

# Dynamic Contracting with Limited Commitment and the Ratchet Effect\*

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## Abstract

We study dynamic contracting with adverse selection and limited commitment. A firm (the principal) and a worker (the agent) interact for potentially infinitely many periods. The worker is privately informed about his productivity and the firm can only commit to short-term contracts. The ratchet effect is in place since the firm has the incentive to change the terms of trade and offer more demanding contracts when it learns that the worker is highly productive.

As the parties become arbitrarily patient, the equilibrium outcome takes one of two forms. If the prior probability of the worker being productive is low, the firm offers a pooling contract and no information is ever revealed. In contrast, if this prior probability is high, the firm fires the unproductive worker at the very beginning of the relationship.

Keywords: Dynamic Contracting; Limited Commitment; Ratchet Effect.  
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# 1 Introduction

Private information is pervasive in long-run relationships. Information revelation enhances efficiency as it helps finding the best plans of action. However, parties involved in long-run relationships often fear that revealing their private information may worsen their future terms of trade. This problem is aggravated when the privately informed party (the agent) contracts with a party with a stronger bargaining position (the principal). These relationships are thus shaped by the principal's desire to elicit information and the agent's reluctance to reveal it.

The phenomenon above, known as the *ratchet effect*, is present in several real-life situations. Roy (1952) documents this effect in his sociological study of working behavior under the piece-rate system in a machine shop of a steel-processing plant. He describes the machine shop's Methods Department, which was in charge of finding out how fast each worker could produce a certain drill. When it became clear that a worker could perform his job faster, the Methods-Department timer would slightly change the size of the drill (or another small specification in the task) and offer lower piece-rates to the worker. The dialog below, extracted from Roy (1952), records the reaction of a senior worker to a novice's desire to earn an average wage of 1.25\$ an hour:

*“Averaging, you say! Averaging? ... Don't you know, that \$1.25 an hour is the most we can make, even when we can make more! ... What do you suppose would happen if I turned in \$1.25 an hour on these pump bodies? ...”*

The new worker replies:

*“They'd have to pay you, wouldn't they? ...”*

The senior worker replies:

*“ Yes! They'd pay me – once! Don't you know that if I turned in \$1.50 an hour on these pump bodies tonight, the whole God-damned Method Department would be down here tomorrow? And they'd retime this job so quick it would make your head swim! And when they retimed it, they'd cut the price in half! And I'd be working for 85 cents instead of \$1.25!”*

Roy concludes that most workers usually engage in “*quota restriction*” in order to avoid worse terms of trade in the future. He estimates that experienced workers thus produce 40% less than their capacity due to the ratchet effect.

In his study of the Soviet economy, Litwack (1991) documents how central planners

would use a firm's past outputs to establish future production targets and future managerial compensation. Achieving the goal one year increased the output requirements in the following years. The study of the implications of government policies on the Shchekino Chemical Combine illustrates this effect well (see Litwack 1991). In 1967, the managers of the Shchekino Chemical Combine argued that it was impossible to achieve the government's goals without 400 additional workers. However, the planners denied their demand and instead offered not to extract any surplus due to productivity increase. In response, the chemical combine increased its productivity by 52%. However, the planners did not keep their promise, and soon thereafter they increased the demands on the combine, which led to its decline and eventual shut down in less than a decade. According to Litwack, firm managers deliberately avoided turning in large surpluses in the Soviet firms for fear of worse terms of trade. Litwack (1991) concludes that this incentive problem was an important reason for the decline of the Soviet Union.

In capitalist economies, regulators of public monopolies are often nominated by politicians and hence shape regulatory policies towards short-run political gains. Not surprisingly, regulatory norms often change when a politician from a new party wins the election; hence, commitment problems are pervasive in regulation (see Laffont and Tirole, 1993).

Motivated by these real-life situations, we build a model to study the dynamics of the ratchet effect, considering the relationship between a worker and a firm as our main interpretation. We assume that the parties interact for potentially infinitely many periods. In each period, the worker can produce a good of quality  $q \in [0, 1]$  at a cost that is linear in  $q$ . At the outset of the relationship, the worker is privately informed about his (persistent) marginal cost which can be either low or high. The firm can only commit to short-term contracts which indicate the payment that the worker is entitled to receive, in the current period, if he turns in a good of a certain specified quality. In each period in which the worker is employed, the firm offers a menu of contracts. Upon being offered a menu, the worker has two options: he can accept one contract in the menu or he can reject all the contracts and quit. If the worker quits, the relationship is terminated and the worker and the firm collect their outside options.

The lack of commitment brings several additional challenges to this problem. Following the usual mechanism-design approach, one could try to construct equilibria in which the firm separates the two types of workers. As pointed out by Laffont and Tirole (1993), this attempt is doomed to fail when the parties are sufficiently patient. Indeed, the firm would

have to pay a large amount of resources to motivate the efficient worker to reveal his type. But this would make very attractive for the inefficient worker to imitate the efficient one for one period, grab this amount, and then quit the relationship in the next period.<sup>1</sup> Therefore, equilibria may involve complicated semi-pooling dynamics for which the literature offers no standard methods. To make progress, we introduce an equilibrium refinement that forces the firm to make the best use of the available information at *every* history. We refer to our solution concept as *ratchet equilibrium*. First, in our view, *ratchet equilibria* capture the economic forces behind the ratchet effect, as the firm completely loses its commitment power to use the revealed information in suboptimal ways. Second, we show that *ratchet equilibria* have a recursive structure, which makes it amenable to characterize and show existence for large discount factors.

In the limit, as the parties become arbitrarily patient, the equilibrium outcome takes one of two forms. If the prior probability of the worker’s cost being low is below a certain threshold  $\hat{p}$ , then in every period the firm offers the most profitable contract that the high-cost worker is willing to accept. Both types accept this contract (i.e., they pool), and no private information is ever revealed. Conversely, if the prior is above  $\hat{p}$ , then, in every period, the firm offers the most profitable contract that the low-cost worker is willing to accept. Clearly, the high-cost worker quits the relationship in the first period, and the firm becomes immediately aware of the worker’s type.

The arguments in this paper shed light on economic forces precluding information revelation in dynamic environments. Unless the firm is willing to fire the less productive worker right away, it will never be able to separate the two types. The logic behind the result is as follows. Assume that the parties are patient and that the prior is low (below  $\hat{p}$ ). Once the firm discovers the actual cost of producing the good, it will leave no rents to the worker. Suppose that there is a period in which the firm offers a menu that induces the low-cost worker to reveal his identity. If the low-cost worker behaves honestly, his expected rents are close to zero (they are positive in the current period, and zero in the future). However, the efficient worker can mimic the inefficient worker’s behavior. This may lower his current payoff, but it makes the firm less optimistic that the cost of producing the good is low. The low-cost worker can continue to imitate the high-cost worker for several periods, until the firm is sufficiently convinced that the cost is high. At that point, the low-cost worker will enjoy a significant rent. We show that as the discount factor goes to one, the discounted

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<sup>1</sup>Laffont and Tirole (1993) call this choice “take the money and run.”

time that it takes to lower the firm's belief and extract the rent is strictly positive. Finally, because of the impossibility of discovering the true cost, the firm gives up learning and implements the most profitable pooling allocation.

Workers in our pooling equilibrium engage in quota restrictions like those described by Roy (1952). Our analysis suggests that those restrictions are likely to emerge in environments in which most workers are not very talented. The analysis also illustrates that firing unproductive workers may be necessary to elicit information from productive workers. In turn, this result may shed light on the inefficiency of firms for which firing is difficult, such as some large public companies, some government agencies, and hierarchies in planned economies. More broadly, our results suggest that the lack of commitment in organizations often leads to inefficiencies due to a transmission failure of relevant private information.

This paper belongs to the literature of repeated adverse-selection with limited commitment pioneered by Freixas, Guesnerie and Tirole (1985), Gibbons (1987), and Laffont and Tirole (1987, 1988). In these seminal papers, the parties interact for two periods. One of the main findings is that there is partial separation of the agent's types in the first period (i.e., the equilibrium is semi-pooling) and full separation in the second and final period. Therefore the outcome of two-period environments presents gradual information revelation. In contrast, our paper shows that when the relationship is infinitely repeated (and the prior is low), no useful information is ever revealed.

Hart and Tirole (1988) analyze a dynamic model in which the seller makes a rental offer to the buyer in every period. The buyer's valuation for the good is private information and can take two values, both of which are larger than the seller's cost of producing the good. As the parties become sufficiently patient, the equilibrium allocation converges to the efficient allocation in which both types of buyer consume the good in every period. Note that for large values of the probability of the low valuation, this pooling allocation coincides with the seller's optimal mechanism under full commitment (i.e., lack of commitment is not detrimental to the seller's payoff). Our work differs from Hart and Tirole (1990) in two respects. First, in our model, the agent's private information is necessary to determine the best course of action and, therefore, pooling allocations are never optimal for the firm under full commitment. Second, we analyze environments in which the ratchet effect leads to inefficiencies.

Bhaskar (2014) studies learning in a two-periods model in which the principal and the agent are ex-ante symmetrically informed about the job's difficulty. When the agent's effort

is unobservable, it is impossible for the principal to design a contract that induces an interior effort level in the first period. Bhaskar and Mailath (2015) generalize the work of Bhaskar (2014) to both an arbitrary finite horizon as well as an infinite horizon (assuming binary actions) and show that long horizons make prohibitively costly for principals to motivate effort. In contrast to these studies, our paper assumes adverse-selection and no exogenous learning and concludes that the ratchet effect imposes constraints on what information is revealed in a dynamic relationship.

Our paper is also related to the literature on durable goods monopoly under limited commitment. In this context, Skreta (2006, 2015) shows that posting a price is the seller’s optimal strategy. Of course, the relationship between the buyer and the seller ends as soon as the durable good is traded while in our model the parties can make a new transaction in every period.

Finally, a number of authors have identified situations in which the ratchet effect is mitigated. Kanemoto and MacLeod (1992) argue that competition for secondhand workers guarantees the existence of efficient piece-rate contracts in long-term relationships. Carmichael and MacLeod (2000) show that in infinitely repeated relationships, it is possible to sustain cooperation between the firm and the workers. In this paper, we rule out such cooperative behavior by refining the solution concept and selecting equilibria which yield, at every history, the largest continuation payoff to the principal.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 reviews the benchmark model with commitment. Section 4 specifies the equilibrium concept (ratchet equilibrium) and discusses related solution concepts. Section 5 analyzes the properties of the ratchet equilibria. Section 6 characterizes the equilibrium outcome when the parties are arbitrarily patient. Section 7 concludes. Most proofs are relegated to a number of Appendices.

## 2 The Model

We study a dynamic principal-agent model with adverse selection and short-term contracts. We interpret the model as the relationship between a firm and a worker.

The worker has private information about his (persistent) type, which is equal to  $L$  with probability  $p_0 \in (0, 1)$ , and equal to  $H$  with probability  $1 - p_0$ . We refer to  $p_0$  as to the prior. The firm and the worker interact for potentially infinitely many periods. In each

period, the worker of type  $i = L, H$  can produce a good of quality  $q \in [0, 1]$  at the cost  $\theta_i q$ , where  $0 < \theta_L < \theta_H$ . We refer to the low type  $L$  (high type  $H$ ) as the efficient or low-cost worker (inefficient or high-cost worker). We write  $\Delta\theta := \theta_H - \theta_L$  to indicate the difference between the marginal costs of the two types. The worker has an outside option that yields a payoff equal to  $\alpha > 0$  in every period.<sup>2</sup>

The firm's valuation of a good of quality  $q$  is  $v(q)$ . The function  $v : [0, 1] \rightarrow \mathbb{R}_+$  is twice continuously differentiable, increasing, and strictly concave, and satisfies  $v(0) = 0$ .

Both parties' preferences are linear in money. In particular, suppose that the worker produces a good of quality  $q$  and the firm makes a transfer equal to  $x$ . Then, the payoff of type  $i = L, H$  is  $x - \theta_i q$ , while the firm's payoff is  $v(q) - x$ .

We let  $q_i^*$ ,  $i = L, H$ , denote the efficient quality produced by type  $i$ :

$$q_i^* = \arg \max_{q \in [0, 1]} v(q) - \theta_i q.$$

To make the problem interesting, we assume

$$v(q_H^*) - \theta_H q_H^* > \alpha,$$

where  $\alpha$  stands for the outside option of the worker. This assumption guarantees that the firm prefers hiring the inefficient worker over collecting the outside option that yields a payoff equal to zero. In other words, we rule out the uninteresting case in which the firm essentially faces no uncertainty about the worker's productivity.<sup>3</sup> Moreover, we assume that  $q_L^* > q_H^*$  and hence the worker's information is necessary to achieve a socially efficient outcome, which is an important driving force behind the ratchet effect.

The firm and the worker play the following game. At the beginning of period  $t = 0, 1, \dots$ , the firm offers a menu  $m_t$  of contracts to the worker. Each contract is of the form  $(x_t, q_t)$  and specifies the transfer  $x_t$  paid by the firm and the quality  $q_t \in [0, 1]$  that the worker must produce. We assume that the quality is verifiable and, thus, each contract is enforceable. After receiving the menu  $m_t$ , the worker has two options: (i) selecting a contract from the menu; (ii) rejecting all the contracts and quitting the relationship. In the first case, the game moves to the next period  $t + 1$ . In the second case, the game ends and each player obtains his outside option (which yields zero for the firm and  $\alpha$  for the

<sup>2</sup>In Section 7, we discuss the robustness of our findings to this assumption.

<sup>3</sup>It is immediate to see that the firm can implement the first best if  $v(q_H^*) - \theta_H q_H^* \leq \alpha$ .

worker).<sup>4</sup> The parties discount future payoffs at the common discount factor  $\delta \in (0, 1)$ .

We let  $\mathcal{M} = \{\mathbb{R} \times [0, 1]\} \cup \{\mathbb{R}^2 \times [0, 1]^2\}$  denote the set of available menus. In words, we assume that the menus can contain one or two contracts. In Section 4, we introduce an equilibrium refinement that selects, after every history, the best continuation equilibrium for the firm. It is possible to show that under our solution concept, it is without loss of generality to assume that the firm offers menus with at most two contracts. In particular, our main results (Theorems 1, 2 and 3) continue to hold if the menus are compact sets of contracts. The argument is similar to the one in Bester and Strausz (2001): the firm cannot profit by endowing the worker with a set of messages larger than his set of types.<sup>5</sup>

When the firm offers the menu  $m_t$ , the set of actions available to the worker is  $m_t \cup \{\emptyset\}$ , where  $\emptyset$  denotes the choice of rejecting all the contracts in  $m_t$  and quitting. We let  $a_t$  denote the agent's decision in period  $t$ .

For every  $t \geq 1$ , a period- $t$  (non-final) public history  $h^t = (m_0, a_0, \dots, m_{t-1}, a_{t-1})$  consists of all the menus offered by the firm in previous periods  $\tau = 0, \dots, t-1$ , as well as all the worker's decisions, provided that he never chose to quit (i.e.,  $a_\tau \neq \emptyset$  for every  $\tau = 0, \dots, t-1$ ). We let  $H^0 = \{h^0\}$  denote the set containing the empty history  $h^0$ . We write  $H^t$  for the set of all period- $t$  public histories. Finally,  $H = \cup_{t=0,1,\dots} H^t$  is set of all (non-final) public histories.

