Delegation with Endogenous States

Dino Gerardi  
(Collegio Carlo Alberto)

Lucas Maestri  
(FGV EPGE)

Ignacio Monzón  
(Collegio Carlo Alberto)

University of Bonn - October 23rd, 2019
Delegation problems are widespread:

- A party with authority to make a decision (*Principal*)
- must rely on a better informed party (*Agent*)
- Should the principal give flexibility to the agent, or instead restrict what the agent can choose?

Some examples:

- CEO selects feasible projects
  Manager (better informed about their profitability) chooses one
- Regulator restricts the prices that a monopolist (better informed about costs) can charge
Introduction

Moral hazard

Before choosing an action, agent can exert effort and affect outcomes

- Effort is typically unobservable
- Agent cannot fully control outcomes

Examples:

- Manager’s effort affects potential profits of various projects
- Monopolist can adopt practices that reduce production costs
Goal of the paper

- How can a principal incentivize the agent to both exert effort and choose appropriate actions?
  - Principal chooses a delegation set
  - Cares about effort and actions
- We characterize the optimal delegation set
  - With aligned and misaligned preferences
  - The optimal delegation set has a simple form: actions below a threshold are excluded
Closely related literature

- Delegation with misaligned preferences, no moral hazard:
  - Holmström (1977, 1984)
  - Alonso and Matouschek (2008)
  - Amador and Bagwell (2013)

- Delegation with Information Acquisition:
  - Szalay (2005)
  - Deimen and Szalay (2018)
The model with no bias.

**The model with no bias. Timing**

Principal selects a delegation set $A \subseteq \mathbb{R}$ ($A$ closed)

Agent exerts effort $e \in [0, \bar{e}]$ at cost $c(e)$

Given effort $e$, the state $\gamma$ is realized according to c.d.f. $F(\gamma, e)$

Agent observes the state $\gamma$ and chooses an action $a \in A$
Distribution of the state

The support of the state distribution is \( \Gamma = [\gamma, \bar{\gamma}] \)

For every \( e \in [0, \bar{e}] \) and every \( \gamma \in \Gamma, f(\gamma, e) > 0 \)

\( F(\cdot, \cdot) \) is smooth

\( F \) satisfies the (strict) monotone likelihood ratio property (MLRP):

\[
\frac{f(\gamma', e')}{f(\gamma, e')} > \frac{f(\gamma', e)}{f(\gamma, e)}
\]

for all \( e' > e \) and \( \gamma' > \gamma \)
The model with no bias

Payoffs

The parties’ payoffs are:

\[ U_P(a, \gamma, e) = u(a, \gamma) + v(e) \]
\[ U_A(a, \gamma, e) = u(a, \gamma) - c(e) \]

Assumptions

- \( v(\cdot) : [0, \bar{e}] \rightarrow \mathbb{R} \) is strictly increasing and strictly concave
- \( c(\cdot) : [0, \bar{e}] \rightarrow \mathbb{R} \) is strictly increasing and strictly convex
The model with no bias

- the common payoff component $u(\cdot, \cdot)$ is $C^2$ and satisfies
  - for every $\gamma \in [\underline{\gamma}, \bar{\gamma}]$, $u(\cdot, \gamma)$ is strictly quasiconcave in $a$ and
    $$\max_a u(a, \gamma) = u(a^*(\gamma), \gamma) = 0$$
  - **Limit condition**: for every $\gamma \in [\underline{\gamma}, \bar{\gamma}]$
    $$\lim_{a \to -\infty} u(a, \gamma) = \lim_{a \to +\infty} u(a, \gamma) = -\infty$$
  - **Single crossing condition**: for all $(a, \gamma) \in \mathbb{R} \times \Gamma$
    $$\frac{\partial u^2(a, \gamma)}{\partial \gamma \partial a} > 0$$
The model with no bias

Expected payoffs

Given a delegation set $A$ and an effort level $e$, the parties’ expected payoffs are:

$$V_P (A,e) = \mathbb{E} \left[ \max_{a \in A} u(a, \gamma) \mid e \right] + v(e)$$

$$V_A (A,e) = \mathbb{E} \left[ \max_{a \in A} u(a, \gamma) \mid e \right] - c(e)$$

Notice that $v(e)$ can be thought as $\mathbb{E} [r(\gamma) \mid e]$ where $r(\cdot)$ is an increasing function
Floor Delegation

