k) &= -\binom{n}{k} qP(I) p^{n-k}, \\ v_2(k) &= \binom{n}{k} (1-q) P(G) p^k. \end{aligned}$$

Then,

$$\begin{aligned} v(k)a(k') - v(k')a(k) &= v_1(k)a_1(k')(1-p)^{k+k'} + v_1(k)a_2(k')(1-p)^{k+n-k'-1} + \\ &v_2(k)a_1(k')(1-p)^{n-k+k'} + v_2(k)a_2(k')(1-p)^{2n-k-k'-1} - v_1(k')a_1(k)(1-p)^{k+k'} \\ &- v_1(k')a_2(k)(1-p)^{k'+n-k-1} - v_2(k')a_1(k)(1-p)^{n-k'+k} - v_2(k')a_2(k)(1-p)^{2n-k-k'-1}. \end{aligned}$$

The smallest power of $(1 - p)$ is $k + n - k' - 1$ (similarly, if we switch k with $k + 1$, the smallest power would be $k + n - k'$).

Consider now the LHS of inequality (2):

$$[v(k+1)a(k') - v(k')a(k+1)][b(k)a(k') - b(k')a(k)].$$

The term with the smallest power of $(1 - p)$ is $v_1(k+1)a_2(k')b_1(k)a_2(k')$ and that power is $2(n - k' - 1 + k)$.

Consider the RHS of inequality (2):

$$[v(k)a(k') - v(k')a(k)][b(k+1)a(k') - b(k')a(k+1)].$$

The term with the smallest power of $(1 - p)$ is $v_1(k)a_2(k')b_1(k+1)a_2(k')$ and that power is $2(n - k' - 1 + k)$.

Thus, the two sides have the same powers and we have to show that

$$v_1(k+1)b_1(k)(a_2(k'))^2 > v_1(k)b_1(k+1)(a_2(k'))^2.$$

We divide both sides by $(a_2(k'))^2$ and compute the value of

$$v_1(k+1)b_1(k) - v_1(k)b_1(k+1)$$

when $p = 1$ (by continuity, the sign of the expression extends to p close to 1).

When $p = 1$,

$$\begin{aligned} v_1(k+1)b_1(k) - v_1(k)b_1(k+1) &= \\ (qP(I))^2 \left[-\binom{n}{k+1}\binom{n-1}{k-1} + \binom{n}{k}\binom{n-1}{k} \right] &= \end{aligned}$$

$$(qP(I))^2 \left[-\frac{n!}{(k+1)!(n-k-1)!} \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{n!}{k!(n-k)!} \frac{(n-1)!}{k!(n-k-1)!} \right] =$$

$$(qP(I))^2 \frac{n!(n-1)!}{(n-k-1)!(n-k)!(k!)^2} \left(-\frac{k}{k+1} + 1 \right) > 0.$$

This concludes the proof of Fact A.

Proof of Fact B

In this section we will prove the following. Consider $k' = k_n (= \frac{n+1}{2}), \dots, n-2$, $k'' = k' + 1, \dots, n-1$ and $k = 1, \dots, \frac{n-1}{2}$. Consider two different courses of action. In the first one, we decrease $\gamma(k')$ by $\eta > 0$ and increase the value of $\gamma(k)$ by $\varepsilon > 0$ to increase the LHS of both constraints by the same (small) number $\delta > 0$. Let $Z(k)$ denote the corresponding change of the value of the objective function (this is the case analyzed in the previous section). In the second course of action, we increase $\gamma(k')$ by $\eta > 0$ and decrease the value of $\gamma(k'')$ by $\varepsilon > 0$ to increase the LHS of both constraints by the same (small) number $\delta > 0$. This will change the value of the objective function by $\bar{Z}(k'')$. We want to show that $\bar{Z}(k'') < Z(k)$. (Recall that $Z(k) < 0$. Thus, the inequality $\bar{Z}(k'') < Z(k)$ will also prove the first part of Fact B.)