A behavior strategy  $\sigma^F$  for the firm is a sequence  $\{\sigma_t^F\}$ , where  $\sigma_t^F$  is a probability transition from  $H^t$  into  $\mathcal{M}$ , mapping the history  $h^t$  into a (possibly random) menu. A behavior strategy  $(\sigma^L, \sigma^H)$  for the worker is a sequence  $\{(\sigma_t^L, \sigma_t^H)\}$ , where  $\sigma_t^i$ ,  $i = L, H$ , associates to every pair  $(h^t, m_t) \in H^t \times \mathcal{M}$  a probability distribution over the set  $m_t \cup \{\emptyset\}$ . We write  $\sigma = (\sigma^F, \sigma^L, \sigma^H)$  for a strategy profile. Finally, we let  $\mu = \{\mu(h^t)\}_{h^t \in H}$  denote the firm's system of beliefs, where  $\mu(h^t) \in [0, 1]$  represents the probability that the firm assigns, at the history  $h^t$ , to the event that the worker's type is equal to  $L$ .

Given a strategy profile  $\sigma$  and a system of beliefs  $\mu$ , for each history  $h^t$  we let  $V_F(h^t; (\sigma, \mu))$  denote the firm's continuation payoff at  $h^t$ . If we use  $\mathcal{T} \in \mathbb{Z}_+ \cup \{\infty\}$  to denote the random period in which the relationship terminates (setting  $\mathcal{T} = \infty$  if the worker remains employed

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<sup>4</sup>Our results extend to the case in which the worker has the additional option to reject all the contracts in the menu and remain in the relationship.

<sup>5</sup>A formal proof of this result is available from the authors upon request.



forever), then we have:

$$V_F(h^t; (\sigma, \mu)) := \mathbb{E}_{(\sigma, \mu)} \left[ (1 - \delta) \sum_{\tau=t}^T \delta^{\tau-t} (v(q_\tau) - x_\tau) \mid h^t \right],$$

where  $\mathbb{E}_{(\sigma, \mu)}[f|h^t]$  represents the conditional expected value (given  $h^t$ ) of the random variable  $f$  given the strategy profile  $\sigma$  and the system of beliefs  $\mu$ . Analogously, for every history  $h^t$  we let  $W_i(h^t; (\sigma, \mu))$  denote the expected continuation *rent* at  $h^t$  of the worker of type  $i = L, H$ . We have:

$$W_i(h^t; (\sigma, \mu)) := \mathbb{E}_{(\sigma, \mu)} \left[ (1 - \delta) \sum_{\tau=t}^T \delta^{\tau-t} (x_\tau - \theta_i q_\tau - \alpha) \mid i, h^t \right].$$

To simplify the notation, we omit the argument  $(\sigma, \mu)$  and write  $V_F(h^t)$  and  $W_i(h^t)$  when there is no ambiguity.

Throughout the rest of the paper, we maintain the following assumption.

**Assumption 1**  $\delta > \frac{1}{1+q_H^*} := \underline{\delta}$ .

When the discount factor  $\delta$  is small, the game is not particularly interesting as the equilibrium outcomes resemble those of the static model (in which the parties interact for only one period). The focus of this paper is on the equilibrium dynamics when the parties care greatly about the future. Consequently, we restrict attention to large values of  $\delta$ .

### 3 Commitment Allocation and the Ratchet Effect

It is useful to start the analysis by quickly reviewing the benchmark model in which the firm can fully commit to a sequence of menus  $(m_0, m_1, \dots)$ . This provides an upper bound to the firm's profits in the game with limited commitment. It is well known that the solution to the firm's commitment problem is to replicate the optimal static mechanism, which is as follows. When the prior  $p_0$  is above a certain threshold  $p^C \in (0, 1)$ , the optimal menu is  $\{(\theta_L q_L^* + \alpha, q_L^*)\}$ . The low-cost worker accepts the contract in the menu while the high-cost worker rejects it. Thus, the firm's profits are equal to:

$$p_0 [v(q_L^*) - \theta_L q_L^* - \alpha].$$

On the other hand, when the prior  $p_0$  is below  $p^C$ , the optimal menu is

$$\{(x_H^C, q_H^C), (x_L^C, q_L^C)\} = \{(\theta_H q_H^C + \alpha, q_H^C), (\theta_L q_L^* + \Delta\theta q_H^C + \alpha, q_L^*)\}$$

for some  $q_H^C \in (0, q_H^*)$ . The high-cost worker accepts the first contract and obtains a rent equal to zero. The low-cost worker is indifferent between the two contracts (therefore, he obtains a rent equal to  $\Delta\theta q_H^C$ ) and accepts the second contract. In this case, the firm's commitment profits are equal to:

$$p_0 [v(q_L^*) - \theta_L q_L^* - \Delta\theta q_H^C - \alpha] + (1 - p_0) [v(q_H^C) - \theta_H q_H^C - \alpha].$$

Let us now consider the game in which the firm can only offer short-term contracts. For  $p_0 \geq p^C$ , the firm implements the commitment solution. In fact, the game admits a unique perfect Bayesian equilibrium in which the firm offers the menu  $\{(\theta_L q_L^* + \alpha, q_L^*)\}$  in every period. In equilibrium, the low-cost worker accepts the contract  $(\theta_L q_L^* + \alpha, q_L^*)$  in every period, while the high-cost worker quits the relationship in the first period.

For  $p_0 < p^C$  it is much more difficult to characterize the whole set of perfect Bayesian equilibria, especially when the parties are sufficiently patient. As mentioned above, it is then impossible for the firm to offer a menu that fully separates the two types of worker and equilibria may involve complicated semi-pooling strategies. To understand the dynamics generated by the ratchet effect, in the next section we develop an equilibrium refinement. Our solution concept has a number of features. First, it reflects the essence of the ratchet effect since it imposes to select, *at every history*, the best continuation payoff of the firm. Implicit in the refinement is the idea that the firm has all the bargaining power and cannot commit to use the available information in suboptimal ways.<sup>6</sup> Second, our refinement has a recursive structure which makes the analysis tractable. In particular, we are able to show existence and provide a complete characterization of the equilibrium outcomes when the parties are infinitely patient. Finally, the solution concept delivers strong predictions.

## 4 Equilibrium Refinement

In this section, we introduce the concept of *ratchet equilibrium*, a refinement of perfect Bayesian equilibrium (see Fudenberg and Tirole, 1991). As mentioned above, our refine-

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<sup>6</sup>It is an open question whether there exist equilibria which do not satisfy the refinement. However, we emphasize that such equilibria, if they exist, would also reflect economic forces different from the ratchet effect.

ment expresses the idea that the principal (in our case, the firm) takes advantage of his bargaining power and chooses how the relationship evolves to maximize his continuation payoff. This occurs at every history  $h^t$ . Of course, the principal has to take into account the worker's incentives to reveal his information.

As we will see below, the set of ratchet equilibria is non-empty (for  $\delta > \underline{\delta}$ ) and is relatively tractable when the parties become arbitrarily patient.

To provide a formal definition of our solution concept, we first need to introduce some additional notation. We let  $p \in [0, 1]$  denote the firm's belief that the worker's type is equal to  $L$ . We also introduce the function  $V : [0, 1] \rightarrow \mathbb{R}_+$  and the correspondence  $\Phi : [0, 1] \rightrightarrows \mathbb{R}_+$ . For every belief  $p$ ,  $V(p)$  denotes the firm's payoff (we also refer to  $V$  as to the value function) while  $\Phi(p)$  denotes the set of rents of the low-cost worker.

We do not need a specific piece of notation to define the set of rents of the high-cost worker. This is because in a ratchet equilibrium, the rent of the high type is always zero. The intuition is very simple. Suppose that the firm offers a menu which yields a rent  $\varepsilon > 0$  to the high type. Then the firm can deviate and lower the payments of all the contracts by  $\varepsilon$ . This does not affect the worker's incentive compatibility constraints, and the individual rationality constraints are still satisfied. Thus, the firm has an incentive to deviate. This, in turn, implies that in a ratchet equilibrium, the high worker accepts a contract if it provides a non-negative rent.<sup>7</sup> However, if all the contracts in the menu yield a negative rent, then the high type quits the relationship.

Suppose that the firm knows that the worker's type is  $i = L, H$ . Then the firm maximizes its payoff by offering the contract  $(\theta_i q_i^* + \alpha, q_i^*)$ . Thus, in what follows we restrict attention to pairs  $(V, \Phi)$  that satisfy the following conditions:

$$\begin{aligned} V(0) &= v(q_H^*) - \theta_H q_H^* - \alpha, & \Phi(0) &= \{(\theta_H - \theta_L) q_H^*\}, \\ V(1) &= v(q_L^*) - \theta_L q_L^* - \alpha, & \Phi(1) &= \{0\}. \end{aligned}$$

Suppose now that  $p \in (0, 1)$ . The firm has four different options to maximize its payoff. The first option is to offer a menu with two contracts  $\{(x_1, q_1), (x_2, q_2)\}$  such that each type accepts an element of the menu with probability one. In this case we say that the firm offers a *surely accepted* menu. Without loss, we can assume that the high type is more

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<sup>7</sup>If both contracts provide positive rents, then the high type chooses the contract with the largest rent (he may randomize when the two contracts provide the same rent).

likely than the low type to accept the first contract  $(x_1, q_1)$ . If we let  $p_i$ ,  $i = 1, 2$ , denote the firm's belief when the contract  $(x_i, q_i)$  is accepted, then we have  $p_1 \leq p \leq p_2$ . Let  $r_i$ ,  $i = 1, 2$ , denote the probability that the contract  $(x_i, q_i)$  is accepted, and notice that  $r_1 + r_2 = 1$  since the menu is surely accepted. By the martingale property of the beliefs, we have

$$p = r_1 p_1 + r_2 p_2 = r_1 p_1 + (1 - r_2) p_2.$$

We can therefore express the probabilities of acceptance  $r_1$  and  $r_2$  as functions of the initial belief  $p$  and the posterior beliefs  $p_1$  and  $p_2$ . In particular, the first contract is accepted with probability  $r_1 = \frac{p_2 - p}{p_2 - p_1}$ , while the second contract is accepted with probability  $r_2 = \frac{p - p_1}{p_2 - p_1}$ .<sup>8</sup>

To find the optimal surely accepted menu, the firm solves the following problem, which we label Problem 1:

$$\begin{aligned} \max_{\{x_i, q_i, p_i, v_i\}_{i=1,2}} & \frac{p_2 - p}{p_2 - p_1} [(1 - \delta)(v(q_1) - x_1) + \delta V(p_1)] + \frac{p - p_1}{p_2 - p_1} [(1 - \delta)(v(q_2) - x_2) + \delta V(p_2)] \\ \text{s.t.} & 0 \leq p_1 \leq p \leq p_2 \leq 1, \\ & x_1 = \theta_H q_1 + \alpha, \\ & x_2 \leq \theta_H q_2 + \alpha \text{ with equality if } p_2 < 1, \\ & (1 - \delta)(x_2 - \theta_L q_2) + \delta v_2 \geq (1 - \delta)(x_1 - \theta_L q_1) + \delta v_1 \text{ with equality if } p_1 > 0, \\ & v_i \in \Phi(p_i) \text{ for } i = 1, 2. \end{aligned}$$

First, let us consider the firm's objective function. With probability  $\frac{p_2 - p}{p_2 - p_1}$ , the first contract is accepted. Then the firm obtains  $v(q_1) - x_1$  in the current period. Also, the belief becomes  $p_1$ , and the firm's continuation payoff is  $V(p_1)$ . With the remaining probability  $\frac{p - p_1}{p_2 - p_1}$ , the second contract is accepted. Again, the firm's total payoff can be decomposed into the current payoff  $v(q_2) - x_2$  and the continuation payoff  $V(p_2)$ .

We now turn to the worker's incentives. Recall that the high type accepts the first contract with positive probability and that his rent is always zero. This immediately implies  $x_1 = \theta_H q_1 + \alpha$ . Also, the high type must (weakly) prefer the first contract over the second. Furthermore, if  $p_2 < 1$ , then he also accepts the second contract with positive probability. In this case, we must have  $x_2 = \theta_H q_2 + \alpha$ . The fourth constraint is the incentive compatibility constraint, which guarantees that the low type is willing to accept the second

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<sup>8</sup>Without loss of generality, we define  $r_1 = r_2 = \frac{1}{2}$  if  $p_1 = p_2$ .

contract. Finally, if  $p_1 > 0$ , then the low type accepts the first contract with positive probability. But then he must be indifferent between the two contracts.

For every  $p \in (0, 1)$ , we let  $Z_1(p)$  denote the set of solutions to Problem 1, with typical element denoted by  $\{(x_i(p), q_i(p), p_i(p), v_i(p))\}_{i=1,2}$ . We also define the function  $V_1 : [0, 1] \rightarrow \mathbb{R}$  by setting

$$V_1(p) = \max_{\{x_i, q_i, p_i, v_i\}_{i=1,2}} \frac{p_2 - p}{p_2 - p_1} [(1 - \delta)(v(q_1) - x_1) + \delta V(p_1)] + \frac{p - p_1}{p_2 - p_1} [(1 - \delta)(v(q_2) - x_2) + \delta V(p_2)],$$

where the maximization is taken over all the tuples  $\{x_i, q_i, p_i, v_i\}_{i=1,2}$  satisfying the constraints of Problem 1.

A surely accepted menu is the first option available to the firm. The remaining three options can be jointly described. The firm offers a menu with only one contract  $(x, q)$ . The low type accepts the contract with probability one. The high type accepts the contract with probability  $r$  and quits with probability  $1 - r$ . Let  $p' := \frac{p}{p + (1-p)r}$  denote the firm's belief when the contract is accepted.<sup>9</sup> We can express the probability that the contract is accepted as  $\frac{p}{p'}$ .

Depending on the value of  $p'$ , we have three different cases. If  $p' = p$ , then both types accept the contract with probability one, and the firm does not update its belief. In this case, we say that the firm offers a *pooling* menu. On the other hand, if  $p' = 1$ , then the firm fires the high type (since he quits with probability one) and keeps the low type. In this case, we say that the firm offers a *firing* menu. Finally, if  $p' \in (p, 1)$ , then the high type randomizes between accepting the contract and quitting. We say that the firm offers a *partially separating* menu.

The optimal menu with a single contract is the solution to the following problem, which we label Problem 2:

$$\begin{aligned} \max_{x, q, p'} \frac{p}{p'} [(1 - \delta)(v(q) - x) + \delta V(p')] \\ \text{s.t. } 0 \leq p \leq p' \leq 1, \\ x \leq \theta_H q + \alpha \text{ with equality if } p' < 1, \\ x \geq \theta_L q + \alpha. \end{aligned}$$

---

<sup>9</sup>Clearly, the firm's belief becomes equal to zero when the worker quits.

The constraints are very intuitive. If the high type accepts the contract with positive probability ( $p' < 1$ ), then we must have  $x = \theta_H q + \alpha$  (recall, his rent is zero). When the high type quits with probability one, the payment  $x$  can be smaller than  $\theta_H q + \alpha$ , but the contract still must satisfy type  $L$ 's individual rationality constraint (this constraint is automatically satisfied when  $p' < 1$ ).

For every  $p \in (0, 1)$ , let  $Z_2(p)$  denote the set of solutions to Problem 2, with typical element denoted by  $(x(p), q(p), \check{p}(p))$ . It is immediate to see that if  $\check{p}(p) < 1$ , then  $(x(p), q(p)) = (\theta_H q_H^* + \alpha, q_H^*)$ . On the other hand, if  $\check{p}(p) = 1$ , then  $(x(p), q(p)) = (\theta_L q_L^* + \alpha, q_L^*)$ .

We also define the function  $V_2 : [0, 1] \rightarrow \mathbb{R}$  by setting

$$V_2(p) = \max_{x, q, p'} \frac{p}{p'} [(1 - \delta)(v(q) - x) + \delta V(p')],$$

where the maximization is taken over all the triples  $(x, q, p')$  satisfying the constraints of Problem 2.