**Definition** A delegation set is a floor if \( A = [a, +\infty) \) for some \( a \in R \).

The agent’s optimal action when the delegation set is a floor is

\[
\hat{a}(\gamma, a) = \arg \max_{a \in [a, +\infty)} u(a, \gamma) = \max \{ a, a^*(\gamma) \}
\]
Interval and floor delegation sets

Proposition 1  i) Let $\tilde{A}$ be an optimal delegation set and let $\tilde{e} > 0$ be the optimal level of effort. For every $\gamma \in \Gamma$ let

$$\tilde{a}(\gamma) = \max_{a \in \tilde{A}} u(a, \gamma)$$

denote the action chosen by the agent when the state is $\gamma$. Then the set

$$\{ a : a = \tilde{a}(\gamma) \text{ for some } \gamma \in (\underline{\gamma}, \bar{\gamma}) \}$$

is convex.

ii) If there is an optimal delegation set, then there is also an optimal floor delegation set.
Results with no bias

Sketch of the proof of Proposition 1

The proof of part i) is by contradiction

Assume that

\[ \int_{\gamma}^{\tilde{\gamma}} u(\tilde{a}(\gamma), \gamma) f(\gamma, \tilde{e}) d\gamma \geq \int_{\gamma}^{\tilde{\gamma}} u(a^*(\gamma), \gamma) f(\gamma, \tilde{e}) d\gamma \]

The other case is similar

By continuity, there exists a unique \( a \in \left[ a^*(\gamma), a^*(\tilde{\gamma}) \right] \) such that

\[ \int_{\gamma}^{\tilde{\gamma}} u(\tilde{a}(\gamma), \gamma) f(\gamma, \tilde{e}) d\gamma = \int_{\gamma}^{\tilde{\gamma}} u(\hat{a}(\gamma, a), \gamma) f(\gamma, \tilde{e}) d\gamma \]
Furthermore, quasiconcavity and single crossing of $u(\cdot,\cdot)$ guarantee that there exists a unique $\hat{\gamma} < (a^*)^{-1}(a)$ such that $u(\tilde{a}(\gamma),\gamma) > u(\hat{a}(\gamma,a),\gamma)$ if and only if $\gamma < \hat{\gamma}$

If the principal adopts the floor delegation set $[a, +\infty)$, the agent prefers $\tilde{e}$ to lower levels of effort.

The difference $u(\hat{a}(\gamma,a),\gamma) - u(\tilde{a}(\gamma),\gamma)$ is negative (positive) below (above) $\hat{\gamma}$.

Thus, it follows from MLRP that
\[
\int_{\tilde{\gamma}}^{\hat{\gamma}} \left[ u(\hat{a}(\gamma,a),\gamma) - u(\tilde{a}(\gamma),\gamma) \right] f(\gamma,\tilde{e}) d\gamma > \\
\int_{\gamma}^{\hat{\gamma}} \left[ u(\hat{a}(\gamma,a),\gamma) - u(\tilde{a}(\gamma),\gamma) \right] f(\gamma,e) d\gamma
\]

for every $e < \tilde{e}$.
Results with no bias

From the optimality of $\tilde{e}$ given $\tilde{A}$ we have:

$$\int_{\gamma}^{\tilde{\gamma}} u(\tilde{a}(\gamma), \gamma) f(\gamma, \tilde{e}) d\gamma - c(\tilde{e}) \geq \int_{\gamma}^{\tilde{\gamma}} u(\tilde{a}(\gamma), \gamma) f(\gamma, e) d\gamma - c(e)$$

Combining the two inequalities we obtain:

$$\int_{\gamma}^{\tilde{\gamma}} u(\hat{a}(\gamma, a), \gamma) f(\gamma, \tilde{e}) d\gamma - c(\tilde{e}) \geq \int_{\gamma}^{\tilde{\gamma}} u(\hat{a}(\gamma, a), \gamma) f(\gamma, e) d\gamma - c(e)$$

for every $e < \tilde{e}$
If $\tilde{e} < \bar{e}$ and the principal adopts the floor delegation set $[a, +\infty), \tilde{e}$ is not optimal (this, again, follows from MLRP)

