Game Theory, Spring 2021
Midterm Solutions

You have 90 minutes to complete this exam. Please answer the following three questions. Be sure to allocate your time in proportion to the points. Always justify your answers by providing a formal proof or a detailed argument. Good luck.

1. [**30 points**] Consider the following oligopoly model in which $n \geq 2$ firms compete by choosing their quantities $q_1, \ldots q_n$. The inverse demand function is $P(q_1 + \ldots + q_n) = \frac{a}{q_1+\ldots+q_n}$, where a is a positive parameter. The cost of firm $i = 1, \ldots, n$ of producing the quantity q_i is cq_i^2, with $c > 0$. The payoff of each firm is equal to its profits.

Construct the symmetric Nash equilibrium of the game.

Consider firm i and let $Q_{-i} = \sum_{j \neq i} q_j$ denote the total quantity produced by i's opponents. Then firm i faces the following problem

$$\max_{q_i \geq 0} \frac{a}{q_i + Q_{-i}} q_i - cq_i^2$$

From the first order conditions we obtain:

$$\frac{aQ_{-i}}{(q_i + Q_{-i})^2} - 2cq_i = 0$$

It is easy to check that the second order conditions are satisfied. In a symmetric equilibrium $q_1^* = \ldots = q_n^* = q^*$ and, thus, $Q_{-i}^* = \sum_{j \neq i} q_j^* = (n-1)\, q^*$. Therefore, we have

$$q^* = \frac{1}{n}\sqrt{\frac{a(n-1)}{2c}}$$

2. [**35 points**] Consider the following all pay auction for an object. Three bidders, 1 and 2 and 3, simultaneously submit non-negative bids. The valuation of the object of bidder 1 is $v_1 = 5$, while the valuations of the remaining bidders are $v_2 = v_3 = 10$.

The object is awarded to the bidder with the largest bid (ties are broken randomly, with equal probabilities) and each bidder pays his own bid (independently on whether he gets the good or not).

If bidder i submits the bid b_i, his payoff is $v_i - b_i$ if he gets the good, and $-b_i$ if he does not get the good.

Find a Nash equilibrium of the game.

(Hint: Not all the bidders have to play a mixed strategy.)

We construct a Nash equilibrium in which bidder 1 bids zero and the other two bidders randomize among all the bids in $[0, 10]$. In particular, bidder 2 and bidder 3 play the same mixed strategy and obtain a payoff equal to zero with any bid in the interval $[0, 10]$.

For any $b \in [0, 10]$ let $F(b)$ denote the probability that the bid of bidder $k \in \{2, 3\}$ is weakly smaller than b. Then F must satisfy

$$F(b)\,10 - b = 0$$

for every b. We have $F(b) = \frac{b}{10}$. Finally, we show that bidding zero is indeed optimal for bidder 1. For any $b \in [0, 5]$ the payoff of bidder 1 is equal to

$$F(b)^2\,5 - b = \left(\frac{b}{10}\right)^2 5 - b$$

The payoff is decreasing in b and is equal to zero for $b = 0$. Finally, any bid larger than 5 yields a negative payoff to bidder 1. Thus, $b = 0$ is optimal for bidder 1.

3. [35 points] Consider a group of $n \geq 3$ students who work on a joint project. Each student can exert either a high or a low level of effort. The project is successful if at least two students exert high effort. If the project is successful (not successful) then each student obtains a reward equal to $v > 0$ (zero). The cost of exerting low effort is zero. Each student $i = 1, \ldots, n$ has private information about his cost c_i of exerting high effort. In particular, c_i is distributed uniformly over the interval $[0, v]$. Furthermore, the students' types are independent. The students choose their effort levels simultaneously. Finally, the payoff of each player is equal to the difference between the reward and his cost of effort.

Construct a symmetric Bayesian Nash equilibrium.

The game admits a simple symmetric equilibrium in which each type of each player exerts low effort. To see this, notice that if $n - 1$ students exert low effort then the project fails for any effort level of the last student. Thus, this student also prefers to exert low effort.[1]
We now construct a symmetric equilibrium under which the project is successful with positive probability. Consider player i and fix the strategy of the opponents. For $k \in \{1, 2\}$ let P_k denote the probability that k or more of i's opponents exert high effort. If player i exerts low effort his payoff is $P_2 v$. If player i exerts high effort and his type is c_i, then his payoff is $P_1 v - c_i$. This immediately implies that i's best response is a cutoff strategy: the types below a certain threshold exert high effort while the types above the threshold exert low effort.

[1] Note that this equilibrium exists for any probability distribution of the students' types.

Consider a symmetric BNE and let $\bar{c}$ denote the threshold of the equilibrium strategy of the players. A player of type $\bar{c}$ must be indifferent between high and low effort. We have

$$\bar{c} = (P_1 - P_2)\, v = (n-1)\left(\frac{\bar{c}}{v}\right)\left(1 - \frac{\bar{c}}{v}\right)^{n-2} v$$

and thus

$$\bar{c} = v\left(1 - \left(\frac{1}{n-1}\right)^{\frac{1}{n-2}}\right)$$