This concludes the description of the options that are available to the firm. Note that we do not allow for the possibility that the low type quits with positive probability. This is without loss of generality. In fact, it is easy to show that the firm cannot maximize its payoff unless the low-cost worker accepts one of the contracts with probability one. Suppose that type  $L$  chooses to quit. But then the firm can increase its profits by offering the contract  $(\theta_L q_L^* + \alpha, q_L^*)$  which is accepted by the low type (and rejected by the high type).

We say that a pair  $(V, \Phi)$  is *optimal* if it satisfies the following two conditions.

i) For every  $p \in (0, 1)$

$$V(p) = \max \{V_1(p), V_2(p)\};$$

ii) For every  $p \in (0, 1)$  and for every  $v \in \Phi(p)$  at least one of the following two requirements is satisfied:

a) if  $V(p) = V_1(p)$ , then there exists a solution  $\{(x_i(p), q_i(p), p_i(p), v_i(p))\}_{i=1,2} \in Z_1(p)$  such that

$$v = (1 - \delta)(x_2(p) - \theta_L q_2(p)) + \delta v_2(p);$$

b) if  $V(p) = V_2(p)$ , then there exist a solution  $(x(p), q(p), \check{p}(p)) \in Z_2(p)$  and  $v' \in \Phi(\check{p}(p))$  such that

$$v = (1 - \delta)(x(p) - \theta_L q(p)) + \delta v'.$$

We are now ready to define formally the notion of ratchet equilibrium. Recall that, given a perfect Bayesian equilibrium  $(\sigma, \mu)$ ,  $V_F(h^t; (\sigma, \mu))$  denotes the firm's continuation payoff at the history  $h^t$ , while  $W_L(h^t; (\sigma, \mu))$  denotes the low type's continuation rent at  $h^t$ . Finally,  $\mu(h^t)$  represents the firm's belief at the history  $h^t$ .

**Definition 1** *A perfect Bayesian equilibrium  $(\sigma, \mu)$  is a ratchet equilibrium if there exists an optimal pair  $(V, \Phi)$  such that for every  $h^t \in H$ ,*

$$V_F(h^t; (\sigma, \mu)) = V(\mu(h^t)),$$

and

$$W_L(h^t; (\sigma, \mu)) \in \Phi(\mu(h^t)).$$

Suppose that the parties play a ratchet equilibrium  $(\sigma, \mu)$  and consider any history  $h^t$  (on and off the equilibrium path). Given the belief  $\mu(h^t)$ , the firm maximizes its continuation payoff. To provide the worker with the appropriate incentives, the firm promises continuation rents which are consistent with the firm's posterior beliefs. In other words, the promises are credible.

Throughout the rest of the paper, we restrict attention to ratchet equilibria which exist under very weak conditions.

**Theorem 1** *Suppose Assumption 1 holds (i.e.,  $\delta > \underline{\delta}$ ). Then there exists a ratchet equilibrium.*

We postpone the proof of Theorem 1 to Appendix C where we construct a ratchet equilibrium. For fixed values of  $\delta$ , we are unable to say whether or not the ratchet equilibrium is unique. However, in the next section we identify the properties that all ratchet equilibria must satisfy. Furthermore, in Section 6, we show that there exists a unique equilibrium outcome in the limit, as the parties become arbitrarily patient.

## 4.1 Related Solution Concepts

As discussed above, we believe that our refinement sheds light on the economic forces behind the ratchet effect. We remark, however, that all our results can also be derived using a different refinement that imposes the following two restrictions to the concept of perfect Bayesian equilibrium. First, after any menu proposed by the firm, at most two options are

selected with positive probability by the worker.<sup>10</sup> Second, the equilibrium has to satisfy a stationarity property similar to the one imposed in several bargaining games (see, for example, Ausubel and Deneckere, 1989). Assume that there exists a history at which the belief is  $p \in (0, 1)$  and the firm offers the menu  $m$ . Suppose that the options selected by the worker with positive probability are  $a$  and  $a'$  (the two options may coincide) and that they lead to the posterior beliefs  $p_1 \leq p$  and  $p_2 \geq p$ . The stationary restriction imposes that every time the firm offers the menu  $m$  and the belief is  $p' \in [p_1, p_2]$ , the options selected with positive probability must remain  $a$  and  $a'$  and, furthermore, they have to lead to the posterior beliefs  $p_1$  and  $p_2$ .<sup>11</sup>

Finally, we point out that the ratchet equilibrium constructed in the proof of Theorem 1 is weak Markovian in the following sense. In every period  $t$ , the firm's decision depends only on the belief  $p_t$  and the menu  $m_{t-1}$  offered in the previous period, while the worker's decision depends only on the belief  $p_t$  and the menu  $m_t$ .<sup>12</sup>

## 5 Properties of the Ratchet Equilibria

In this section, we characterize the equilibrium behavior by establishing a number of facts that hold in all ratchet equilibria. We then use these facts to investigate the limiting equilibrium outcome.

We start the analysis by providing a lower bound to the firm's continuation payoff. First, the firm can always choose to offer a pooling menu and leave the belief unchanged. Thus, its payoff is bounded below by the payoff of the optimal pooling menu  $(\theta_H q_H^* + \alpha, q_H^*)$ .<sup>13</sup> Formally, for any  $p \in (0, 1)$ , the triple  $(\theta_H q_H^* + \alpha, q_H^*, p)$  satisfies all the constraints of Problem 2. Thus, we have:

$$V(p) \geq V_2(p) \geq (1 - \delta)(v(q_H^*) - \theta_H q_H^* - \alpha) + \delta V(p),$$

which implies that the firm's payoff is bounded below by the payoff of the optimal pooling menu  $\{(\theta_H q_H^* + \alpha, q_H^*)\}$ :

$$V(p) \geq v(q_H^*) - \theta_H q_H^* - \alpha.$$

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<sup>10</sup>This restriction is without loss of generality for the concept of ratchet equilibrium since this refinement imposes to select, after every history, the highest continuation payoff of the firm.

<sup>11</sup>This restriction can never be satisfied when  $p' \notin [p_1, p_2]$  as it would violate Bayes' rule.

<sup>12</sup>The dependence of the firm's menu  $m_t$  on  $m_{t-1}$  occurs only after an (unprofitable) deviation of the firm in period  $t - 1$ . On the equilibrium path, the firm's decision in period  $t$  depends only on the belief  $p_t$ .

<sup>13</sup>We point out that this result holds for the whole class of perfect Bayesian equilibria.



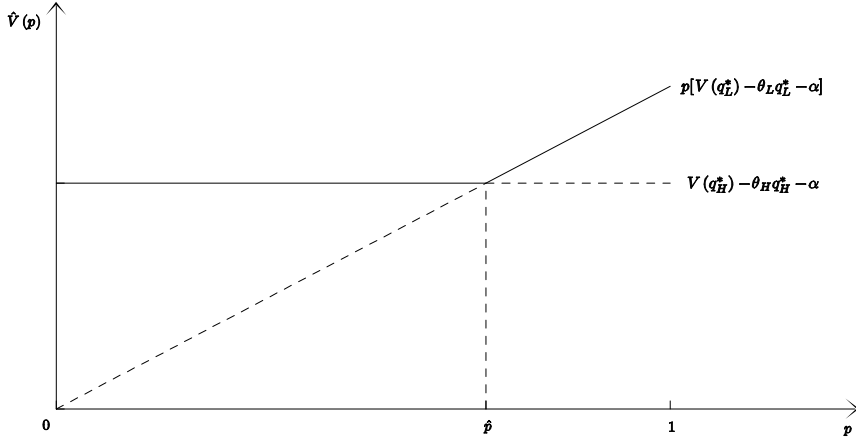


Figure 1: Lower bound to the firm's continuation payoff

Another lower bound can be derived by assuming that the firm offers the optimal firing menu  $(\theta_L q_L^* + \alpha, q_L^*)$ . Again, for any  $p \in (0, 1)$ , the triple  $(\theta_L q_L^* + \alpha, q_L^*, 1)$  satisfies all the constraints of Problem 2. This, in turn, implies:

$$V(p) \geq V_2(p) \geq p[(1 - \delta)(v(q_L^*) - \theta_L q_L^* - \alpha) + \delta V(1)] = p(v(q_L^*) - \theta_L q_L^* - \alpha).$$

We summarize these simple findings in Lemma 1 below. To do so, we first define the function  $\hat{V} : [0, 1] \rightarrow \mathbb{R}_+$  by setting

$$\hat{V}(p) := \max \{v(q_H^*) - \theta_H q_H^* - \alpha, p(v(q_L^*) - \theta_L q_L^* - \alpha)\}, \quad (1)$$

for every  $p \in [0, 1]$ . For future reference, we let  $\hat{p} := \frac{v(q_H^*) - \theta_H q_H^* - \alpha}{v(q_L^*) - \theta_L q_L^* - \alpha}$  denote the smallest belief  $p$  at which  $\hat{V}(p) = p(v(q_L^*) - \theta_L q_L^* - \alpha)$ .

**Lemma 1** *Suppose that  $(V, \Phi)$  is an optimal pair. Then  $V(p) \geq \hat{V}(p)$  for every  $p \in [0, 1]$ .*

Figure 1 illustrates the lower bound to the firm's payoff.

Our next result shows that the value function  $V$  is convex. This result has a number of important implications. First, it provides an upper bound to the firm's payoff. In fact,  $V(p) \leq pV(1) + (1 - p)V(0)$  for every  $p \in [0, 1]$ . Second, it follows from the convexity of  $V$  that the value of information is positive. In other words, the firm would be willing to pay

a positive price to get a signal that refines its beliefs. In contrast to single-person decision problems, where this property is generally easily obtained, our analysis is considerably complicated by strategic interactions. In particular, the firm must provide non-trivial intertemporal incentives to the worker.<sup>14</sup>

**Lemma 2** *Suppose that  $(V, \Phi)$  is an optimal pair. Then the function  $V$  is convex.*

The proof of Lemma 2 is relegated to Appendix A.<sup>15</sup> The proof has a recursive structure that reflects the decomposition of the firm's payoff into the current payoff and the continuation payoff. Fix the belief  $p \in (0, 1)$  and suppose that  $V_1(p) \geq V_2(p)$  (thus, the firm's payoff can be calculated by solving Problem 1). Consider the solution to Problem 1 when the initial belief is  $p$ . The solution yields an expected current payoff, say equal to  $\bar{\pi}$ , and induces a probability distribution over the beliefs  $p_1$  and  $p_2$ , with probabilities  $r_1$  and  $r_2$ , respectively. Suppose now that the firm faces a lottery between two initial beliefs,  $p'$  and  $p''$ , such that the expected belief from this lottery is equal to  $p$ . We use the solution to Problem 1 (when the belief is  $p$ ) to design, both at  $p'$  and  $p''$ , a random selection of tuples that satisfy the constraints of either Problem 1 or Problem 2. The (random) tuples that we select are not necessarily optimal at  $p'$  or  $p''$ . However, our selection has two desirable properties. First, it yields an expected current payoff weakly greater than  $\bar{\pi}$  (the current payoff associated to  $p$ ). Second, the selection induces a probability distribution  $(r_{11}, \dots, r_{1k_1}, r_{21}, \dots, r_{2k_2})$  over  $k_1 + k_2$  (not necessarily distinct) beliefs  $\{p_{11}, \dots, p_{1k_1}, p_{21}, \dots, p_{2k_2}\}$ .<sup>16</sup> This distribution satisfies the following conditions. For every  $i = 1, 2$ , there are at most two beliefs  $p_{ij}$ ,  $j = 1, \dots, k_i$ , different from  $p_i$  (thus,  $k_i \leq 3$ ), and the following equalities hold:

$$\frac{r_{i1}p_{i1} + \dots + r_{ik_i}p_{ik_i}}{r_{i1} + \dots + r_{ik_i}} = \frac{r_{i1}p_{i1} + \dots + r_{ik_i}p_{ik_i}}{r_i} = p_i.$$

In the proof, we also construct a random selection with similar properties for values of  $p$  with  $V_1(p) < V_2(p)$ .

This concludes our proof. In fact, we demonstrate that the comparison between the belief  $p$  and the lottery over  $p'$  and  $p''$  can be decomposed into a number of comparisons.

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<sup>14</sup>Our result is related to the celebrated *cav(u)* Theorem for zero-sum games (see Aumann and Maschler, 1995). The non-zero-sum nature of our game makes the analysis considerably different.

<sup>15</sup>Appendix A also contains the proofs of the remaining results in this section.

<sup>16</sup>Both the expected current payoff and the probability distribution over the posterior beliefs are computed ex-ante, before the realization of the lottery between  $p'$  and  $p''$  is realized.

The first of these is about the payoffs in the current period, and we show that the firm prefers the lottery. All other comparisons take place in the following period. Each of these comparisons is between a deterministic belief and a lottery over beliefs with an expected value equal to the deterministic belief. Therefore, it is analogous to the initial comparison between  $p$  and the lottery over  $p'$  and  $p''$ . Recall that, as far as the current payoff is concerned, the firm prefers the lottery. We therefore conclude that the value function  $V(\cdot)$  is convex.

Our next result shows that in a ratchet equilibrium, the firm never offers a partially separating menu.

**Lemma 3** *Let  $(V, \Phi)$  be an optimal pair and fix  $p \in (0, 1)$ . For every  $\check{p} \in (p, 1)$ ,*

$$V(p) > \frac{p}{\check{p}} [(1 - \delta)(v(q_H^*) - \theta_H - \alpha) + \delta V(\check{p})].$$

This result is another consequence of the convexity of  $V$ . Suppose the belief is equal to  $p$  and the firm offers a partially separating menu - that is, a menu containing the contract  $(\theta_H q_H^* + \alpha, q_H^*)$  which is accepted with probability  $\frac{p}{\check{p}}$  for some  $\check{p} \in (p, 1)$ . This implies that  $\check{p}$  is a maximizer of the function  $g : [p, 1] \rightarrow \mathbb{R}_+$  defined by:

$$g(p') = \frac{p}{p'} [(1 - \delta)(v(q_H^*) - \theta_H - \alpha) + \delta V(p')]. \quad (2)$$

However, using the convexity of  $V$ , we are able to show that if  $\check{p} < 1$  is an interior extremizer of  $g$ , then it must be a minimizer. We thus conclude that it is suboptimal to offer a partially separating menu.

Our next goal is to characterize the firm's behavior when the firm is sufficiently optimistic that the worker is efficient. Recall that when the belief is larger than  $p^C$ , the solution to the commitment problem is to offer the contract  $(\theta_L q_L^* + \alpha, q_L^*)$ , which is accepted only by the low type.<sup>17</sup> Of course,  $V$  is bounded above by the payoff of the commitment solution. Therefore, there exists a threshold  $\bar{p} \in [\hat{p}, p^C]$ , above which it is optimal for the firm to offer the firing menu. Lemma 4 contains a stronger statement. For any belief  $p$  above  $\bar{p}$ , offering the firing menu is the unique solution to the firm's optimization problem. Any other menu yields a total payoff smaller than  $V(p)$ .

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<sup>17</sup>It is easy to see that, for any  $p \in (0, p^C)$ , the firm's payoff from the commitment solution is strictly larger than  $\hat{V}(p)$ .