Thus, the optimal effort level $e'$ must be larger than $\tilde{e}$. We have

$$V_A (a, e') > V_A (a, \tilde{e}) = V_A (\tilde{A}, \tilde{e})$$
$$V_P (a, e') > V_P (\tilde{A}, \tilde{e})$$

If $\tilde{e} = \bar{e}$, then the agent will continue to choose $\bar{e}$ even if the principal adopts the floor delegation set $[a - \varepsilon, +\infty)$ for some small $\varepsilon > 0$. Again, the original delegation set $\tilde{A}$ is not optimal
Existence

**Proposition 2**  *There exists an optimal delegation set.*

We restrict attention to floor delegation sets and show that the principal’s optimization problem admits a solution.
Comparative Statics

Given the floor delegation set $[a, +\infty)$, let $BR(a)$ denote the set of optimal effort levels.

**Proposition 3** i) If $a > a'$ then $e \geq e'$ for every $(e, e') \in BR(a) \times BR(a')$.

ii) Consider two benefit functions, $v_1(\cdot)$ and $v_2(\cdot)$ with $v'_1(e) > v'_2(e)$ for every $e$. Let $e_i$ be an optimal level of effort for the model in which $v = v_i$ for $i = 1, 2$. Then $e_1 \geq e_2$.

iii) Consider two cost functions, $c_1(\cdot)$ and $c_2(\cdot)$ with $c_1(0) = c_2(0) = 0$ and $c'_1(e) \leq c'_2(e)$ for every $e$. Let $V^i_p$, $i = 1, 2$, denote the principal’s payoff of the optimal delegation set when the cost is $c_i(\cdot)$. Then $V^1_p \geq V^2_p$. 
The model with bias

Quadratic payoff function and uniform distributions with shifting support

Agent is biased towards some action:

\[
\begin{align*}
    u_P(a, \gamma) &= - (\gamma + \beta - a)^2 \\
    u_A(a, \gamma) &= - (\gamma - a)^2
\end{align*}
\]

\(\beta > 0\) \((\beta < 0)\): the principal prefers higher (lower) actions than the agent
Consider a simple family of probability distributions

When the effort is $\gamma \geq 0$ the state is uniformly distributed in the unit interval $[\gamma, \gamma + 1]$

Cost function is quadratic: $c(\gamma) = \frac{\gamma^2}{2}$

The benefit function $v(\gamma)$ is concave
The delegation set $A$ and the effort level $\gamma$ induce expected payoffs:

$$V_P(A, \gamma) = - \int_{\gamma}^{\gamma+1} (\tilde{\gamma} + \beta - \hat{a}(\tilde{\gamma}, A))^2 d\gamma + v(\gamma)$$

$$V_A(A, \gamma) = - \int_{\gamma}^{\gamma+1} (\tilde{\gamma} + \beta - \hat{a}(\tilde{\gamma}, A))^2 d\gamma - \frac{\gamma^2}{2}$$

where $\hat{a}(\tilde{\gamma}, A) = \arg \max_{a \in A} - (\tilde{\gamma} - a)^2$
Necessary conditions for optimal effort

Given a delegation set $A$, the agent solves the following problem:

$$\max_{\gamma \geq 0} \int_\gamma^{\gamma+1} \left[ \max_{a \in \tilde{A}} - (\tilde{\gamma} - a)^2 \right] d\tilde{\gamma} - \frac{\gamma^2}{2} =$$

$$\max_{\gamma \geq 0} \int_\gamma^{\gamma+1} - (\tilde{\gamma} - \hat{a} (\tilde{\gamma}, \tilde{A}))^2 d\tilde{\gamma} - \frac{\gamma^2}{2}$$

First-order conditions for interior $\gamma$:

$$(\gamma - \hat{a} (\gamma, \tilde{A}))^2 - (\gamma + 1 - \hat{a} (\gamma + 1, \tilde{A}))^2 = \gamma$$

In general, the first-order conditions are not sufficient (the problem is not necessarily concave)
Concavity under interval delegation

Lemma 1 Suppose that the delegation set is an interval $[a, \bar{a}]$ for some $a \leq \bar{a}$. For every $\gamma$, let $z(\gamma)$ denote the agent’s expected payoff if the effort is $\gamma$:

$$z(\gamma) = - \int_\gamma^{\gamma+1} \left[ \max_{a \in [a, \bar{a}]} u_A(a, \tilde{\gamma}) \right] d\tilde{\gamma} - \frac{\gamma^2}{2}$$

The function $z(\cdot)$ is concave.
Optimal interval delegation

Proposition 4 Let $\gamma > 0$ be an optimal level of effort and $\tilde{A}$ denote the smallest optimal delegation set. Then $\tilde{A}$ is convex. Moreover, either $\tilde{A} \subset [\gamma, \gamma + 1]$ or $\tilde{A} = \{\bar{a}\}$ with $\bar{a} > \gamma + 1$.