Consider the second course of action. We need to solve the following system of equations:

$$-a(k'')\varepsilon + a(k')\eta = \delta,$$

$$-b(k'')\varepsilon + b(k')\eta = \delta.$$

The solution is

$$\varepsilon = \frac{a(k')-b(k')}{b(k')a(k'')-b(k'')a(k')} \delta,$$

$$\eta = \frac{a(k'')}{a(k')} \frac{a(k')-b(k')}{b(k')a(k'')-b(k'')a(k')} \delta + \frac{1}{a(k')} \delta.$$

It is simple to check that when p is close to 1 both ε and η are positive. Notice also that the denominator of ε is negative.

The total effect on the objective function $\bar{Z}(k'')$ is equal to

$$\bar{Z}(k'') = v(k')\eta - v(k'')\varepsilon =$$

$$\left(v(k') \frac{a(k'')}{a(k')} - v(k'') \right) \frac{a(k')-b(k')}{b(k')a(k'')-b(k'')a(k')} \delta + \frac{v(k')}{a(k')} \delta.$$

Recall that $Z(k)$ is equal to

$$Z(k) = \left(v(k) - v(k') \frac{a(k)}{a(k')} \right) \frac{a(k') - b(k')}{b(k) a(k') - b(k') a(k)} \delta + \frac{v(k')}{a(k')} \delta.$$

We subtract $\frac{v(k')}{a(k')} \delta$ from both $\bar{Z}(k'')$ and $Z(k)$ and multiply both by $\delta \frac{a(k')}{a(k')-b(k')} > 0$. It remains to show that

$$\frac{v(k) a(k') - v(k') a(k)}{b(k) a(k') - b(k') a(k)} > \frac{v(k') a(k'') - v(k'') a(k')}{b(k') a(k'') - b(k'') a(k')}.$$

We multiply both sides by $[b(k) a(k') - b(k') a(k)] [b(k') a(k'') - b(k'') a(k')] > 0$ and get

$$\begin{aligned} & [v(k) a(k') - v(k') a(k)] [b(k') a(k'') - b(k'') a(k')] > \\ & [v(k') a(k'') - v(k'') a(k')] [b(k) a(k') - b(k') a(k)]. \end{aligned} \quad (3)$$

For each term inside the square brackets we now identify the element with the smallest power of $(1-p)$.

We already know from the previous section that for $[b(k) a(k') - b(k') a(k)]$ we select $b_1(k) a_2(k') (1-p)^{n-k'+k-2}$.

In a similar way, for $[b(k') a(k'') - b(k'') a(k')]$ we select $b_1(k') a_2(k'') (1-p)^{n-k''+k'-2}$.

Consider now $[v(k) a(k') - v(k') a(k)]$. We select $v_1(k) a_2(k') (1-p)^{k+n-k'-1}$.

Finally, consider $[v(k') a(k'') - v(k'') a(k')]$. We select

$$[v_2(k') a_2(k'') - v_2(k'') a_2(k')] (1-p)^{2n-k'-k''-1}.$$

Thus for p close to 1, inequality (3) is satisfied if and only if the following inequality is satisfied:

$$\begin{aligned} & v_1(k) a_2(k') b_1(k') a_2(k'') (1-p)^{2n-k''+k-3} > \\ & [v_2(k') a_2(k'') - v_2(k'') a_2(k')] b_1(k) a_2(k') (1-p)^{3n+k-2k'-k''-3}. \end{aligned}$$

The exponent of the RHS is strictly smaller than the exponent of the LHS. Thus, it suffices to show

$$[v_2(k') a_2(k'') - v_2(k'') a_2(k')] b_1(k) a_2(k') < 0.$$

Notice that for p close to 1, $b_1(k) a_2(k') < 0$. We now evaluate the difference $v_2(k') a_2(k'') - v_2(k'') a_2(k')$ at $p = 1$ and show that it is positive. By continuity, the above inequality will be satisfied when p is close to 1.

