Online Appendix: Omitted Proofs (Not for Publication)

Appendix A: Proof of Proposition 1 and Theorem 2.1

A.1. Proof of Proposition 1 (Remaining Details)

In the proof, Lemma 3 only covers the special case in which cost and value functions are step functions with two steps only. This appendix covers the general case in which no such assumption is made. The proof is in two steps. First, an induction on the number of steps is made to generalize Lemma 3 to an arbitrary number of steps. Second, a limiting argument is used to establish the result for arbitrary (not necessarily step) functions c and v.

Lemma 6 *If c and v are step functions, and (x, p) is an allocation that is implementable in the full commitment program, and such that, for all $t \in T$,*

$$B(t) = \int_t^1 (x(s)(v(s) - p(s))ds \geq 0,$$

then (x, p) is also implementable in the veto-incentive compatible program.

Proof. Since c and v are step functions, we may equivalently describe the environment as finite: there are N types, with cost and values

$$c_1 \leq c_2 \leq \cdots \leq c_N, \text{ and } v_1 \leq v_2 \leq \cdots \leq v_N.$$

To avoid some trivial but distracting complications, we shall assume that the inequalities involving costs are strict: $\forall i < n$, $c_i < c_{i+1}$. The probability of each type (*i.e.*, the length of each step) is denoted q_i.[23] An allocation, then, reduces to a pair of vectors $x = (x_1, \ldots, x_N)$,

[23] More precisely, the number of types N is the number of types $t_i \in T$ for which either c or v (or both) has a discontinuity. The length of the interval refers to the intervals defined by the corresponding partition of T.

$p = (p_1, \ldots, p_N)$.

The hypothesis that $B(t) \geq 0$ for all $t \in T$ implies that, for all $J = 1, \ldots, N$,

$$\sum_{i=J}^{N} x_i q_i v_i \geq \sum_{i=J}^{N} x_i q_i p_i. \tag{13}$$

We shall show that any incentive-compatible, individually rational allocation satisfying this condition can be implemented in the veto-incentive compatible program, using N prices. The proof is by induction on the number of types (uniformly over all cost, values and probabilities).

Note that this is true for $N = 1$. In that case, the buyer's individual rationality constraint implies $p_1 \leq v_1$ (which trivially implies our hypothesis), while the seller's individual rationality constraint implies $p_1 \geq c_1$. Note then that any such allocation (x_1, p_1) with $p_1 \in [c_1, v_1]$ satisfies the veto-incentive compatibility constraint: conditional on p_1, the buyer assigns probability one to the (unique) type 1, and since $v_1 \geq p_1$, his payoff conditional on this event is positive.

Assume then that, whenever there are N types, and for any collection of costs, values and probabilities $\{(c_1, v_1, q_1), \ldots, (c_N, v_N, q_N)\}$, any incentive compatible, individually rational allocation $\{(x_1, p_1), \ldots, (x_N, p_N)\}$ that satisfies (13) can be implemented in the veto-incentive compatible program with N (not necessarily distinct) prices. Consider the case of $N+1$ types, with cost, values and probabilities $\{c_i, v_i, q_i\}_{i=1}^{N+1}$. Fix some incentive compatible, individually rational allocation

$$\{(x_1, p_1), \ldots, (x_{N+1}, p_{N+1})\},$$

satisfying (13). The argument is divided into three steps.

Step 1. Note that, by (13) with $J = N + 1$, $p_{N+1} \leq v_{N+1}$. Also, incentive compatibility implies that $p_N \leq p_{N+1}$.[24] It follows that there exists $z \in [0, x_{N+1}/x_N]$ such that

$$zx_N p_N + (x_{N+1} - zx_N)\, v_{N+1} = x_{N+1} p_{N+1}. \tag{14}$$

To see this, note that, for $z = 0$, the left-hand side reduces to $x_{N+1}v_{N+1}$, which is at least as large as the right-hand side, while for $z = x_{N+1}/x_N$, the left-hand side reduces to $x_{N+1}p_N$, which is at most as large as the right-hand side. Fix some z satisfying (28).