To incentive the agent to exert high effort levels the principal may allow only one action: $\tilde{A} = \{\bar{a}\}$ with $\bar{a} > \gamma + 1$. 
Optimal interval delegation: sketch of the proof

Step 1: If $\tilde{A} \cap (\gamma, \gamma + 1) = \emptyset$, then $\tilde{A}$ is a singleton.

The delegation set $A'$ yields to the principal a larger payoff than $A$. 

\[ A \]
\[ \gamma - x \quad \gamma \quad \gamma + 1 \quad \gamma + 1 + \nu \]

\[ A' \]
\[ \gamma - x \quad \gamma \quad \gamma + x \quad \gamma + 1 - \nu \quad \gamma + 1 \quad \gamma + 1 + \nu \]
We work with a relaxed problem: the agent’s level of effort has to satisfy the first-order conditions.

**Step 2:** Let $\tilde{A}$ denote the smallest optimal delegation set and let $a$ denote the smallest element of $\tilde{A}$. Then either $\tilde{A}$ is a singleton or $a \geq \gamma$.

The delegation set $A'$ yields to the principal a larger payoff than $A$. 

- Diagram:
  - $A$: $\gamma - x$, $\gamma$, $\gamma + x$, $\gamma + 1$
  - $A'$: $\gamma - x$, $\gamma$, $\gamma + x$, $\gamma + 1$
**Step 3:** Let $\tilde{A}$ denote the smallest optimal delegation set and let $\tilde{a}$ denote the largest element of $\tilde{A}$. Then either $\tilde{A}$ is a singleton or $\tilde{a} \leq \gamma + 1$.

**Step 4:** Suppose that the optimal delegation set $\tilde{A}$ is not a singleton.

$\tilde{A}$ solves the relaxed problem. Therefore, $\tilde{A} \subseteq [\gamma, \gamma + 1]$

Suppose that $\tilde{A}$ has a gap. The principal’s payoff increases if the gap is filled.

The interval delegation set induces the same effort level as $\tilde{A}$ (it satisfied the same first-order conditions and the problem is concave).
Floor Delegation

**Proposition 5**  Let $\gamma > 0$ be the optimal level of effort and $\tilde{A}$ the smallest optimal delegation set. If $\tilde{A} \subseteq [\gamma, \gamma + 1]$ then $\tilde{A} = [a, \gamma + 1]$ for some $a > \gamma$.

Notice that in this case the floor delegation set $[a, \infty]$ is also optimal.
Results with bias

The optimal delegation set is

\[ \tilde{A} = [\hat{a} (\gamma, \tilde{A}), \hat{a} (\gamma + 1, \tilde{A})] \subseteq [\gamma, \gamma + 1] \]

The first-order conditions imply

\[ (\gamma - \hat{a} (\gamma, \tilde{A}))^2 - (\gamma + 1 - \hat{a} (\gamma + 1, \tilde{A}))^2 = \gamma \]

imply

\[ |\gamma - \hat{a} (\gamma, \tilde{A})| > |\hat{a} (\gamma + 1, \tilde{A}) - (\gamma + 1)| \]
If $\hat{a}(\gamma + 1, \tilde{A}) < \gamma + 1$ then it is possible to increase $\hat{a}(\gamma + 1, \tilde{A})$ and decrease $\hat{a}(\gamma, \tilde{A})$ simultaneously preserving

$$(\gamma - \hat{a}(\gamma, \tilde{A}))^2 - (\gamma + 1 - \hat{a}(\gamma + 1, \tilde{A}))^2 = \gamma$$

This change restricts the set of states in which the agent takes a suboptimal action, increasing payoffs.
Results with bias

Discretion and level of effort

Lemma 2 Suppose that the optimal effort level \( \gamma \) is interior. Let \( \tilde{A} \) denote the smallest optimal delegation set. If \( \gamma < 1 \), then

\[
\tilde{A} = [\gamma + \sqrt{\gamma}, \gamma + 1]
\]