When $p = 1$,

$$\begin{aligned} & v_2(k') a_2(k'') - v_2(k'') a_2(k') = \\ & ((1-q) P(G))^2 \left[\binom{n}{k'} \binom{n-1}{k''-1} \frac{k''-n}{k''} - \binom{n}{k''} \binom{n-1}{k'-1} \frac{k'-n}{k'} \right] = \\ & ((1-q) P(G))^2 \frac{n!(n-1)!}{(n-k')!(n-k'')!k'!k''!} (k'' - k') > 0. \end{aligned}$$

This concludes the proof of Fact B.

Proof of Fact C

Consider $k = 1, \dots, k_n - 1 (= \frac{n-1}{2})$ and $k' = k_n (= \frac{n+1}{2}), \dots, n-1$. Suppose that we want to decrease the value of $\gamma(k)$ and increase the value of $\gamma(k')$ to increase the

LHS of both constraints by the same positive amount δ . We now show that this is impossible.

If the change described above is possible then there exist $\varepsilon > 0$ and $\eta > 0$ that solve the following system

$$\begin{aligned} -a(k)\varepsilon + a(k')\eta &= \delta, \\ -b(k)\varepsilon + b(k')\eta &= \delta. \end{aligned}$$

The solution is

$$\begin{aligned} \varepsilon &= \frac{a(k')-b(k')}{b(k')a(k)-b(k)a(k')} \delta, \\ \eta &= \frac{a(k)}{a(k')} \frac{a(k')-b(k')}{b(k')a(k)-b(k)a(k')} \delta + \frac{1}{a(k')} \delta. \end{aligned}$$

Notice that $a(k') - b(k') < 0$. Moreover, we know from the analysis above that for p close to 1 the sign of

$$b(k')a(k) - b(k)a(k')$$

coincides with the sign of $-b_1(k)a_2(k')$, which is positive. Thus, ε and η must be negative.

Proof of Fact D

Consider $k = k_n, \dots, n-1$. Suppose that we want to increase both the value of $\gamma(k)$ and the value of $\gamma(0)$ to increase the LHS of both constraints by the same positive amount δ . We now show that this is impossible.

If the change described above is possible then there exist $\varepsilon > 0$ and $\eta > 0$ that solve the following system

$$\begin{aligned} a(0)\varepsilon + a(k)\eta &= \delta, \\ b(0)\varepsilon + b(k)\eta &= \delta. \end{aligned}$$

The solution is

$$\begin{aligned} \varepsilon &= \frac{a(k)-b(k)}{b(0)a(k)-b(k)a(0)} \delta, \\ \eta &= -\frac{a(0)}{a(k)} \frac{a(k)-b(k)}{b(0)a(k)-b(k)a(0)} \delta + \frac{1}{a(k)} \delta. \end{aligned}$$

Notice that $a(k) - b(k) < 0$. We now show that $b(0)a(k) - b(k)a(0)$ is positive, which implies that ε is negative.

Recall that

$$a(0) = -f(1; n) = qP(I)p^{n-1}(1-p) - (1-q)P(G)p(1-p)^{n-1}$$

and that

$$b(0) = f(0; n) = -qP(I)p^n + (1-q)P(G)(1-p)^n.$$

For p close to 1 the sign of $b(0)a(k) - b(k)a(0)$ coincides with the sign of $-qP(I)a_2(k)$ which is positive.

Proof of Fact E

Consider $k = k_n, \dots, n - 1$. Suppose that we want to decrease both the value of $\gamma(k)$ and the value of $\gamma(n)$ to increase the LHS of both constraints by the same positive amount δ . We now show that this is impossible.

If the change described above is possible then there exist $\varepsilon > 0$ and $\eta > 0$ that solve the following system

$$\begin{aligned} -a(k)\varepsilon - a(n)\eta &= \delta, \\ -b(k)\varepsilon - b(n)\eta &= \delta. \end{aligned}$$

The solution is

$$\begin{aligned} \varepsilon &= \frac{a(n)-b(n)}{b(n)a(k)-b(k)a(n)}\delta, \\ \eta &= -\frac{a(k)}{a(n)}\frac{a(n)-b(n)}{b(n)a(k)-b(k)a(n)}\delta - \frac{1}{a(n)}\delta. \end{aligned}$$

Recall that

$$a(n) = f(n; n) = -qP(I)(1-p)^n + (1-q)P(G)p^n$$

and that

$$b(n) = -f(n-1; n) = qP(I)p(1-p)^{n-1} - (1-q)P(G)p^{n-1}(1-p).$$

Define $a_1(n) = -qP(I)$ and $a_2(n) = (1-q)P(G)p^n$. Also, define $b_1(n) = qP(I)p$ and $b_2(n) = -(1-q)P(G)p^{n-1}$.