Step 2. Consider the game in which there are N types, with costs and values $\{\hat{c}_i, \hat{v}_i, \hat{q}_i\}_{i=1}^N$, defined as follows. Costs are unchanged: $\hat{c}_i := c_i$, all $i = 1, \ldots, N$. Values are given by

$$\hat{v}_i := v_i \text{ for } i < N, \text{ and } \hat{v}_N := \frac{q_N v_N + q_{N+1} z v_{N+1}}{q_N + q_{N+1} z},$$

(note that $\hat{v}_N \geq v_N > \hat{c}_N$), while probabilities are

$$\hat{q}_i := \frac{q_i}{\sum_{j \leq N} q_j + q_{N+1} z} \text{for } i < N, \text{ and } \hat{q}_N := \frac{q_N + q_{N+1} z}{\sum_{i \leq N} q_i + q_{N+1} z}.$$

We claim that the allocation $\{(x_i, p_i)\}_{i=1}^N$ (derived from $\{(x_i, p_i)\}_{i=1}^{N+1}$) is implementable, in this new environment, in the veto-incentive compatible program.

First, because costs are the same in this environment as in the original environment, individual rationality and incentive compatibility for all seller's types is implied by the fact that these were satisfied by the allocation $\{(x_i, p_i)\}_{i=1}^{N+1}$ in the original environment.

[24]The argument is standard: considering the two incentive compatibility conditions involving types N and $N + 1$ only, it follows that $x_N \geq x_{N+1}$ and $p_N \leq p_{N+1}$.

Therefore, to show that this allocation is implementable in the veto-incentive compatible program, given the induction hypothesis, it suffices to show that, for all $J \leq N$,

$$\sum_{i=J}^{N} x_i \hat{q}_i \hat{v}_i \geq \sum_{i=J}^{N} x_i \hat{q}_i p_i.$$

(Note that individual rationality for the buyer is the special case $J = 1$.) Simplifying,

$$\sum_{i=J}^{N} x_i \hat{q}_i \left(\hat{v}_i - p_i\right) = \frac{1}{\sum_{i \leq N} q_i + q_{N+1} z} \left[\sum_{i=J}^{N-1} x_i q_i \left(v_i - p_i\right) + q_N x_N \left(v_N - p_N\right) + q_{N+1} x_N z \left(v_{N+1} - p_N\right) \right].$$

Adding and subtracting $\left(x_{N+1} - x_N z\right) v_{N+1}$ to the expression inside the square brackets yield

$$\sum_{i=J}^{N} x_i \hat{q}_i \left(\hat{v}_i - p_i\right) = \frac{1}{\sum_{i \leq N} q_i + q_{N+1} z} \left[\begin{array}{c} \sum_{i=J}^{N-1} x_i q_i \left(v_i - p_i\right) + q_N x_N \left(v_N - p_N\right) + \\ q_{N+1} \left(x_{N+1} v_{N+1} - x_N z p_N - \left(x_{N+1} - x_N z\right) v_{N+1}\right) \end{array} \right].$$

Using the definition of z, we finally obtain

$$\sum_{i=J}^{N} x_i \hat{q}_i \left(\hat{v}_i - p_i\right) = \frac{1}{\sum_{i \leq N} q_i + q_{N+1} z} \left[\sum_{i=J}^{N+1} x_i q_i \left(v_i - p_i\right) \right] \geq 0,$$

where the last inequality uses that, by assumption, the allocation satisfies (13).

Therefore, by the induction hypothesis, the allocation $\{(x_i, p_i)\}_{i=1}^{N}$ is implementable in the veto-incentive compatible program, in this new environment, with N prices. Let $\{\hat{r}_1, \ldots, \hat{r}_N\}$ be the prices that implement this allocation in the veto-incentive compatible program, and $\{\hat{x}_1(r), \ldots, \hat{x}_N(r)\}_{r \in \{\hat{r}_1, \ldots, \hat{r}_N\}}$ be the probabilities assigned to these prices.

Step 3. We now construct a set of prices $\{r_1, \ldots, r_{N+1}\}$ and probabilities $\{x_1(r), \ldots, x_{N+1}(r)\}$, $r \in \{r_1, \ldots, r_{N+1}\}$, that implement $\{(x_1, p_1), \ldots, (x_{N+1}, p_{N+1})\}$ in the veto-incentive compatible program, in the original environment.