**Lemma 4** *Let  $(V, \Phi)$  be an optimal pair. There exists  $\bar{p} \in [\hat{p}, p^C]$  such that  $V(p) = p(v(q_L^*) - \theta_L q_L^* - \alpha)$  if and only if  $p \geq \bar{p}$ . Furthermore,  $V(p) > V_1(p)$  for every  $p > \bar{p}$ .*

Because of Lemma 3, it is enough to show that above  $\bar{p}$  the firing menu performs strictly better than any surely accepted menu. To see that this is indeed the case, assume that for a certain belief  $p' > \bar{p}$ , it is optimal to offer the surely accepted menu  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$ , with  $p_1 < p' < p_2$ . For simplicity, assume that  $p_2 < 1$  (the proof also deals with the case  $p_2 = 1$ ). Notice that both  $V$  and the payoff of the surely accepted menu  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$  are linear in the initial belief over the interval  $(p' - \varepsilon, p_2)$ , for some small  $\varepsilon > 0$ . This implies that  $V$  and the payoff of the surely accepted menu must coincide over the interval and that  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$  must be optimal for any  $p \in (p' - \varepsilon, p_2)$ . However, it is easy to see that the surely accepted menu cannot be optimal for a belief  $p$  sufficiently close to  $p_2$ . When  $p$  is close to  $p_2$  and the firm offers  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$ , the worker accepts the second contract  $(x_2, q_2)$  with probability close to one. Therefore, the firm's current payoff is at most  $(v(q_H^*) - \theta_H q_H^* - \alpha)$ , while the continuation payoff is close to  $V(p_2) = p_2(v(q_L^*) - \theta_L q_L^* - \alpha)$ . This is strictly less than the payoff of the firing menu since  $p_2 > \hat{p}$ .

The last finding in this section investigates what happens when the firm offers a surely accepted menu (which may be optimal when the belief is small). Lemma 5 links the distance between the initial and the posterior beliefs to the difference between the payoff  $V(p)$  of the surely accepted menu and the payoff  $(v(q_H^*) - \theta_H q_H^* - \alpha)$  of the optimal pooling menu. Intuitively, in the current period the payoff of the surely accepted menu is at most  $(v(q_H^*) - \theta_H q_H^* - \alpha)$ . Therefore, if a surely accepted menu is much more attractive than the pooling menu, it must be because it allows the firm to learn a lot about the worker's type. Formally, we have the following result.

**Lemma 5** *Let  $(V, \Phi)$  be an optimal pair. Fix  $p$  such that  $V(p) = V_1(p)$ , and let  $\{x_i, q_i, p_i, v_i\}_{i=1,2}$  denote the corresponding solution to Problem 1. Then we have:*

$$\min \{p - p_1, p_2 - p\} \geq \frac{(1 - p^C)}{(v(q_L^*) - \theta_L q_L^* - \alpha)} (1 - \delta) [V(p) - (v(q_H^*) - \theta_H q_H^* - \alpha)].$$

Note that the term  $\frac{(1 - p^C)}{(v(q_L^*) - \theta_L q_L^* - \alpha)}$  does not depend on the discount factor; thus, the lower bound on the distance between the beliefs (i.e., the right hand side of the above expression) is linear in  $\delta$ . This will play a crucial role in the characterization of the limiting equilibrium outcome.

## 6 Limiting Equilibrium Outcome

The goal of this section is to investigate the implications of the ratchet effect when the parties are patient. As we will see below, in the limit the equilibrium outcome takes a very simple form. In general, the firm finds it too expensive to learn about the worker's ability while keeping both types in the relationship. As a consequence, if the prior  $p_0$  is below  $\hat{p}$ , then the firm gives up to learning and offers the optimal pooling menu. On the other hand, if the prior  $p_0$  is above  $\hat{p}$ , then the firm fires the inefficient worker right away and interacts only with the efficient type.

It is easy to see that for  $\delta > \underline{\delta}$ , it is impossible to find a menu  $\{(x_L, q_L), (x_H, q_H)\}$  that fully separates the agent's types (i.e., a menu such that type  $i$  chooses the contract  $(x_i, q_i)$  with probability one). In fact, in a ratchet equilibrium, the firm must offer the contract  $(\theta_i q_i^* + \alpha, q_i^*)$  as soon as it becomes aware that the agent's type is equal to  $i = L, H$ . This implies that  $x_L$  must be at least equal to  $x_H + \theta_L (q_L - q_H) + \frac{\delta}{1-\delta} \Delta \theta q_H^*$  so that the low-cost worker has an incentive to reveal his identity. However, the fact that  $\delta > \underline{\delta}$  implies that it is strictly profitable for the high-cost worker to use the so called "take the money and run" strategy - that is, to accept the contract  $(x_L, q_L)$  and then quit the relationship.

Of course, full separation is only one way (specifically, the simplest way) in which the firm can learn about the agent's type. One could imagine more sophisticated learning processes in which, for example, the firm's belief evolves gradually over time. Below, we show that no matter how complex the process, learning cannot take place when the parties are sufficiently patient.

Our first result shows that as the discount factor goes to one, the critical value of the belief above which the firm offers the firing menu converges to  $\hat{p}$ .

**Theorem 2** *Consider a sequence  $\{\delta_n\}_{n=1,\dots}$  of discount factors converging to one and let  $(V_n, \Phi_n)$  be an optimal pair for  $\delta_n$ . For every  $n$ , let  $\bar{p}_n \in [\hat{p}, p^C]$  denote the smallest belief  $p$  for which  $V_n(p) = p(v(q_L^*) - \theta_L q_L^* - \alpha)$ . We have:*

$$\lim_{n \rightarrow \infty} \bar{p}_n = \hat{p}$$

The proof of Theorem 2 is in Appendix B. To simplify the exposition and focus on the basic intuition behind the theorem, below we sketch the proof of a weaker result (Corollary 1), which follows from Theorem 2.

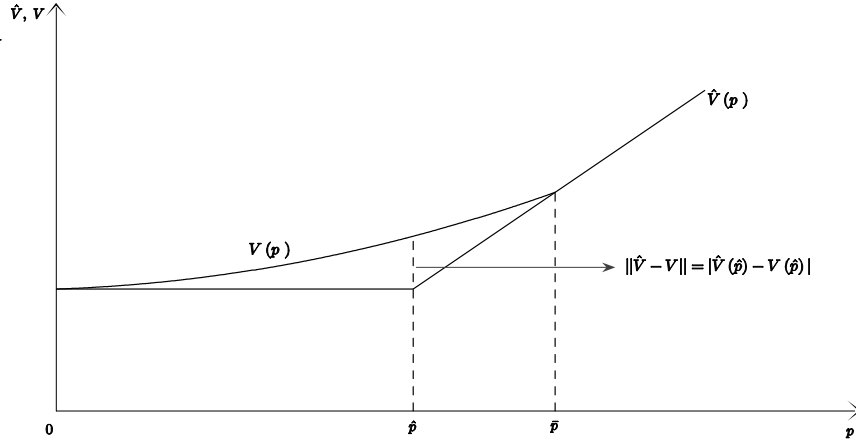


Figure 2: The firm's limiting payoff

An implication of Theorem 2 is that as the parties become more patient, the firm's payoff converges to the lower bound  $\hat{V}$  obtained in Lemma 1. We measure the distance  $\|V - \hat{V}\|$  between a value function  $V$  and  $\hat{V}$  by taking the maximum distance between the two functions:

$$\|V - \hat{V}\| := \max_{p \in [0,1]} |V(p) - \hat{V}(p)| = V(\hat{p}) - \hat{V}(\hat{p}).$$

The distance  $V(p) - \hat{V}(p)$  is maximized at  $\hat{p}$ . This is because  $V$  is convex and bounded below by  $\hat{V}$ , and  $V(0) - \hat{V}(0) = V(1) - \hat{V}(1) = 0$ . Given Theorem 2, it is easy to check that the following result must hold.

**Corollary 1** *Consider a sequence  $\{\delta_n\}_{n=1,\dots}$  of discount factors converging to one and let  $(V_n, \Phi_n)$  be an optimal pair for  $\delta_n$ . We have:*

$$\lim_{n \rightarrow \infty} \|V_n - \hat{V}\| = \lim_{n \rightarrow \infty} V_n(\hat{p}) - \hat{V}(\hat{p}) = 0.$$

To provide some intuition, let us assume that, for arbitrarily large values of the discount factor, there are ratchet equilibria that yield a value function  $V$  that is bounded away from  $\hat{V}$  (see Figure 2).<sup>18</sup> Let  $\bar{p}$  denote the belief above which the firm offers the firing menu.

<sup>18</sup>For simplicity, here we assume that the value functions of the different equilibria are all equal to  $V$ . Of course, in the proof of Theorem 2 we assume that there is a sequence of value functions.

Since  $V$  is bounded away from  $\hat{V}$ , we can find  $p' < \hat{p}$  such that  $V(p) > v(q_H^*) - \theta_H q_H^* - \alpha$  for every  $p > p'$ . This tells us that the firm offers a surely accepted menu whenever the belief  $p$  is in the interval  $(p', \bar{p})$ . Furthermore, Lemma 5 provides a lower bound  $\varepsilon(1 - \delta)$  (for some  $\varepsilon > 0$ ) to the distance between the initial and the posterior beliefs. This has two important consequences. First, if the belief  $p$  is smaller than  $\bar{p}$  but sufficiently close to it (in particular,  $p > \bar{p} - \varepsilon(1 - \delta)$ ) and the low type accepts the second contract of the surely accepted menu, then his total rent is at most  $(1 - \delta)\Delta\theta$ . In fact, once the worker accepts the second contract, the posterior belief jumps to the right of  $\bar{p}$  and the firm offers the firing menu (which yields a rent equal to zero). Thus, the low-cost worker obtains a rent only in the current period, and this is bounded above by  $(1 - \delta)\Delta\theta$ .

Second, suppose that the initial belief is  $p \in (\bar{p} - \varepsilon(1 - \delta), \bar{p})$ . The firm offers the surely accepted menu. Suppose that the low type accepts the first contract. This will move the belief down and the firm will offer another surely accepted menu. Again, suppose that the low type accepts the first contract. This continues until the firm's belief becomes weakly smaller than  $p'$ .<sup>19</sup> The lower bound  $\varepsilon(1 - \delta)$  on the distance between the initial and the posterior beliefs gives us an upper bound  $T_\delta = \frac{\bar{M}}{1 - \delta}$  (for some  $\bar{M} > 0$ ) to the number of periods that it takes to move the belief from  $p$  to a value smaller than  $p'$ . Notice that as  $\delta$  converges to one,  $T_\delta$  goes to infinity at the same speed as  $\frac{1}{1 - \delta}$ . It is easy to check that as  $\delta$  goes to one, the discounted value  $\delta^{T_\delta}$  of one dollar due in  $T_\delta$  periods is bounded away from zero.

Finally, we provide a lower bound  $u > 0$  to the low type's rent when the belief is below  $p'$ . This is intuitive; the low type can always get a non-negative rent by imitating the high type. If the low type's rent is close to zero, the firm must necessarily ask the high type to produce goods of very low quality. However, the firm's payoff is then bounded above by a value close to  $p'(v(q_L^*) - \theta_L q_L^* - \alpha)$ ; in other words, the firm essentially makes profits only with the efficient worker. However,  $p'(v(q_L^*) - \theta_L q_L^* - \alpha) < v(q_H^*) - \theta_H q_H^* - \alpha$ , since  $p' < \hat{p}$ , which implies that the firm's behavior is not optimal.

We have now derived a contradiction. When the belief is  $p \in (\bar{p} - \varepsilon(1 - \delta), \bar{p})$ , the firm offers a surely accepted menu and the low type should accept the second contract with positive probability. However, as  $\delta$  goes to one, the low type's rent from accepting the second contract goes to zero. Furthermore, the low type can take another course of action (accepting the first contract of a surely accepted menu until the belief goes below  $p'$ ), which

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<sup>19</sup>Notice that the low type obtains a non-negative rent when he accepts the first contract.

yields, in the limit, a strictly positive payoff. Therefore, for  $\delta$  sufficiently close to one, the low type has an incentive to deviate and to reject the second contract when the belief is  $p$ .

The proof of Theorem 2 must also rule out the more challenging case in which the sequence  $\{V_n\}$  converges to  $\hat{V}$  while  $\{\bar{p}_n\}$  converges (by taking a subsequence if necessary) to  $\bar{p} > \hat{p}$ . The proof sketched above does not work in this case because we cannot provide a strictly positive lower bound to the discounted time that it takes to the low-cost worker to move the belief down from (a value close to)  $\bar{p}$  to a belief smaller than  $p' < \hat{p}$ . This is because as  $n$  grows large, the value  $V_n(p')$  becomes arbitrarily close to  $v(q_H^*) - \theta_H q_H^* - \alpha$ . To avoid this problem, we take  $p'$  larger than  $\hat{p}$  and bounded away from it. This gives us the desired lower bound to the expected time necessary to move the belief down. However, a more involved argument is needed to show that the efficient worker's rent remains bounded away from zero when the belief is below  $p'$  and  $p'$  is larger than  $\hat{p}$ .

So far we have focused on the firm's payoff. We now turn to the limiting equilibrium allocation. This allows us to understand how the relationship evolves when the parties are patient. In particular, we can investigate how the inability to commit to long-term contracts affects the firm's learning process. Furthermore, given the equilibrium allocation, we can easily compute the efficient worker's rent.

First, consider the case in which the prior  $p_0$  is above  $\hat{p}$ . Theorem 2 implies that for every  $p_0 > \hat{p}$ , there exists  $\bar{\delta} < 1$  such that for every  $\delta > \bar{\delta}$ , all ratchet equilibria have the following outcome. The firm offers the firing menu (i.e., the menu containing only the contract  $(\theta_L q_L^* + \alpha, q_L^*)$ ) in every period  $t = 0, 1, \dots$ , and the worker accepts the contract if and only if his type is low. In other words, the firm immediately discovers the worker's type and the worker's rent is equal to zero.

Consider now the case  $p_0 < \hat{p}$ . An allocation  $\mathbf{A}$  specifies, for every period  $t = 0, 1, \dots$ , the transfer  $x_t$  paid by the firm and the quality  $q_t$  produced by the worker. If the worker accepts the contract  $(x_t, q_t)$  in period  $t = 0, \dots, T - 1$  and quits the relationship in period  $T$ , then we set  $x_\tau = q_\tau = 0$  for every  $\tau \geq T$ . We let  $\mathbf{A}_H^*$  denote the allocation in which  $x_t = \theta_H q_H^* + \alpha$  and  $q_t = q_H^*$  for every  $t \geq 0$ . In words,  $\mathbf{A}_H^*$  is the allocation that occurs if the firm offers the optimal pooling menu in every period.

We define the distance  $\mathbf{m}(\mathbf{A}, \mathbf{A}')$  between the allocations  $\mathbf{A} = \{x_t, q_t\}_{t=0, \dots}$  and  $\mathbf{A}' = \{x'_t, q'_t\}_{t=0, \dots}$  as

$$\mathbf{m}(\mathbf{A}, \mathbf{A}') = (1 - \delta) \sum_{t=0}^{\infty} \delta^t \|(x_t, q_t) - (x'_t, q'_t)\|.$$



Theorem 3 shows that as the parties become more patient, the equilibrium allocation converges to the allocation  $\mathbf{A}_H^*$  induced by the optimal pooling menu. Recall that given a ratchet equilibrium  $(\sigma, \mu)$ , we use  $\mathbb{E}_{(\sigma, \mu)}[f|i, h^t]$  to denote the conditional value of  $f$  given type  $i = L, H$  and the public history  $h^t$ . To ease the notation, when the history is  $h^0$  we suppress it and write  $\mathbb{E}_{(\sigma, \mu)}[f|i]$  for  $\mathbb{E}_{(\sigma, \mu)}[f|i, h^0]$ . Thus,  $\mathbb{E}_{(\sigma, \mu)}[\mathbf{m}(\mathbf{A}, \mathbf{A}_H^*) | i]$  represents the expected distance between the random equilibrium allocation  $\mathbf{A}$  and the allocation  $\mathbf{A}_H^*$  when the worker's type is equal to  $i$ .