The numerator of ε is positive. We now show that the denominator of ε is negative.

We have to show $b(n)a(k) - b(k)a(n) < 0$ for p large. Notice that (after some simplifications)

$$\begin{aligned} b(n)a(k) - b(k)a(n) &= b_1(n)a_2(k)(1-p)^{2n-k-2} + b_2(n)a_1(k)(1-p)^{k+1} \\ &\quad - b_1(k)a_2(n)(1-p)^{k-1} - b_2(k)a_1(n)(1-p)^{2n-k}. \end{aligned}$$

The smallest power of $(1-p)$ is $k-1$, and thus for p close to 1 the sign of $b(n)a(k) - b(k)a(n)$ coincides with the sign of $-b_1(k)a_2(n)$ which is negative.

Remark 3 *Suppose that there exists a cost c' such that the optimal device takes the form*

$$\begin{aligned} \gamma(0) &= 0, \quad \gamma(1) = \dots = \gamma(k_n - 1) = 1, \quad \gamma(k_n) = \dots = \gamma(k' - 1) = 0 \\ \gamma(k') &= \alpha \quad \gamma(k' + 1) = \dots = \gamma(n) = 1 \end{aligned} \tag{4}$$

then $k' = n - 1$ and $\alpha < 1$.

Similarly, suppose that there exists a cost c'' such that the optimal device takes the form

$$\begin{aligned}\bar{\gamma}_n(0) = \dots = \bar{\gamma}_n(k'' - 1) = 0, \quad \bar{\gamma}_n(k'') = \beta, \quad \bar{\gamma}_n(k'' + 1) = \dots = \bar{\gamma}_n(k_n - 1) = 1, \\ \bar{\gamma}_n(k_n) = \dots = \bar{\gamma}_n(n - 1) = 0, \quad \bar{\gamma}_n(n) = 1,\end{aligned}$$

then $k'' = 1$ and $\beta > 0$.

An implication of the first part of the remark is the following. Suppose k' were smaller than $n - 1$, and consider a cost c above c' . To satisfy the constraints, we could increase the value of $\gamma(k')$ and decrease the value of $\gamma(k)$ for some $k = k' + 1, \dots, n - 1$. On the other hand, if $k' = n - 1$ as claimed then it is impossible to modify the mechanism in order to satisfy both constraints. A similar implication follows from the second part of the remark and therefore the optimal device must take the form specified in Proposition 3.

Proof of Remark 3

We provide the proof for the first claim. The proof for the second claim is analogous.

To see that $k' = n - 1$ when p is close to 1, consider the device described in (4). Both constraints are satisfied with equality. Thus,

$$\begin{aligned}f(1; n) - \binom{n-1}{\frac{n-1}{2}} f\left(\frac{n+1}{2}\right) + \alpha \binom{n-1}{k'-1} f(k'; n) + (1 - \alpha) \binom{n-1}{k'} f(k' + 1; n) = \\ -f(0; n) + \binom{n-1}{\frac{n-1}{2}} f\left(\frac{n-1}{2}\right) - \alpha \binom{n-1}{k'-1} f(k' - 1; n) - (1 - \alpha) \binom{n-1}{k'} f(k'; n)\end{aligned}$$

(and both sides are equal to c'). Notice that as p approaches 1 the RHS of the equality converges to $qP(I)$ (since $-f(0; n)$ contains the term $qP(I)p^n$ and every other term contains $(1 - p)^r$ for some $r > 0$). If $k' < n - 1$, the LHS converges to zero (since each term contains $(1 - p)^r$ for some $r > 0$) and the equality cannot be satisfied.