The prices are given by

$$\{r_1, \ldots, r_{N+1}\} = \{\hat{r}_1, \ldots, \hat{r}_N\} \cup \{v_{N+1}\}.$$

The probabilities are given by, for $i < N + 1$,

$$x_i (r) = \hat{x}_i (r), \forall r \in \{\hat{r}_1, \ldots, \hat{r}_N\}, \text{ and } x_i (v_{N+1}) = 0,$$

and

$$x_{N+1} (r) = z\hat{x}_N (r) \forall r \in \{\hat{r}_1, \ldots, \hat{r}_N\}, \text{ and } x_{N+1} (v_{N+1}) = x_{N+1} - zx_N.$$

It is immediate to see that, conditional on any given $r \in \{\hat{r}_1, \ldots, \hat{r}_N\}$, the conditional value is the same as in the modified environment, so that the buyer's veto-incentive compatibility constraint holds. This is also true if $r = v_{N+1}$, because the only seller's type trading at this price is type $N + 1$. Furthermore, by construction, buyer i trades with probability x_i and receives an average price p_i. This completes the proof. ■

Finally, we can show sufficiency for arbitrary cost and value functions.

Lemma 7 *If (x, p) is an individually rational and incentive compatible allocation such that, for all $t \in T$,*

$$B(t) = \int_t^1 x(s)[v(s) - p(s)]ds \geq 0,$$

then (x, p) is implementable in the veto-incentive compatible program.

Proof. Fix an allocation (x, p) that satisfies the assumptions of the lemma. Consider a sequence of partitions $\mathcal{P}^n = \{t_1^n, \ldots, t_n^n\}$, with $t_1^n = 0, t_n^n = 1$, $\max_i |t_i^n - t_{i+1}^n| < K/n$ for some constant K independent of n, and such that $D \subseteq \mathcal{P}^n$, where D is the set of discontinuities of either v or c (without loss of generality, assume that n is large enough to include this finite set).

We now define a sequence of functions $c^n, v^n : T \to \mathbb{R}_+$ as follows: for all $t < 1$, set $c^n(t) := c(t_j^n)$ for $t \in \left[t_j^n, t_{j+1}^n\right)$, $c^n(1) := c(t_{n-1}^n)$, as well as, for all $t < 1$, $v^n(t) := v(t_{j+1}^n)$ for $t \in \left[t_j^n, t_{j+1}^n\right)$, $v^n(1) := v(t_n^n)$.

Further, define the sequence of allocations x^n, p^n as follows: for all $t \in T$, set $x^n(t) := x(t_j^n)$, and $p^n(t) := p(t_j^n)$ for $t \in \left[t_j^n, t_{j+1}^n\right)$, $j < n-1$ ($t \in \left[t_j^n, t_{j+1}^n\right]$ if $j = n$.)[25]

Note that the allocation (x^n, p^n) is incentive compatible and individually rational for the seller given the functions (c^n, v^n) (because the choices of the types in the set $\mathcal{P}^n$ are incentive compatible and individually rational given the original allocation (x, p)). Define

$$B_j^n := \int_{t_j^n}^1 x^n(s)[v^n(s) - p^n(s)]ds.$$

Because $x(t_{j+1}^n) \leq x(t) \leq x(t_j^n)$ and $p(t_{j+1}^n) \leq p(t) \leq p(t_j^n)$ (by incentive compatibility) for $t \in [t_j^n, t_j^n + 1)$, $j < i - 1$, we can pick these sequences such that, because $B(t_j^n) \geq 0$ (the lemma's hypothesis), it is also the case that also $B_j^n \geq 0$ for all j (clearly, $B_n^n = 0$). Therefore, the allocation (x^n, p^n) is individually rational for the buyer given (c^n, v^n) and further, given Lemma 7, this allocation is veto-incentive compatible in the game with cost and value functions (c^n, v^n). Let μ^n denote the corresponding mechanism. The mechanism μ^n defines a function x^n specifying the probability of trade given some message t, and a joint distribution $\tilde{\mu}^n$ on $T \times \mathbb{R}_+$ in case that there is a trade for each type.[26] Let $\hat{\mu}^n$ denote the product distribution whose marginals coincide with those of $\tilde{\mu}^n$. Note that incentive compatibility and veto-incentive compatibility are restrictions on the marginal distributions only, so that any mechanism inducing the pair x^n and $\hat{\mu}^n$ also implements (x^n, p^n). Note that, by construction, (x^n, p^n) converge (pointwise) to (x, p), and similarly, (c^n, v^n) converge pointwise to (c, v). Also, since we can replace the set of prices $\mathbb{R}_+$ by the compact interval $[0, v(1)]$ (because $v(1)$ is an upper bound on the price that can be in

[25]Note that the functions v^n, c^n as well as the allocations x^n, p^n are right-continuous.