If \( \gamma \geq 1 \), then

\[
\tilde{A} = \left\{ \frac{3\gamma + 1}{2} \right\}
\]
Suppose that the optimal delegation set is $\tilde{A} = [a, \gamma + 1]$ for some $\gamma < a < \gamma + 1$

The effort level $\gamma$ satisfies the first order conditions:

$$(\gamma - a)^2 = \gamma$$

which imply

$$a = \gamma + \sqrt{\gamma} < \gamma + 1$$

and, thus,

$$\gamma < 1$$
On the other hand, if \( \tilde{A} = \{a\} \) for some \( a \geq \gamma + 1 \) then

\[
(a - \gamma)^2 - (a - \gamma - 1)^2 = \gamma
\]

which yields

\[
a = \frac{3\gamma + 1}{2} > \gamma + 1
\]

and, thus,

\[
\gamma > 1
\]
The optimal level of effort

For every $\gamma > 0$ let $V_P(\gamma)$ denote the principal's payoff if he offers the optimal delegation set that induces the effort level $\gamma$.

We have

$$V_P(\gamma) = - \int_{\gamma}^{\sqrt{\gamma}} (\gamma + \sqrt{\gamma} - (s + \beta))^2 ds - \int_{\gamma + \sqrt{\gamma}}^{\gamma + 1} \beta^2 ds + v(\gamma)$$

for $\gamma < 1$, and

$$V_P(\gamma) = - \int_{\gamma}^{\gamma + 1} \left( \frac{3\gamma + 1}{2} - (s + \beta) \right)^2 ds + v(\gamma)$$

for $\gamma \geq 1$.
We compute the derivative of $V_P$:

$$V'_P(\gamma) = \beta - \frac{1}{2} \sqrt{\gamma} + v'(\gamma)$$

for $\gamma < 1$, and

$$V'_P(\gamma) = \beta - \frac{1}{2} \gamma + v'(\gamma)$$

for $\gamma \geq 1$

$V_P$ is concave (recall $v$ is concave) and $V'_P$ is continuous everywhere

We set $V'_P(\gamma) = 0$ and obtain a unique solution
Proposition 6 Assume that the optimal level of effort is strictly positive.

If $\beta - \frac{1}{2} + \nu'(1) < 0$, then the optimal delegation set is $[\gamma^* + \sqrt{\gamma^*}, \gamma^* + 1]$ where the optimal level of effort $\gamma^* < 1$ satisfies

$$\beta - \frac{1}{2} \sqrt{\gamma^*} + \nu'(\gamma^*) = 0$$

If $\beta - \frac{1}{2} + \nu'(1) \geq 0$, then the optimal delegation set is $\left\{ \frac{3\gamma^* + 1}{2} \right\}$ where $\gamma^* \geq 1$ satisfies

$$\beta - \frac{1}{2} \gamma^* + \nu'(\gamma^*) = 0$$
Results with bias

Corner solution

If $\beta \geq 0$, it is not optimal for the principal to induce an effort level equal to zero.

The delegation set $[\eta, 1 + \eta]$, with $\eta > 0$ and small, yields a strictly larger payoff than the delegation set $[0, 1]$.

If $\beta < 0$, the optimal delegation set that induces zero effort coincides with the optimal delegation set $(-\infty, \tilde{a}], \tilde{a} < 1$, when the state is uniformly distributed over the unit interval (no moral hazard).
Comparative Statics

Proposition 7 (For $\beta < 0$ assume $\gamma^* > 0$)

i) The optimal level of effort $\gamma^*$ and the principal’s payoff are increasing in $\beta$

ii) Suppose that $c(\gamma) = \frac{1}{2}k\gamma^2$ for $k > 0$. Both $\gamma^*$ and the principal’s payoff are decreasing in $k$

iii) Suppose that $v(\gamma) = \alpha h(\gamma)$, with $\alpha > 0$ and $h(\cdot)$ increasing and concave. Then $\frac{\partial \gamma^*}{\partial \alpha} > 0$
Conclusions

- We introduce endogenous states in the canonical delegation model.
- When effort is desirable, the optimal mechanism can be implemented by a floor delegation set.
- With bias, the agent loses discretion when the principal cares a lot about effort.
- Floor and distortions increase with the principal bias.