2 Distortionary Mechanisms when N is Fixed and p is Close to 1

In Proposition 2 we fix q , $P(I)$, p and let N go to infinity. In Proposition 3 and the notes above, we fix N and let p approach 1. The following Proposition extends Proposition 2 and provides conditions for the optimal extended mechanism to involve distortions when, indeed, N is fixed and p is large.

Proposition 2* *Fix N , q and $P(I)$ and assume that either $qP(I) > 2(1 - q)P(G)$ or $qP(I) < \frac{1}{2}(1 - q)P(G)$. There exists $\tilde{p} < 1$ such that for every $p > \tilde{p}$ the following holds. For any $n = 2, \dots, N$, suppose that the Bayesian device with n agents is admissible. Then there exists an admissible distortionary device with $n + 1$ agents that yields greater expected utility than $\hat{V}(n)$.*

Proof of Proposition 2*

To simplify the notation, we define $D \equiv qP(I)$ and $E \equiv (1 - q)P(G)$. The proof depends on which of the two cases specified in the proposition holds and on whether n is even or odd. We present the proof for the case $D > 2E$ and n odd (so that $n \geq 3$). The other three cases follow analogously.

When p is close to 1, and n is odd, then $k_n = \frac{n+1}{2}$. Moreover, $z(n)$ is strictly larger than $\frac{n}{2}$ but very close to $\frac{n}{2}$. In particular, $k_n - z(n) < \frac{1}{2}$.

We now adapt the proof of Proposition 2. Clearly, when p is close to 1, the inequalities used in the proof of Proposition 2: $k_n - 1 \geq n(1 - p)$ and $k_n \leq np$, are satisfied. As in the proof of Proposition 2 we need to show that $\alpha_2 < \alpha^*$ and $\alpha_2 < \alpha_1$, where

$$\alpha_1 = \frac{\binom{n}{k_n} f(k_n + 1; n + 1) - \binom{n-1}{k_n-1} f(k_n; n)}{\binom{n}{k_n} f(k_n + 1; n + 1) - \binom{n}{k_n-1} f(k_n; n + 1)},$$

$$\alpha_2 = \frac{\binom{n-1}{k_n-1} f(k_n; n) + \binom{n}{k_n} f(k_n; n + 1)}{\binom{n}{k_n} f(k_n; n + 1) - \binom{n}{k_n-1} f(k_n - 1; n + 1)},$$

and

$$\alpha^* = \frac{n - k_n + 1}{n + 1}.$$

The denominators of α_1 and α_2 are positive. We begin with the inequality $\alpha^* > \alpha_2$. We need to show

$$\left(n - \frac{n+1}{2} + 1\right) \left[\binom{n}{\frac{n+1}{2}} f\left(\frac{n+1}{2}; n + 1\right) - \binom{n}{\frac{n+1}{2}-1} f\left(\frac{n+1}{2} - 1; n + 1\right) \right] >$$

$$(n + 1) \left[\binom{n-1}{\frac{n+1}{2}-1} f\left(\frac{n+1}{2}; n\right) + \binom{n}{\frac{n+1}{2}} f\left(\frac{n+1}{2}; n + 1\right) \right].$$

The easiest way to show that the inequality is satisfied for p close to 1 is to identify, for each term $f(k'; n')$, the component with the smallest power of $(1 - p)$.

For $f\left(\frac{n+1}{2}; n + 1\right)$ we select $-D(1 - p)^{\frac{n+1}{2}} p^{\frac{n+1}{2}} + E(1 - p)^{\frac{n+1}{2}} p^{\frac{n+1}{2}}$.

For $f\left(\frac{n-1}{2}; n + 1\right)$ we select $-D(1 - p)^{\frac{n-1}{2}} p^{\frac{n+3}{2}}$.

For $f\left(\frac{n+1}{2}; n\right)$ we select $E(1 - p)^{\frac{n-1}{2}} p^{\frac{n+1}{2}}$.