[26]More precisely, $x = \mu(\cdot)\,[1, \mathbb{R}_+]$, as defined in Section 2, and the distribution $\tilde{\mu}$ is the joint distribution $\nu((1, \cdot), \cdot)$, where ν is the conditional distribution defined in Section 2 as well.

the support of any mechanism that is veto-incentive compatible), a subsequence of the sequence $\{\hat{\mu}^n\}$ (without loss of generality the sequence itself) must converge weakly to some distribution $\hat{\mu}$. It follows from Theorem 3.2 of Billinsgley (1968) that $\hat{\mu}$ must itself be a product distribution, and that the marginals of $\hat{\mu}^n$ converge weakly to the marginals of $\hat{\mu}$. Therefore, for all prices p, the marginal distribution $\hat{\mu}^n(\cdot \mid p)$ converges weakly to $\hat{\mu}(\cdot \mid p)$, and so it follows that, for all p,

$$\int_T \hat{\mu}(t \mid p)(v(t) - p)dt \geq 0,$$

which is precisely the requirement of veto-incentive compatibility. Therefore, along with x, $\hat{\mu}$ defines a veto-incentive compatible mechanism. (Incentive compatibility and individual rationality are satisfied by hypothesis, given the limiting allocation (x, p).) ■

Note that Lemma 2 and 7 immediately imply Proposition 1.

A.2. Proof of Theorem 2.1

Here, we prove the three claims stated in Section 4.1.2. Assume that $v > c$. (Simple changes have to be made otherwise.) Given (x, p), let $\bar{t} := \sup\{t \in T : x(t) > 0\}$.[27]

Clearly, $(0, 0)$ is an extreme point of this set, and because it is achieved by the allocation $(x, p) = (0, 0)$, the claims are trivially valid for this point. We further divide this boundary into $\Pi_-^V := \{(\pi^S, \pi^B) \in \mathbb{R}^2 : \pi^B = \max_{(\pi_1,\pi_2)\in\Pi^V} \pi_2 \text{ s.t. } \pi_1 \leq \pi^S\}$ and $\Pi_+^V := \{(\pi^S, \pi^B) \in \mathbb{R}^2 : \pi^B = \max_{(\pi_1,\pi_2)\in\Pi^V} \pi_2 \text{ s.t. } \pi_1 \geq \pi^S\}$. As will be clear, Π_+^V intersects the axis $\{(\pi^S, 0) : \pi^S \in \mathbb{R}\}$, so that $\Pi^V = \text{co}\ \{(0, 0)\} \cup \Pi_+^V \cup \Pi_-^V$, where, given any set A, co A denotes the convex hull of A.

Let us establish three claims for $\Pi_+^V \cup \Pi_-^V$ at once. If (x, p) achieves $\pi \in \Pi_+^V \cup \Pi_-^V$, then

1. $\lim_{s\downarrow t} \pi^S(s \mid t) = \pi^S(t)$ for all t. Suppose that this is not the case. First, consider the case in which the payoff is in Π_+^V. Take the supremum over $\hat{t}$ such that $\pi^S(\hat{t}) > \lim_{s\downarrow t} \pi^S(s \mid \hat{t})$. Clearly, $\hat{t}$ is a point of discontinuity of $c(t)$ and $x(t)$. Consider then the following alternative

[27] Not to be confused with $\bar{t}$ as defined in (9).

allocation (x', p'), defined by

$$\begin{aligned} x'(t) = x(t) + \varepsilon \text{ if } t \in [\hat{t}, \hat{t}+\varepsilon),\ x'(t) = x(t) &\quad \text{otherwise;} \\ \bar{p}'(t) = \bar{p}(t) + \varepsilon c(t+\varepsilon) \text{ if } t \in [\hat{t}, \hat{t}+\varepsilon),\ \bar{p}'(t) = \bar{p}(t) &\quad \text{otherwise.} \end{aligned}$$