**Theorem 3** *Assume that  $p_0 < \hat{p}$ . Consider a sequence  $\{\delta_n\}_{n=1, \dots}$  of discount factors converging to one and let  $(\sigma_n, \mu_n)$  be a ratchet equilibrium when the discount factor is equal to  $\delta_n$ . For  $i = L, H$ , we have:*

$$\lim_{n \rightarrow \infty} \mathbb{E}_{(\sigma_n, \mu_n)}[\mathbf{m}(\mathbf{A}, \mathbf{A}_H^*) | i] = 0.$$

The reader can find the proof of Theorem 3 in Appendix B. Intuitively, in the limit the allocation must be close to the pooling allocation  $\mathbf{A}_H^*$  as long as the firm offers surely accepted menus that lead to two posterior beliefs both smaller than  $\hat{p}$ . This is because the firm's current payoff is at most  $v(q_H^*) - \theta_H q_H^* - \alpha$  (since both contracts are accepted with positive probability by the high type), while its continuation payoff is close to  $v(q_H^*) - \theta_H q_H^* - \alpha$  (recall that the value function  $V$  converges to  $\hat{V}$  as  $\delta$  approaches one). Therefore, the equilibrium allocation can remain bounded away from  $\mathbf{A}_H^*$  only if the discounted value of the time in which the firm fires the inefficient worker is bounded away from zero. When the parties are patient, the low types prefers to avoid the event of the firm offering a firing menu, since his continuation payoff is close to zero. However, the firm is willing to fire the inefficient worker only if the belief is close to or above  $\hat{p}$ . Since the prior  $p_0$  is smaller than  $\hat{p}$ , this means that a firing menu is an event that is more likely to occur when the worker is efficient than when he is inefficient. We conclude that the low type has a strict incentive to deviate and to mimic the high type's behavior.

## 7 Concluding Remarks

History tells us that, under the piece-rate system, workers often choose to limit their production. It has been suggested that this behavior is induced by the fear that good performances may lead to worse terms of trade. History also tells us that the lack of

commitment to incentive schemes was an important explanation for the underperformance of Soviet economies. This paper shows that these findings are natural predictions of a fully dynamic model in which the principal is unable to commit not to use the acquired information in his best interest. Outside the realm of these applications, the main message of this paper is that the ratchet effect leads to large inefficiencies due to a persistent failure of the transmission of private information in institutions and markets plagued by adverse-selection and commitment problems.

We have assumed throughout the paper that both types have the same outside option and that  $\alpha$  is strictly positive. This simplifies the notation, but our results do not hinge on these assumptions. In particular, our findings continue to hold provided that the efficient worker's option is not much larger than the inefficient worker's. It is also possible to show that our main results extend to the case  $\alpha = 0$ .

We have also restricted attention to a two-type model. This assumption is somewhat standard in the literature on dynamic contracting with limited commitment and is made mainly for tractability reasons. We leave the challenging extension to several (or a continuum of) types for future work.

Finally, we have analyzed a model in which the parties interact for potentially infinitely many periods. A different approach would be to assume a finite horizon. We do not know whether the equilibrium outcome of the finite horizon model converges, as the number of periods grows and the parties become more patient, to the limiting outcome of the infinite horizon model. Of course, a model with finitely many periods loses the recursive structure and the analysis becomes less tractable. This is another challenging direction for future research.

## Appendix A: Proofs of the Results in Section 5

### Proof of Lemma 2.

Fix two beliefs,  $p'$  and  $p''$ , with  $0 \leq p' < p'' \leq 1$ , and let  $p$  denote a convex combination of them:

$$p := \lambda p' + (1 - \lambda) p'',$$

for some  $\lambda \in (0, 1)$ . We need to show that

$$V(p) \leq \lambda V(p') + (1 - \lambda) V(p'').$$

We distinguish between two cases depending on whether  $V_1(p) \geq V_2(p)$  or  $V_1(p) < V_2(p)$ .

**Case a.** We start with the case of  $V_1(p) \geq V_2(p)$ .

To simplify the notation, we let  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$  denote the solution to Problem 1 when the initial belief is  $p$ . Recall that

$$V(p) = r_1 [(1 - \delta)(v(q_1) - x_1) + \delta V(p_1)] + r_2 [(1 - \delta)(v(q_2) - x_2) + \delta V(p_2)],$$

where  $r_1 = \frac{p_2 - p}{p_2 - p_1}$ , and  $r_2 = \frac{p - p_1}{p_2 - p_1}$ .

We let

$$\bar{\pi} := r_1 (v(q_1) - x_1) + r_2 (v(q_2) - x_2)$$

denote the firm's *current* payoff when the initial belief is  $p$ .

We associate to  $p'$  a (possibly degenerate) lottery between  $\{(x'_i, q'_i, p'_i, v'_i)\}_{i=1,2}$  (a tuple which satisfies the constraints of Problem 1) and  $(x', q', p')$  (a triple which satisfies the constraints of Problem 2). We associate a similar lottery to  $p''$ . We show that our random selection yields an expected current payoff of at least  $\bar{\pi}$ . Furthermore, as mentioned in Section 5, the selection induces a probability distribution  $(r_{11}, \dots, r_{1k_1}, r_{21}, \dots, r_{2k_2})$  over  $k_1 + k_2$  (not necessarily distinct) beliefs  $\{p_{11}, \dots, p_{1k_1}, p_{21}, \dots, p_{2k_2}\}$  such that for every  $i = 1, 2$ , there are at most two beliefs  $p_{ij}$ ,  $j = 1, \dots, k_i$ , different from  $p_i$ , and the following equalities hold:

$$\frac{r_{i1}p_{i1} + \dots + r_{ik_i}p_{ik_i}}{r_{i1} + \dots + r_{ik_i}} = \frac{r_{i1}p_{i1} + \dots + r_{ik_i}p_{ik_i}}{r_i} = p_i.$$

The exact form of the random selection depends on the comparisons between  $p'$  and  $p_1$  and between  $p''$  and  $p_2$ . Below, we deal with all possible cases when  $p_2 < 1$ . This simplifies the notation. However, a similar argument holds when  $p_2 = 1$  and we omit the details.

Notice that when  $p_2 < 1$ ,  $x_i = \theta_H q_i + \alpha$  for  $i = 1, 2$ . Thus, the current payoff  $\bar{\pi}$  is bounded above by  $v(q_H^*) - \theta_H q_H^* - \alpha$ .

**Case a1:**  $p' \leq p_1$  and  $p'' \geq p_2$ .

We associate the triple  $(\theta_H q_H^* + \alpha, q_H^*, p')$  to  $p'$  and the triple  $(\theta_H q_H^* + \alpha, q_H^*, p'')$  to  $p''$ . In words, in both cases the firm offers the optimal pooling menu. This yields a current payoff of  $v(q_H^*) - \theta_H q_H^* - \alpha \geq \bar{\pi}$ . Furthermore, our selection induces the probability distribution  $(\lambda_1 r_1, (1 - \lambda_1) r_1, \lambda_2 r_2, (1 - \lambda_2) r_2)$  over the beliefs  $\{p', p'', p', p''\}$ , where  $\lambda_i \in [0, 1]$ ,  $i = 1, 2$ , is the solution to the following equation:

$$p_i = \lambda_i p' + (1 - \lambda_i) p''.$$

**Case a2:**  $p' > p_1$  and  $p'' < p_2$ .

We associate  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$  (the solution to Problem 1 at  $p$ ) to  $p'$  and  $p''$ . It is easy to check that this yields a current payoff equal to  $\bar{\pi}$ . In fact, we have:

$$\begin{aligned} & \left[ \frac{\lambda(p_2 - p')}{p_2 - p_1} + \frac{(1 - \lambda)(p_2 - p'')}{p_2 - p_1} \right] (v(q_1) - x_1) + \left[ \frac{\lambda(p' - p_1)}{p_2 - p_1} + \frac{(1 - \lambda)(p'' - p_1)}{p_2 - p_1} \right] (v(q_2) - x_2) = \\ & \frac{p_2 - p}{p_2 - p_1} (v(q_1) - x_1) + \frac{p - p_1}{p_2 - p_1} (v(q_2) - x_2) = r_1 (v(q_1) - x_1) + r_2 (v(q_2) - x_2) = \bar{\pi}. \end{aligned}$$

Our selection induces the probability distribution  $\left( \frac{\lambda(p_2 - p')}{p_2 - p_1} + \frac{(1 - \lambda)(p_2 - p'')}{p_2 - p_1}, \frac{\lambda(p' - p_1)}{p_2 - p_1} + \frac{(1 - \lambda)(p'' - p_1)}{p_2 - p_1} \right)$  over the beliefs  $\{p_1, p_2\}$ . Note that

$$\begin{aligned} \frac{\lambda(p_2 - p')}{p_2 - p_1} + \frac{(1 - \lambda)(p_2 - p'')}{p_2 - p_1} &= \frac{p_2 - p}{p_2 - p_1} = r_1, \\ \frac{\lambda(p' - p_1)}{p_2 - p_1} + \frac{(1 - \lambda)(p'' - p_1)}{p_2 - p_1} &= \frac{p - p_1}{p_2 - p_1} = r_2. \end{aligned}$$

**Case a3:**  $p' > p_1$  and  $p'' = p_2$ .

We associate  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$  to  $p'$  and the triple  $(\theta_H q_H^* + \alpha, q_H^*, p'')$  to  $p''$ . This yields a current payoff equal to:

$$\begin{aligned} & \frac{\lambda(p_2 - p')}{p_2 - p_1} (v(q_1) - x_1) + \frac{\lambda(p' - p_1)}{p_2 - p_1} (v(q_2) - x_2) + (1 - \lambda) (v(q_H^*) - \theta_H q_H^* - \alpha) \geq \\ & \frac{\lambda(p_2 - p')}{p_2 - p_1} (v(q_1) - x_1) + \frac{\lambda(p' - p_1)}{p_2 - p_1} (v(q_2) - x_2) + (1 - \lambda) (v(q_2) - x_2) = \bar{\pi}. \end{aligned}$$

Furthermore, the selection induces the probability distribution  $\left( \frac{\lambda(p_2 - p')}{p_2 - p_1}, \frac{\lambda(p' - p_1)}{p_2 - p_1} + \frac{(1 - \lambda)(p'' - p_1)}{p_2 - p_1} \right)$  over the beliefs  $\{p_1, p_2\}$ . Note that  $\frac{\lambda(p_2 - p')}{p_2 - p_1} = r_1$ .

**Case a4:**  $p' > p_1$  and  $p'' > p_2$ .

Let  $\beta \in (0, 1)$  be the solution to the following equation:

$$\lambda\beta\frac{p_2 - p'}{p_2 - p_1} = \frac{p_2 - p}{p_2 - p_1} = r_1.$$

We propose the following selection. At  $p'$  the firm chooses  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$  with probability  $\beta$ , and the triple  $(\theta_H q_H^* + \alpha, q_H^*, p')$  with probability  $1 - \beta$ . At  $p''$  the firm chooses the triple  $(\theta_H q_H^* + \alpha, q_H^*, p'')$  (with probability one). The firm's current payoff is equal to:

$$\begin{aligned} \lambda\beta\frac{p_2 - p'}{p_2 - p_1}(v(q_1) - x_1) + \left(1 - \lambda\beta\frac{p_2 - p'}{p_2 - p_1}\right)(v(q_H^*) - \theta_H q_H^* - \alpha) = \\ r_1(v(q_1) - x_1) + (1 - r_1)(v(q_H^*) - \theta_H q_H^* - \alpha) \geq \bar{\pi}. \end{aligned}$$

Our selection induces the probability distribution  $(\lambda\beta\frac{p_2 - p'}{p_2 - p_1}, \lambda(1 - \beta), \lambda\beta\frac{p' - p_1}{p_2 - p_1}, 1 - \lambda)$  over the beliefs  $\{p_1, p', p_2, p''\}$ . It is easy to check that

$$\frac{\lambda(1 - \beta)p' + \lambda\beta\frac{p' - p_1}{p_2 - p_1}p_2 + (1 - \lambda)p''}{\lambda(1 - \beta) + \lambda\beta\frac{p' - p_1}{p_2 - p_1} + 1 - \lambda} = \frac{\lambda(1 - \beta)p' + \lambda\beta\frac{p' - p_1}{p_2 - p_1}p_2 + (1 - \lambda)p''}{r_2} = p_2.$$

**Case a5:**  $p' = p_1$  and  $p'' < p_2$ .

This case is similar to Case a3 and we omit the proof.

**Case a6:**  $p' < p_1$  and  $p'' < p_2$ .

This case is similar to Case a4 and we omit the proof.

**Case b.** We now turn to the case of  $V_1(p) < V_2(p)$ .

We let  $(x, q, \check{p})$  denote the solution to Problem 2 when the initial belief is  $p$ . Recall that

$$V(p) = \frac{p}{\check{p}}[(1 - \delta)(v(q) - x) + \delta V(\check{p})],$$

where  $\frac{p}{\check{p}}$  denotes the probability that the worker accepts the contract  $(x, q)$ . Also, recall that with probability  $1 - \frac{p}{\check{p}}$  the worker quits the relationship.

Similarly to Case a, we associate to  $p'$  and  $p''$  two triples,  $(x', q', \check{p}')$  and  $(x'', q'', \check{p}'')$ , which satisfy the constraints of Problem 2. We show that when the firm uses these triples, its current payoff is at least  $\frac{p}{\check{p}}(v(q) - x)$ . Furthermore, the worker quits the relationship with probability  $1 - \frac{p}{\check{p}}$ . Finally, the firm's posterior belief becomes  $\check{p}_1$  with probability  $\check{r}_1$  and  $\check{p}_2$  with probability  $\check{r}_2$  ( $\check{p}_1$  and  $\check{p}_2$  are not necessarily distinct). We show that the values  $\check{p}_1, \check{p}_2, \check{r}_1$ , and  $\check{r}_2$  are such that

$$\frac{\check{r}_1\check{p}_1 + \check{r}_2\check{p}_2}{\check{r}_1 + \check{r}_2} = \frac{\check{r}_1\check{p}_1 + \check{r}_2\check{p}_2}{\frac{p}{\check{p}}} = \check{p}.$$

As in Case a, the triples that we associate to  $p'$  and  $p''$ , depend both on the solution  $(x, q, \check{p})$  (at  $p$ ) and the comparison between  $p''$  and  $\check{p}$ . Below we deal with all possible cases.

**Case b1:**  $(x, q, \check{p}) = (\theta_H q_H^* + \alpha, q_H^*, p)$ .

In this case, we associate  $(\theta_H q_H^* + \alpha, q_H^*, p')$  to  $p'$  and  $(\theta_H q_H^* + \alpha, q_H^*, p'')$  to  $p''$ . The firm's current payoff is equal to

$$\lambda (v(q_H^*) - \theta_H q_H^* - \alpha) + (1 - \lambda) (v(q_H^*) - \theta_H q_H^* - \alpha) = \frac{p}{\check{p}} (v(q) - x).$$

The firm's posterior belief is equal to  $p'$  with probability  $\lambda$  and equal to  $p''$  with probability  $1 - \lambda$ .