Thus, when p is sufficiently close to 1, the above inequality is satisfied if and only if

$$\frac{n + 1}{2} \binom{n}{\frac{n-1}{2}} D > (n + 1) \binom{n-1}{\frac{n-1}{2}} E$$

which is equivalent to

$$\frac{n}{n + 1} D > E.$$

Clearly, if $D > 2E$ then the inequality is satisfied for every $n \geq 3$.

Consider now the inequality $\alpha_1 > \alpha_2$. We need to show (recall the denominators are positive):

$$\left[\binom{n}{\frac{n+1}{2}} f\left(\frac{n+3}{2}; n+1\right) - \binom{n-1}{\frac{n-1}{2}} f\left(\frac{n+1}{2}; n\right) \right] \left[\binom{n}{\frac{n+1}{2}} f\left(\frac{n+1}{2}; n+1\right) - \binom{n}{\frac{n-1}{2}} f\left(\frac{n-1}{2}; n+1\right) \right] > \\ \left[\binom{n-1}{\frac{n-1}{2}} f\left(\frac{n+1}{2}; n\right) + \binom{n}{\frac{n+1}{2}} f\left(\frac{n+1}{2}; n+1\right) \right] \left[\binom{n}{\frac{n+1}{2}} f\left(\frac{n+3}{2}; n+1\right) - \binom{n}{\frac{n-1}{2}} f\left(\frac{n+1}{2}; n+1\right) \right]$$

We proceed as above and identify the components with the smallest power of $(1-p)$.

For $f\left(\frac{n+3}{2}; n+1\right)$ we select $E(1-p)^{\frac{n-1}{2}} p^{\frac{n+3}{2}}$.

For $f\left(\frac{n+1}{2}; n+1\right)$ we select $-D(1-p)^{\frac{n+1}{2}} p^{\frac{n+1}{2}} + E(1-p)^{\frac{n+1}{2}} p^{\frac{n+1}{2}}$.

For $f\left(\frac{n-1}{2}; n+1\right)$ we select $-D(1-p)^{\frac{n-1}{2}} p^{\frac{n+3}{2}}$.

For $f\left(\frac{n+1}{2}; n\right)$ we select $E(1-p)^{\frac{n-1}{2}} p^{\frac{n+1}{2}}$.

Thus, we need to show

$$E \left[\binom{n}{\frac{n+1}{2}} - \binom{n-1}{\frac{n-1}{2}} \right] (1-p)^{\frac{n-1}{2}} D \binom{n}{\frac{n-1}{2}} (1-p)^{\frac{n-1}{2}} > \\ E \binom{n-1}{\frac{n-1}{2}} (1-p)^{\frac{n-1}{2}} E \binom{n}{\frac{n+1}{2}} (1-p)^{\frac{n-1}{2}}.$$

We divide both sides by $E(1-p)^{n-1}$ yielding

$$D \left[\binom{n}{\frac{n+1}{2}} - \binom{n-1}{\frac{n-1}{2}} \right] \binom{n}{\frac{n-1}{2}} > E \binom{n-1}{\frac{n-1}{2}} \binom{n}{\frac{n+1}{2}}$$

Notice that $\binom{n}{\frac{n+1}{2}} = \frac{n}{\frac{n+1}{2}} \binom{n-1}{\frac{n-1}{2}}$ and that $\binom{n}{\frac{n-1}{2}} = \binom{n-1}{\frac{n-1}{2}}$. Therefore, the inequality above translates into

$$D \binom{n-1}{\frac{n-1}{2}} \left[\frac{n}{\frac{n+1}{2}} - 1 \right] \binom{n-1}{\frac{n-1}{2}} > E \binom{n-1}{\frac{n-1}{2}} \binom{n-1}{\frac{n+1}{2}}.$$

We divide both sides by $\binom{n-1}{\frac{n-1}{2}} \binom{n-1}{\frac{n-1}{2}}$ and get

$$D \left(\frac{2n}{n+1} - 1 \right) > E$$

The inequality is satisfied for every odd $n \geq 3$ provided that $D > 2E$.

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