**Case b2:**  $(x, q, \check{p}) = (\theta_H q_H^* + \alpha, q_H^*, \check{p})$  with  $p < \check{p} \leq p''$ .

Let  $\hat{p} \in (p', \check{p})$  be the solution to the following equation:

$$\lambda \left(1 - \frac{p'}{\hat{p}}\right) = 1 - \frac{p}{\check{p}}.$$

We associate the triple  $(\theta_H q_H^* + \alpha, q_H^*, \hat{p})$  to  $p'$  and  $(\theta_H q_H^* + \alpha, q_H^*, p'')$  to  $p''$ . The corresponding current payoff is equal to

$$\left(1 - \lambda \left(1 - \frac{p'}{\hat{p}}\right)\right) (v(q_H^*) - \theta_H q_H^* - \alpha) = \frac{p}{\check{p}} (v(q) - x).$$

The firm's posterior belief is equal to  $\hat{p}$  with probability  $\lambda \frac{p'}{\hat{p}}$  and equal to  $p''$  with probability  $1 - \lambda$ . It is easy to check that

$$\frac{\left(\lambda \frac{p'}{\hat{p}}\right) \hat{p} + (1 - \lambda) p''}{\left(\lambda \frac{p'}{\hat{p}}\right) + (1 - \lambda)} = \check{p}.$$

**Case b3:**  $(x, q, \check{p}) = (\theta_H q_H^* + \alpha, q_H^*, \check{p})$  with  $p'' < \check{p} < 1$ .

In this case, we associate the same triple  $(\theta_H q_H^* + \alpha, q_H^*, \check{p})$  to  $p'$  and  $p''$ . The current payoff is equal to

$$\left(\lambda \frac{p'}{\check{p}} + (1 - \lambda) \frac{p''}{\check{p}}\right) (v(q_H^*) - \theta_H q_H^* - \alpha) = \frac{p}{\check{p}} (v(q) - x).$$

Moreover, the firm's posterior belief is equal to  $\check{p}$  with probability  $\frac{p}{\check{p}}$ .

**Case b4:**  $(x, q, \check{p}) = (\theta_L q_L^* + \alpha, q_L^*, 1)$ .

Again, we associate the same triple  $(\theta_L q_L^* + \alpha, q_L^*, 1)$  to  $p'$  and  $p''$ . The current payoff is equal to

$$(\lambda p' + (1 - \lambda) p'') (v(q_L^*) - \theta_L q_L^* - \alpha) = \frac{p}{\check{p}} (v(q) - x),$$

and the posterior belief is equal to one with probability  $p$ . This concludes the proof of Lemma 2. ■

### Proof of Lemma 3.

Fix  $p \in (0, 1)$  and suppose that there exists  $\check{p} \in (p, 1)$  such that

$$V(p) = \frac{p}{\check{p}} (1 - \delta) (v(q_H^*) - \theta_H - \alpha) + \delta V(\check{p}).$$

Recall the definition of the function  $g : [p, 1] \rightarrow \mathbb{R}_+$  in equation (2). Our contradiction hypothesis implies that  $\check{p}$  is a maximizer of  $g$ . Therefore, the following condition must necessarily hold:

$$\delta \xi \check{p} - (1 - \delta) (v(q_H^*) - \theta_H - \alpha) - \delta V(\check{p}) = 0 \quad (3)$$

for some  $\xi \in \partial V(\check{p})$ , where  $\partial V(\check{p})$  is the subdifferential of  $V$  at  $\check{p}$ .<sup>20</sup>

It follows from the generalized version of the mean value theorem that there exist  $p' \in (\check{p}, 1)$  and  $\xi' \in \partial V(\check{p})$  such that

$$g(1) - g(\check{p}) = \frac{p(1-\check{p})[\delta \xi' p' - (1-\delta)(v(q_H^*) - \theta_H - \alpha) - \delta V(p')]}{(p')^2} = \frac{p(1-\check{p})\delta[\xi' p' - \xi \check{p} + V(\check{p}) - V(p')]}{(p')^2}, \quad (4)$$

where the last equality follows from equation (3).

The function  $V$  is convex and, therefore, absolutely continuous on the interval  $[p, 1]$ .<sup>21</sup> Thus, we can write

$$V(p') - V(\check{p}) = \int_{\check{p}}^{p'} \varphi(s) ds, \quad (5)$$

where  $\varphi$  is a non-decreasing function and  $\varphi(s) \leq \xi'$  for every  $s < p'$ . Substituting equation

<sup>20</sup>For every  $p'$ , the subdifferential  $\partial V(p')$  is non-empty since  $V$  is convex.

<sup>21</sup> $V$  is bounded above by the payoff of the commitment solution, and this payoff is equal to  $p'(v(q_L^*) - \theta_L q_L^* - \alpha)$  if the belief  $p'$  is above a certain threshold  $p^C \in (0, 1)$ . Thus,  $V(p') = p'(v(q_L^*) - \theta_L q_L^* - \alpha)$  for  $p' > p^C$ .

(5) into equation (4) yields

$$g(1) - g(\check{p}) = \frac{p(1-\check{p})\delta \left[ \xi' p' - \xi \check{p} - \int_{\check{p}}^{p'} \varphi(s) ds \right]}{(p')^2} \geq$$

$$\frac{p(1-\check{p})\delta \left[ \xi' p' - \xi \check{p} - \int_{\check{p}}^{p'} \varphi(s) ds \right]}{(p')^2} = \frac{p(1-\check{p})\delta \left[ \int_{\check{p}}^{p'} (\xi' - \varphi(s)) ds \right]}{(p')^2} \geq 0.$$

We have therefore derived the following contradiction:

$$V(p) = g(\check{p}) \leq g(1) < p(v(q_L^*) - \theta_L q_L^* - \alpha),$$

which concludes the proof of Lemma 3. ■

#### Proof of Lemma 4.

By contradiction, suppose that there exists  $p' > \bar{p}$  such that  $V(p') = V_1(p')$ . Let  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$ , with  $p_1 < p' < p_2$ , denote the solution to Problem 1 (recall that  $x_1 = \theta_H q_1 + \alpha$ ). Fix  $\varepsilon \in (0, p' - \max\{p_1, \bar{p}\})$ . For every  $p \in (p' - \varepsilon, p_2)$ , let

$$L(p) := \left( \frac{p_2 - p}{p_2 - p_1} \right) [(1 - \delta)(v(q_1) - \theta_H q_1 - \alpha) + \delta V(p_1)] +$$

$$\left( \frac{p - p_1}{p_2 - p_1} \right) [(1 - \delta)(v(q_2) - x_2) + \delta V(p_2)]$$

denote the firm's payoff at  $p$  when it offers the surely accepted menu  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$ .

Note that both  $L$  and  $V$  are linear on the interval  $(p' - \varepsilon, p_2)$  and  $L(p') = V(p')$ . This implies that  $L(p) = V(p)$  for every  $p \in (p' - \varepsilon, p_2)$  (otherwise, we would have  $L(p) > V(p)$  for some  $p$  close to  $p'$ ).

We now distinguish between the case  $p_2 < 1$  and the case  $p_2 = 1$ .

Consider  $p_2 < 1$  and notice that this implies  $x_2 = \theta_H q_2 + \alpha$ . If we take the limit of  $L$  as  $p$  converges to  $p_2$  (from below), we get

$$\lim_{p \uparrow p_2} L(p) = (1 - \delta)(v(q_2) - \theta_H q_2 - \alpha) + \delta V(p_2) < V(p_2),$$

since  $V(p_2) = p_2(v(q_L^*) - \theta_L q_L^* - \alpha)$  and  $p_2 > \hat{p}$ . But then  $L$  and  $V$  cannot coincide for values of  $p$  close to  $p_2$ .

Consider now  $p_2 = 1$ , and recall that the low type's rent is equal to zero when the firm's belief is equal to one (thus  $x_2 \geq \theta_L q_2 + \alpha$ ). This immediately implies  $v(q_2) - x_2 = v(q_L^*) - \theta_L q_L^* - \alpha$ . In fact, if this equality is violated, then  $\lim_{p \uparrow 1} L(p) < V(1)$  and we get the same contradiction as in the case above.



Suppose that when the belief is  $p'$  the firm offers the surely accepted menu  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$ , with  $p_1 < p' < p_2 = 1$ . If the low type accepts the contract  $(x_2, q_2) = (\theta_L q_L^* + \alpha, q_L^*)$ , his total rent is equal to zero. Then  $(x_1, q_1)$  must be equal to  $(\alpha, 0)$  and the high type will never produce a positive quality in the future. However, this implies that  $L(p')$  is strictly smaller than  $V(p') = p'(v(q_L^*) - \theta_L q_L^* - \alpha)$ . ■

**Proof of Lemma 5.**

Fix  $p$  such that  $V(p) = V_1(p) > v(q_H^*) - \theta_H q_H^* - \alpha$  and let  $\{(x_i, q_i, p_i, v_i)\}_{i=1,2}$  (with  $p_1 < p < p_2$ ) denote the corresponding solution to Problem 1. Recall that  $r_1 = \frac{p_2 - p}{p_2 - p_1}$  and  $r_2 = \frac{p - p_1}{p_2 - p_1}$ .

First, we assume that  $p_2 < 1$ . In this case, for  $i = 1, 2$ ,  $v(q_i) - x_i \leq v(q_H^*) - \theta_H q_H^* - \alpha$  since  $x_i = \theta_H q_i + \alpha$ . Then, it follows from

$$V(p) = \sum_{i=1}^2 r_i [(1 - \delta)(v(q_i) - \theta_H q_i - \alpha) + \delta V(p_i)]$$

that

$$\begin{aligned} (1 - \delta)[V(p) - (v(q_H^*) - \theta_H q_H^* - \alpha)] = \\ \sum_{i=1}^2 r_i \{(1 - \delta)[(v(q_i) - \theta_H q_i - \alpha) - (v(q_H^*) - \theta_H q_H^* - \alpha)] + \delta[V(p_i) - V(p)]\} \leq \\ \sum_{i=1}^2 r_i \delta [V(p_i) - V(p)]. \end{aligned}$$

The function  $V$  is non-decreasing, since it is convex and bounded below by the non-decreasing function  $\hat{V}$ , and  $V(0) = \hat{V}(0)$ . Thus,  $V(p_1) \leq V(p)$ . Also,  $v(q_L^*) - \theta_L q_L^* - \alpha$  is the largest subgradient of  $V$  and, thus,  $V(p_2) - V(p) \leq (p_2 - p)(v(q_L^*) - \theta_L q_L^* - \alpha)$ . These facts imply

$$\frac{(1 - \delta)[V(p) - (v(q_H^*) - \theta_H q_H^* - \alpha)]}{\delta(v(q_L^*) - \theta_L q_L^* - \alpha)} \leq r_2(p_2 - p) = r_1(p - p_1),$$

where the equality follows from the martingale property of the beliefs. For the case  $p_2 < 1$ , we conclude that

$$\min\{p - p_1, p_2 - p\} \geq \frac{(1 - \delta)[V(p) - (v(q_H^*) - \theta_H q_H^* - \alpha)]}{(v(q_L^*) - \theta_L q_L^* - \alpha)}.$$

Suppose now that  $p_2 = 1$  and notice that  $p \leq \bar{p} \leq p^C$  since  $V(p) = V(p_1)$ . The firm's payoff  $V(p)$  is bounded above by

$$r_1 [(1 - \delta)(v(q_1) - x_1) + \delta V(p)] + r_2 (v(q_L^*) - \theta_L q_L^* - \alpha),$$

where we used the fact  $V(p_1) \leq V(p)$  since  $V$  is non-decreasing. This, in turn, implies

$$\begin{aligned} & (1 - \delta) [V(p) - (v(q_H^*) - \theta_H q_H^* - \alpha)] \leq \\ & r_1 (1 - \delta) [(v(q_1) - x_1) - (v(q_H^*) - \theta_H q_H^* - \alpha)] + r_2 (v(q_L^*) - \theta_L q_L^* - \alpha) \leq \\ & \frac{r_2 (v(q_L^*) - \theta_L q_L^* - \alpha)(1 - p)}{1 - p^C}, \end{aligned}$$

where the last inequality follows from  $p \leq p^C$  and from the fact that  $v(q_1) - x_1 \leq v(q_H^*) - \theta_H q_H^* - \alpha$  since  $x_1 = \theta_H q_1 + \alpha$ . Thus, we have

$$\frac{(1 - p^C)(1 - \delta) [V(p) - (v(q_H^*) - \theta_H q_H^* - \alpha)]}{(v(q_L^*) - \theta_L q_L^* - \alpha)} \leq r_2 (1 - p) = r_2 (p_2 - p) = r_1 (p - p_1),$$

which concludes the proof. ■

## Appendix B: Limit Results

### Proof of Theorem 2.

To simplify the notation, throughout the proof we let  $\pi_L^* := v(q_L^*) - \theta_L q_L^* - \alpha$  denote the firm's payoff when the low-cost worker produces the efficient quality  $q_L^*$ . Also, notice that  $\pi_L^*$  is the largest subgradient of the value function. Furthermore, we define

$$M := \frac{(1 - p^C) [(\frac{1}{2}\bar{p} + \frac{1}{2}\hat{p}) \pi_L^* - (v(q_H^*) - \theta_H q_H^* - \alpha)]}{\pi_L^*}. \quad (6)$$

The proof is by contradiction. By taking a subsequence if necessary, assume that there exist a sequence  $\{\delta_n, V_n, \Phi_n, \bar{p}_n\}_{n=1, \dots}$  and  $\bar{p} > \hat{p}$  such that  $\{\delta_n\}_{n=1, \dots}$  converges to one,  $(V_n, \Phi_n)$  is an optimal pair given  $\delta_n$ , and

$$\lim_{n \rightarrow \infty} \bar{p}_n = \bar{p} > \hat{p}.$$

Recall that  $\bar{p}_n \in [\hat{p}, p^C]$  denotes the smallest belief  $p$  for which  $V_n(p) = p(v(q_L^*) - \theta_L q_L^* - \alpha)$ . Without loss of generality, we assume that  $\bar{p}_n > \frac{3}{4}\bar{p} + \frac{1}{4}\hat{p}$  for every  $n$ .

For every  $n$  and  $p \in [0, 1]$ , we define  $\phi^n(p)$  to be equal to

$$\phi_n(p) := \inf \{u \in \Phi_n(p') : p' \leq p\}.$$

**Claim 1** *We have*

$$\liminf_{n \rightarrow \infty} \phi_n \left( \frac{1}{2} \bar{p} + \frac{1}{2} \hat{p} \right) > 0.$$

**Proof of Claim 1.**

Consider the optimal pair  $(V_n, \Phi_n)$  and suppose that the prior is equal to  $p$ . To every  $u \in \Phi_n(p)$  there corresponds a (random) sequence of solutions to the firm's optimization problem which yields the expected rent  $u$  to the low type. This defines a probability distribution  $\Pr_n(\cdot | (p, u))$  over the firm's belief  $p^t$  in period  $t = 1, 2, \dots$ . Of course, the martingale property of the beliefs implies that  $\mathbb{E}(p^t | (p, u)) = p$  for every  $t$ .

Recall that for every  $n$ , the firm never chooses a firing menu when the belief is smaller than  $\bar{p}_n$  (and  $\bar{p}_n > \frac{3}{4}\bar{p} + \frac{1}{4}\hat{p}$ ). Using this fact, it is easy to see that if Claim 1 is false, then there exist  $\bar{n}$  and  $v > 0$  such that for every  $n > \bar{n}$  the following two requirements are satisfied:

i)

$$\Lambda_n := \left\{ (p_n, u_n) : p_n \leq \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} \text{ and } u_n \in \Phi_n(p_n) \cap \left[ 0, \frac{1}{2}\Delta\theta q_H^* \right] \right\} \neq \emptyset;$$

ii) For any pair  $(p_n, u_n) \in \Lambda_n$  there exists a positive integer  $T(p_n, u_n)$  such that

$$\min \left\{ (\delta_n)^{T(p_n, u_n)}, \Pr_n \left( p^{T(p_n, u_n)} \in \left[ \frac{3}{4}\bar{p} + \frac{1}{4}\hat{p}, 1 \right] \mid (p_n, u_n) \right) \right\} \geq v.$$

Consider  $n > \bar{n}$  and take a pair  $(p_n, u_n) \in \Lambda_n$ . We define

$$z_n := \Pr_n \left( p^{T(p_n, u_n)} \in \left[ 0, p_n - v \left( \frac{\bar{p} - \hat{p}}{8} \right) \right] \mid (p_n, u_n) \right).$$

We now derive a lower bound to  $z_n$ . It follows from the martingale property of the beliefs that

$$p_n \geq v \left( \frac{3}{4}\bar{p} + \frac{1}{4}\hat{p} \right) + (1 - v - z_n) \left[ p_n - v \left( \frac{\bar{p} - \hat{p}}{8} \right) \right].$$

This and the fact that  $p_n \leq \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p}$  imply

$$z_n > \frac{v(\bar{p} - \hat{p})}{4(\bar{p} + \hat{p})}.$$

Fix the prior  $p_n$  and consider the low type who faces the sequence of menus which yields the expected rent  $u_n$ . The low type can deviate and mimic the behavior of the high type

with probability  $1 - p_n$  (with probability  $p_n$ , the low type follows his own behavior). This implies that

$$u_n \geq z_n (\delta_n)^{T(p_n, u_n)} \phi_n \left( p_n - v \left( \frac{\bar{p} - \hat{p}}{8} \right) \right) > \frac{v^2 (\bar{p} - \hat{p})}{4(\bar{p} + \hat{p})} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} - v \left( \frac{\bar{p} - \hat{p}}{8} \right) \right).$$

Notice that the above inequality holds for any pair  $(p_n, u_n) \in \Lambda_n$ . Thus, we have

$$\liminf_{n \rightarrow \infty} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} \right) \geq \frac{v^2 (\bar{p} - \hat{p})}{4(\bar{p} + \hat{p})} \liminf_{n \rightarrow \infty} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} - v \left( \frac{\bar{p} - \hat{p}}{8} \right) \right).$$

We now proceed inductively. Either  $\liminf_{n \rightarrow \infty} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} - v \left( \frac{\bar{p} - \hat{p}}{8} \right) \right) > 0$  or we apply the same argument once again. However, since the belief is non-negative we can apply the argument only finitely many times. This shows that  $\liminf_{n \rightarrow \infty} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} \right) > 0$ . ■

Notice that for every  $n$  and every  $p \in \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p}, \bar{p}_n \right)$ , the solution to the firm's optimization problem is a surely accepted menu, which we denote by  $\{(x_{i,n}(p), q_{i,n}(p), p_{i,n}(p), v_{i,n}(p))\}_{i=1,2}$ .

Recall the definition of  $M$  in equation (6), and note that  $M$  is strictly positive. For each  $n$ , fix  $p_n^0 \in \left( \max \left\{ \frac{3}{4}\bar{p} + \frac{1}{4}\hat{p}, \bar{p}_n - (1 - \delta_n)M \right\}, \bar{p}_n \right)$  and construct the sequence  $\{(x_n^t, q_n^t, p_n^t, v_n^t)\}_{t=1, \dots, T_n}$  by letting

$$(x_n^t, q_n^t, p_n^t, v_n^t) = (x_{1,n}(p_n^{t-1}), q_{1,n}(p_n^{t-1}), p_{1,n}(p_n^{t-1}), v_{1,n}(p_n^{t-1})),$$

for  $t = 1, \dots, T_n$  and choosing  $T_n$  such that

$$p_n^{T_n} \leq \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} < p_n^{T_n-1}.$$

Note that  $T_n$  is finite for every  $n$ , since

$$p_n^t - p_n^{t-1} \geq (1 - \delta_n)M, \tag{7}$$

for every  $t = 1, \dots, T_n$  (see Lemma 5).

Therefore, for every  $n = 1, \dots$ , we have

$$\begin{aligned} (1 - \delta_n) (x_{2,n}(p_n^0) - \theta_L q_{2,n}(p_n^0) - \alpha) + \delta_n v_{2,n}(p_n^0) &\geq \\ (1 - \delta_n) \sum_{t=1}^{T_n} (\delta_n)^{t-1} (x_n^t - \theta_L q_n^t - \alpha) + (\delta_n)^{T_n} v_n^{T_n}. &\end{aligned} \tag{8}$$

The left-hand side of the above inequality denotes the low type's rent if he accepts the contract  $(x_{2,n}(p_n^0), q_{2,n}(p_n^0))$  when the belief is equal to  $p_n^0$ . The right-hand side instead

represents the low type's rent if he accepts the contract  $(x_{1,n}(p_n^{t-1}), q_{1,n}(p_n^{t-1}))$  - that is, the contract that moves the belief down, in period  $t = 1, \dots, T_n$ .

It follows from Lemma 5 that

$$p_{2,n}(p_n^0) \geq p_n^0 + (1 - \delta_n)M > \bar{p}_n,$$

which implies  $v_{2,n}(p_n^0) = 0$  since  $V_n(p_{2,n}(p_n^0)) = p_{2,n}(p_n^0)\pi_L^*$ . Also,  $x_{2,n}(p_n^0) \leq \theta_H q_{2,n}(p_n^0) + \alpha$ . We conclude that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} [(1 - \delta_n)(x_{2,n}(p_n^0) - \theta_L q_{2,n}(p_n^0) - \alpha) + \delta_n v_{2,n}(p_n^0)] = \\ & \limsup_{n \rightarrow \infty} [(1 - \delta_n)(x_{2,n}(p_n^0) - \theta_L q_{2,n}(p_n^0) - \alpha)] \leq \lim_{n \rightarrow \infty} (1 - \delta_n) \Delta\theta = 0. \end{aligned} \quad (9)$$

We now provide a lower bound to the right-hand side of inequality (8). First, for every  $n$  and every  $t = 1, \dots, T_n$ , we have  $x_n^t = \theta_H q_n^t + \alpha$ , and, thus,  $x_n^t - \theta_L q_n^t - \alpha \geq 0$ .

Second, it follows from inequality (7) that

$$T_n \leq \frac{(1 - \frac{1}{2}\bar{p} - \frac{1}{2}\hat{p})}{(1 - \delta_n)M},$$

for every  $n$ . Putting together these observations we get

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left[ (1 - \delta_n) \sum_{t=1}^{T_n} (\delta_n)^{t-1} (x_n^t - \theta_L q_n^t - \alpha) + (\delta_n)^{T_n} v_n^{T_n} \right] \geq \\ & \liminf_{n \rightarrow \infty} \left( (\delta_n)^{T_n} v_n^{T_n} \right) \geq \liminf_{n \rightarrow \infty} \left( (\delta_n)^{\left\lceil \frac{(1 - \frac{1}{2}\bar{p} - \frac{1}{2}\hat{p})}{(1 - \delta_n)M} \right\rceil} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} \right) \right) = \\ & \lim_{n \rightarrow \infty} \left( (\delta_n)^{\left\lceil \frac{(1 - \frac{1}{2}\bar{p} - \frac{1}{2}\hat{p})}{(1 - \delta_n)M} \right\rceil} \right) \liminf_{n \rightarrow \infty} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} \right) \\ & = e^{-\left\lceil \frac{(1 - \frac{1}{2}\bar{p} - \frac{1}{2}\hat{p})}{M} \right\rceil} \liminf_{n \rightarrow \infty} \phi_n \left( \frac{1}{2}\bar{p} + \frac{1}{2}\hat{p} \right) > 0. \end{aligned} \quad (10)$$

Equations (9) and (10) imply that inequality (8) cannot hold for large values of  $n$ , which establishes the desired contradiction. ■

### Proof of Theorem 3.

Given a path  $\omega$  of the game, we let  $(x_t(\omega), q_t(\omega))$  denote the contract accepted by the worker in every period  $t$  in which he is employed. We also let  $h^t(\omega)$  denote the period- $t$  public history associated to  $\omega$ .

Consider a sequence  $\{\delta_n\}_{n=1,\dots}$  of discount factors converging to one, and let  $(\sigma_n, \mu_n)$  be a ratchet equilibrium when the discount factor is equal to  $\delta_n$ .<sup>22</sup> For every  $n$ , we let  $(V_n, \Phi_n)$  denote the optimal pair associated to the equilibrium  $(\sigma_n, \mu_n)$ , and  $\bar{p}_n$  denote the smallest value of  $p$  for which  $V_n(p) = p(v(q_L^*) - \theta_L q_L^* - \alpha)$ .

For every  $n$  and for every path  $\omega$ , define the set  $S_n(\omega)$  as follows:

$$S_n(\omega) := \left\{ t : \mu_n(h^t(\omega)) < \bar{p}_n \text{ and there exists } (m_t, a_t) \text{ such that } \sigma_{n,t}^F(m_t|h^t(\omega)) > 0, \right. \\ \left. \sigma_{n,t}^L(a_t|h^t(\omega), m_t) > 0 \text{ and } \mu_n((h^t(\omega), (m_t, a_t))) \geq \bar{p}_n \right\},$$

where  $\mu_n(h^t(\omega))$  denotes the firm's belief at the history  $h^t(\omega)$  under the equilibrium  $(\sigma_n, \mu_n)$ . In words,  $t$  belongs to  $S_n(\omega)$  if the firm's belief at the beginning of the period (i.e., after the history  $h^t(\omega)$ ) is smaller than  $\bar{p}_n$ , and the menu offered by the firm (under the equilibrium  $(\sigma_n, \mu_n)$ ) contains a contract that, if accepted, moves the belief above  $\bar{p}_n$ . Note that the fact that  $t \in S_n(\omega)$  does not imply that  $\mu_n(h^{t+1}(\omega)) \geq \bar{p}_n$ , as the path  $\omega$  may specify that the worker accepts the contract that moves the belief down, below  $\mu_n(h^t(\omega))$ .

We set  $T_n(\omega) = \min_{t \in S_n(\omega)} t$  if  $S_n(\omega) \neq \emptyset$ , and  $T_n(\omega) = \infty$  if  $S_n(\omega) = \emptyset$ . Finally, we set  $\delta_n^\infty$  equal to zero, and for each  $\eta \in (0, 1)$  define

$$T_n^\eta(\omega) := \begin{cases} T_n(\omega) & \text{if } \delta_n^{T_n(\omega)} \geq \eta, \\ \max_{\{t: \delta_n^t \geq \eta\}} t & \text{if } \delta_n^{T_n(\omega)} < \eta. \end{cases}$$

We also let  $T_n$  and  $T_n^\eta$  denote two random variables, which take the values  $T_n(\omega)$  and  $T_n^\eta(\omega)$ , respectively.

Fix the equilibrium  $(\sigma_n, \mu_n)$ . The expected rent  $W_L(h^0; (\sigma_n, \mu_n))$  of the efficient worker can be written as

$$W_L(h^0; (\sigma_n, \mu_n)) = \mathbb{E}_{(\sigma_n, \mu_n)} \left[ (1 - \delta_n) \sum_{t=0}^{T_n^\eta(\omega)-1} \delta_n^t \Delta \theta q_t(\omega) + \delta_n^{T_n^\eta(\omega)} W_L(h^{T_n^\eta(\omega)}(\omega); (\sigma_n, \mu_n)) \mid L \right].$$

The following claim is useful in the rest of the proof.

---

<sup>22</sup>For every  $n$ , we denote the firm's strategy by  $\sigma_n^F = \{\sigma_{n,t}^F\}$  and the worker's strategy by  $(\sigma_n^L, \sigma_n^H) = \{(\sigma_{n,t}^L, \sigma_{n,t}^H)\}$ .

**Claim 2** For every  $\eta \in (0, 1)$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_{(\sigma_n, \mu_n)} \left[ \left( (1 - \delta_n) \sum_{t=0}^{T_n^\eta(\omega)-1} \delta_n^t \Delta \theta |q_t(\omega) - q_H^*| \right) |L \right] = 0.$$

**Proof of Claim 2.**

By removing elements of the sequence if necessary, assume that there exists  $\varepsilon > 0$  such that

$$\mathbb{E}_{(\sigma_n, \mu_n)} \left[ \left( (1 - \delta_n) \sum_{t=0}^{T_n^\eta(\omega)-1} \delta_n^t \Delta \theta |q_t(\omega) - q_H^*| \right) |L \right] \geq \varepsilon$$

for every  $n$ . Notice that  $x_t(\omega) = \theta_H q_t(\omega) + \alpha$  for every  $\omega$  and for every  $t \leq T_n^\eta(\omega) - 1$ . This and the fact that the function  $v$  is strictly concave imply that there exists  $\hat{\varepsilon} > 0$  such that for every  $n$  we have

$$\begin{aligned} \mathbb{E}_{(\sigma_n, \mu_n)} \left[ \left( (1 - \delta_n) \sum_{t=0}^{T_n^\eta(\omega)-1} \delta_n^t (v(q_t(\omega)) - x_t(\omega)) \right) |L \right] \leq \\ (1 - \delta_n) \sum_{t=0}^{T_n^\eta(\omega)-1} \delta_n^t (v(q_H^*) - \theta_H q_H^* - \alpha) - \hat{\varepsilon}. \end{aligned}$$

Then it follows that for every  $n$ , the firm's expected payoff  $V_F(h^0; (\sigma_n, \mu_n))$  is bounded above by

$$\mathbb{E}_{(\sigma_n, \mu_n)} \left[ (1 - \delta_n) \sum_{t=0}^{T_n^\eta(\omega)-1} \delta_n^t (v(q_H^*) - \theta_H q_H^* - \alpha) + \delta_n^{T_n^\eta(\omega)} V_F(h^{T_n^\eta(\omega)}(\omega); (\sigma_n, \mu_n)) \right] - p_0 \hat{\varepsilon}.$$

It follows from  $\lim_{n \rightarrow \infty} \bar{p}_n = \hat{p}$  that

$$\lim_{n \rightarrow \infty} \mathbb{E}_{(\sigma_n, \mu_n)} \left[ V_F(h^{T_n^\eta(\omega)}(\omega); (\sigma_n, \mu_n)) \right] = v(q_H^*) - \theta_H q_H^* - \alpha.$$

Finally, this implies that for some  $n$  sufficiently large

$$V_F(h^0; (\sigma_n, \mu_n)) < v(q_H^*) - \theta_H q_H^* - \alpha,$$

which leads to a contradiction and proves Claim 2. ■

Fix  $\eta \in (0, 1)$ . Of course, for every  $\omega$  such that  $T_n^\eta(\omega) = T_n(\omega)$ , we have

$$W_L(h^t(\omega); (\sigma_n, \mu_n)) \leq (1 - \delta_n) \Delta\theta. \quad (11)$$

This implies that for every  $\varepsilon > 0$ , there exists  $n'$  such that for every  $n > n'$  and every  $\omega$  with  $T_n^\eta(\omega) = T_n(\omega)$ , we have

$$\mathbb{E}_{(\sigma_n, \mu_n)} \left[ \left[ (1 - \delta_n) \sum_{t=T_n^\eta(\omega)}^{\mathcal{T}} \delta_n^{t-T_n^\eta(\omega)} (v(q_t) - x_t) \right] | H, h^t(\omega) \right] < \varepsilon,$$

where  $\mathcal{T}$  denotes the period in which the relationship ends. The firm's continuation payoff at  $h^t(\omega)$  conditional on the type being high must be close to zero (when  $n$  is large). Intuitively, if this is not the case, then it is not rational for the low-cost worker to accept at  $h^t(\omega)$  the contract that moves the belief above  $\bar{p}_n$ .

This, in turn, implies the following claim.

**Claim 3** *Fix  $\eta \in (0, 1)$ . For every  $\varepsilon > 0$ , there exists  $n''$  such that for every  $n > n''$  and every  $\omega$  with  $T_n^\eta(\omega) = T_n(\omega)$ , we have*

$$\hat{p} - \varepsilon < \mu_n(h^t(\omega)) < \hat{p} + \varepsilon.$$

We are now ready to conclude the proof of Theorem 3. Given Claim 2, assume towards a contradiction that there exist  $\varrho > 0$  and  $\eta' > 0$  such that for every  $\eta \in (0, \eta')$ , the following inequality holds:

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{(\sigma_n, \mu_n)} \left[ \mathbf{1}_{\{T_n^\eta = T_n\}} \delta_n^{T_n^\eta} \right] \geq \varrho,$$

where  $\mathbf{1}_{\{T_n^\eta = T_n\}}$  denotes the indicator function of the event  $T_n^\eta = T_n$ .

This implies that there exist  $\bar{\varrho} > 0$  and  $\eta'' > 0$  such that for every  $\eta \in (0, \eta'')$

$$\limsup_{n \rightarrow \infty} \max \{ \Pr_{n,L}(T_n^\eta = T_n), \Pr_{n,H}(T_n^\eta = T_n) \} \geq 2\bar{\varrho},$$

where  $\Pr_{n,i}(T_n^\eta = T_n)$  denotes the probability, under the equilibrium  $(\sigma_n, \mu_n)$ , that  $T_n^\eta = T_n$  when the worker's type is equal to  $i = L, H$ .

Recall that  $W_L(h^0; (\sigma_n, \mu_n))$  denotes the low-cost worker's expected rent under the equilibrium  $(\sigma_n, \mu_n)$ . We let  $\tilde{W}_L(h^0; (\sigma_n, \mu_n))$  denote the efficient worker's rent if he mimics the high type's behavior and follows the strategy  $\sigma_n^H$ .



Fix a positive  $\varepsilon < \varrho \bar{\varrho} \left(1 - \frac{p_0(1-\hat{p})}{\hat{p}(1-p_0)}\right) \Delta\theta q_H^*$ . It follows from Claim 2 and inequality (11) that there exist  $\bar{\eta} \in (0, \min(\eta', \eta''))$  and  $\hat{n}$  such that, for every  $n > \hat{n}$ , we have:

$$\begin{aligned} W_L(h^0; (\sigma_n, \mu_n)) &\leq \mathbb{E}_{(\sigma_n, \mu_n)} \left[ \mathbf{1}_{\{T_n^{\bar{\eta}} < T_n\}} \Delta\theta q_H^* | L \right] + \\ &\mathbb{E}_{(\sigma_n, \mu_n)} \left[ \mathbf{1}_{\{T_n^{\bar{\eta}} = T_n\}} (1 - \delta_n) \Delta\theta q_H^* \sum_{t=0}^{T_n^{\bar{\eta}}-1} \delta_n^t | L \right] + \frac{\varepsilon}{8}, \end{aligned}$$

and

$$\begin{aligned} \tilde{W}_L(h^0; (\sigma_n, \mu_n)) &\leq \mathbb{E}_{(\sigma_n, \mu_n)} \left[ \mathbf{1}_{\{T_n^{\bar{\eta}} < T_n\}} \Delta\theta q_H^* | H \right] + \\ &\mathbb{E}_{(\sigma_n, \mu_n)} \left[ \mathbf{1}_{\{T_n^{\bar{\eta}} = T_n\}} (1 - \delta_n) \Delta\theta q_H^* \sum_{t=0}^{T_n^{\bar{\eta}}-1} \delta_n^t | H \right] - \frac{\varepsilon}{8}. \end{aligned}$$

Next, it follows from Claim 3 that we can find  $\bar{n} \geq \hat{n}$  such that, for every  $n \geq \bar{n}$ , we have:

$$\begin{aligned} W_L(h^0; (\sigma_n, \mu_n)) &\leq (1 - \Pr_{n,L}(T_n^{\bar{\eta}} = T_n)) \Delta\theta q_H^* + \\ \Pr_{n,L}(T_n^{\bar{\eta}} = T_n) \mathbb{E}_{(\sigma_n, \mu_n)} &\left[ (1 - \delta_n) \Delta\theta q_H^* \sum_{t=0}^{T_n^{\bar{\eta}}-1} \delta_n^t | T_n^{\bar{\eta}} = T_n \right] + \frac{\varepsilon}{4}, \end{aligned}$$

and

$$\begin{aligned} \tilde{W}_L(h^0; (\sigma_n, \mu_n)) &\leq (1 - \Pr_{n,H}(T_n^{\bar{\eta}} = T_n)) \Delta\theta q_H^* + \\ \Pr_{n,H}(T_n^{\bar{\eta}} = T_n) \mathbb{E}_{(\sigma_n, \mu_n)} &\left[ (1 - \delta_n) \Delta\theta q_H^* \sum_{t=0}^{T_n^{\bar{\eta}}-1} \delta_n^t | T_n^{\bar{\eta}} = T_n \right] - \frac{\varepsilon}{4}, \end{aligned}$$

By taking a subsequence if necessary, we can assume that

$$\lim_{n \rightarrow \infty} \mathbb{E}_{(\sigma_n, \mu_n)} \left[ \delta_n^{T_n^{\bar{\eta}}} | T_n^{\bar{\eta}} = T_n \right] \geq \frac{\varrho}{2}.$$

We may also assume that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr_{n,L}(T_n^{\bar{\eta}} = T_n) &= P_L > 0 \\ \lim_{n \rightarrow \infty} \Pr_{n,H}(T_n^{\bar{\eta}} = T_n) &= P_H > 0, \end{aligned}$$

and  $\max\{P_L, P_H\} \geq \bar{\varrho}$ .

Finally, Claim 3 implies that

$$\frac{p_0 P_L}{p_0 P_L + (1 - p_0) P_H} = \hat{p} > p_0,$$

and, thus,

$$(P_L - P_H) = P_L \left( 1 - \frac{p_0(1 - \hat{p})}{\hat{p}(1 - p_0)} \right) \geq \bar{\varrho} \left( 1 - \frac{p_0(1 - \hat{p})}{\hat{p}(1 - p_0)} \right).$$

We conclude that

$$\lim_{n \rightarrow \infty} \tilde{W}_L(h^0; (\sigma_n, \mu_n)) - W_L(h^0; (\sigma_n, \mu_n)) \geq \frac{1}{2} \left( \bar{\varrho} \left( 1 - \frac{p_0(1 - \hat{p})}{\hat{p}(1 - p_0)} \right) \Delta \theta q_H^* - \varepsilon \right) > 0,$$

which shows that for  $n$  sufficiently large, the low type has an incentive to deviate and play the strategy  $\sigma_n^H$ . ■

## Appendix C: Proof of Theorem 1

**Proof of Theorem 1.** In this section, we construct a ratchet equilibrium  $(\sigma, \mu)$  for any  $\delta > \underline{\delta}$ .<sup>23</sup> Fix  $\delta$  and consider an arbitrary history  $h^t$ . The firm's equilibrium strategy  $\sigma^F$  is to offer the pooling menu  $\{(\theta_H q_H^* + \alpha, q_H^*)\}$  if  $\mu(h^t) < \hat{p}$ , and the firing menu  $\{(\theta_L q_L^* + \alpha, q_L^*)\}$  if  $\mu(h^t) > \hat{p}$ . The firm also offers  $\{(\theta_H q_H^* + \alpha, q_H^*)\}$  if  $\mu(h^t) = \hat{p}$  and the prior  $p_0$  is at most  $\hat{p}$ . We postpone the case  $\mu(h^t) = \hat{p} < p_0$  until we describe the worker's strategy and the system of beliefs.

The high-cost worker's equilibrium behavior is very simple. In every period, he chooses the contract in the menu that yields the largest current rent, provided that the rent is non-negative (if the menu contains two contracts and type  $H$  is indifferent between them, then he picks the first one). If all the contracts in the menu yield negative rents, then the high-cost worker quits the relationship.

The low-cost worker's equilibrium behavior and the way the equilibrium beliefs evolve depend on the firm's beginning-of-period belief  $\mu(h^t)$ . Depending on the history  $h^t$ , the belief  $\mu(h^t)$  can take the value of the prior  $p_0$ , the value one or the value  $\hat{p}$  (this cases arises only if  $p_0 \geq \hat{p}$ ). Below, we discuss all possible cases.

**Case 1:**  $\mu(h^t) = 1$ .

We assume that  $\mu(h^\tau) = 1$  for any  $\tau > t$  and for any (non-final) history  $h^\tau$  that follows  $h^t$  (in words, the firm never changes its belief once it becomes convinced that the worker is efficient). Notice that, given  $\sigma^F$ , this implies that the firm will offer the firing menu  $\{(\theta_L q_L^* + \alpha, q_L^*)\}$  after any history  $h^\tau$  that follows  $h^t$ .

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<sup>23</sup>The equilibrium does depend on the value of the discount factor. However, to simplify the notation, we suppress the dependence on  $\delta$  and write  $(\sigma, \mu)$  for any  $\delta > \underline{\delta}$ .

Suppose that the firm offers the menu  $m_t$ . Type  $L$  accepts the contracts that yields the largest current rent, provided that it is non-negative (if indifferent between two contracts, type  $L$  picks the first one). If all the contracts in the menu yields negative rents, then type  $L$  quits the relationship.

**Case 2:**  $\mu(h^t) \leq \hat{p}$ .

Let  $m_t$  denote the menu offered after the history  $h^t$ . We need to distinguish between two cases, depending on type  $H$ 's equilibrium behavior. First, assume that type  $H$  is required to accept the contract  $(x_i, q_i)$ . Then the low type's equilibrium strategy also prescribes acceptance of the contract  $(x_i, q_i)$ . The firm does not update its belief when the worker accepts  $(x_i, q_i)$  (formally,  $\mu(h^t, m_t, (x_i, q_i)) = \mu(h^t)$ ). On the other hand, if the worker remains in the relationship but does not accept  $(x_i, q_i)$ , then the firm's belief jumps to one (and the firm will offer the firing menu in all future periods). It is immediate to check that, since  $\delta > \underline{\delta}$ , it is indeed optimal for type  $L$  to accept  $(x_i, q_i)$ .

Second, suppose that type  $H$  is required to quit. Type  $L$  chooses the action that maximizes his current payoff. Finally, if the worker does not quit, the firm's belief becomes equal to one. Again, it is easy to check that the low type's behavior is optimal.

**Case 3:**  $\mu(h^t) = p_0 > \hat{p}$ .

Suppose that the firm offers the menu  $m_t$ , and type  $H$  is required to accept the contract  $(x_i, q_i)$ .<sup>24</sup> First, assume that either  $m_t = \{(x_i, q_i)\}$  or  $m_t = \{(x_i, q_i), (x_j, q_j)\}$  and  $x_i - \theta_L q_i \geq x_j - \theta_L q_j$ . In this case, type  $L$  is required to accept  $(x_i, q_i)$ . The firm's belief in period  $t + 1$  is equal to  $\mu(h^t) = p_0$  if the worker accepts  $(x_i, q_i)$ , and is equal to one otherwise.

Second, assume that  $m_t = \{(x_i, q_i), (x_j, q_j)\}$  and  $x_i - \theta_L q_i < x_j - \theta_L q_j$ . In this case, type  $L$  is required to accept the contract  $(x_i, q_i)$  with probability  $\frac{\hat{p}(1-p_0)}{p_0(1-\hat{p})}$  and the contract  $(x_j, q_j)$  with probability  $1 - \frac{\hat{p}(1-p_0)}{p_0(1-\hat{p})}$ . It is easy to check that the firm's belief in period  $t + 1$  is equal to  $\hat{p}$  if the worker accepts  $(x_i, q_i)$ . The firm's belief in period  $t + 1$  is equal to one if the worker accepts  $(x_j, q_j)$ .

Continue to assume that  $m_t = \{(x_i, q_i), (x_j, q_j)\}$  and  $x_i - \theta_L q_i < x_j - \theta_L q_j$ . The fact that  $x_i - \theta_H q_i \geq x_j - \theta_H q_j$  and  $\delta > \underline{\delta}$  implies that there exists  $\chi \in [0, 1]$  such that

$$(1 - \delta)(x_i - \theta_L q_i - \alpha) + \chi \delta \Delta \theta q_H^* = (1 - \delta)(x_j - \theta_L q_j - \alpha).$$

Then, in period  $t + 1$ , after the history  $(h^t, m_t, (x_i, q_i))$  the firm offers the pooling menu

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<sup>24</sup>Notice that this implies  $x_i - \theta_H q_i - \alpha \geq 0$ . Furthermore,  $x_i - \theta_H q_i \geq x_j - \theta_H q_j$  if the menu  $m_t$  contains another contract  $(x_j, q_j)$  (the inequality is strict if  $i = 2$  and  $j = 1$ ).

$\{(\theta_H q_H^* + \alpha, q_H^*)\}$  with probability  $\chi$  and the firing menu  $\{(\theta_L q_L^* + \alpha, q_L^*)\}$  with probability  $1 - \chi$ . Furthermore, the firm offers  $\{(\theta_H q_H^* + \alpha, q_H^*)\}$  (with probability one) after any history  $h^\tau$ ,  $\tau \geq t + 2$ , that follows  $(h^t, m_t, (x_i, q_i), \{(\theta_H q_H^* + \alpha, q_H^*)\})$  and such that  $\mu(h^\tau) = \hat{p}$ . Of course, the value  $\chi$  is appropriately chosen to make it optimal for type  $L$  to randomize between the two contracts in period  $t$ .

Finally, assume that  $m_t$  contains only contracts that yield negative rents to the high type (thus, in equilibrium type  $H$  is required to quit). In this case,  $L$  chooses the action that maximizes his current payoff. Again, if the worker does not quit the firm's belief jumps to one.

This completes the description of the strategy profile  $\sigma$  and the system of beliefs  $\mu$ . Given our analysis above, it is straightforward to check that  $(\sigma, \mu)$  is a perfect Bayesian equilibrium. It remains to show that  $(\sigma, \mu)$  is also a ratchet equilibrium.

Note that under the equilibrium  $(\sigma, \mu)$ , the low type's continuation rent is equal to  $\Delta\theta q_H^*$  if the belief is strictly smaller than  $\hat{p}$ . The continuation rent belongs to the interval  $[0, \Delta\theta q_H^*]$  if the belief is equal to  $\hat{p}$ . Finally, the continuation rent is equal to zero if the belief is strictly larger than  $\hat{p}$ . Thus, consider the correspondence  $\hat{\Phi} : [0, 1] \rightrightarrows \mathbb{R}_+$  defined by

$$\hat{\Phi}(p) = \begin{cases} \{\Delta\theta q_H^*\} & \text{if } p < \hat{p} \\ [0, \Delta\theta q_H^*] & \text{if } p = \hat{p} \\ \{\Delta\theta q_H^*\} & \text{if } p > \hat{p} \end{cases} .$$

Recall the definition of the function  $\hat{V}$  in equation (1). To complete the proof of theorem 1 it is enough to check that the pair  $(\hat{V}, \hat{\Phi})$  is optimal for any  $\delta > \underline{\delta}$ . To see that this is indeed the case, notice that given  $(\hat{V}, \hat{\Phi})$  and  $\delta > \underline{\delta}$ , for  $p \leq \hat{p}$  (respectively,  $p > \hat{p}$ ) there is no tuple  $\{x_i, q_i, p_i, v_i\}_{i=1,2}$  with  $p_2 > \hat{p}$  (respectively,  $p_1 < \hat{p}$ ) that satisfies the constraints of Problem 1 in Section 4.<sup>25</sup> ■

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<sup>25</sup>Also recall from Lemma 3 that it is never optimal for the firm to offer a partially separating menu.

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