It is straightforward to see that, for small enough $\varepsilon > 0$, this is incentive-compatible, satisfies $B(t) \geq 0$ for all t and strictly improves the buyer's payoff, while weakly improving the seller's payoff. Consider next the case in which the payoff of (x, p) belongs to Π^V_-. Take the supremum over $\hat{t}$ such that $\pi\left(\hat{t}\right) > \lim_{s \downarrow t} \pi\left(s \mid \hat{t}\right)$. Clearly, $\hat{t}$ is a point of discontinuity of $c(t)$. Thus consider the alternative allocation (x', p'), defined by

$$\begin{aligned} x'(t) = x(t) &\quad \text{for all } t \in [0,1], \\ \bar{p}'(t) = \bar{p}(t) - \varepsilon \text{ if } t \in [0, \hat{t});\ \bar{p}'(t) = \bar{p}(t) &\quad \text{otherwise.} \end{aligned}$$

It is straightforward to check that for small $\varepsilon > 0$ this allocation is implementable. Moreover, it decreases the seller's payoff and increases the buyer's payoff, which contradicts the assumption that the payoff is in Π^V_-.

2. $\pi^S(\bar{t}_-) = 0$, where $\bar{t} := \sup\{t \leq 1 : x(t) > 0\}$ is the highest seller's type that trades with positive probability. Suppose towards a contradiction that this is not the case. Consider first the case in which the payoff is in Π^V_-. Modify the allocation by decreasing $p(t)$ (for all t such that $x(t) > 0$) by some small $\varepsilon > 0$, contradicting that $\pi \in \Pi^V_-$. Suppose next that $\pi \in \Pi^V_+$. Fix some small $\eta > 0$ and let $t^* := \sup\{t : x(t) - x(\bar{t}_-) > \eta\}$. Since the allocation is right-continuous, $x(t^*) \leq x(\bar{t}_-) + \eta$. Thus, define $\hat{p}$ such that $x(t^*)(p(t^*) - c(t^*)) = [x(\bar{t}_-) + \eta](\hat{p} - c(t^*))$, and consider the alternative allocation

$$\begin{aligned} \hat{x}(t) = x(\bar{t}) + \eta \text{ if } t \in [t^*, \bar{t}),\ \hat{x}(t) = x(t) &\quad \text{otherwise;} \\ \hat{p}(t) = \hat{p} \text{ if if } t \in [t^*, \bar{t}),\ \hat{p}(t) = p(t) &\quad \text{otherwise.} \end{aligned}$$

The payoff of each seller's type weakly decreases in this alternative allocation, while the buyer's payoff strictly increases (since c is piecewise continuous and $v(t) > c(t)$ the allocation remains implementable for small η). If the seller's payoff remains constant, we are done, so assume it decreases by $\alpha > 0$. There exists $\varepsilon > 0$ such that $\int_0^{\bar{t}} \varepsilon dt = \alpha$. Thus, increase all prices by ε, so that the seller's overall payoff does not change.[28] This is incentive compatible and increases the buyer's payoff. Thus, since the surplus increase goes to the buyer, it is enough to show that $B(t) \geq 0$, all t. Note that the buyer's *ex ante* payoff changes by

$$\begin{aligned} \Delta B(0) &= \int_0^{\bar{t}} \left(\Delta x(t)\left(v(t) - c(t)\right)\right) dt - \int_0^{\bar{t}} \left(\Delta \bar{p}(t) - \Delta x(t)\, c(t)\right) dt \\ &= \int_0^{\bar{t}} \left(\Delta x(t)\left(v(t) - c(t)\right)\right) dt > 0, \end{aligned}$$

where $\Delta x(t) := x'(t) - x(\bar{t})$ and $\Delta \bar{p}(t) := \bar{p}'(t) - \bar{p}(t)$. Furthermore,

$$\Delta x(t)\left(v(t) - c(t)\right) + \left(\Delta \bar{p}(t) - \Delta x(t)\, c(t)\right) < 0,$$

if and only if $t < t^{**} \in [t^*, \bar{t})$. Thus $\Delta B(t) \geq 0$ for all t, which completes the argument.

3. $x(0) = 1$. Suppose towards a contradiction that $x(0) < 1$. Since the cost function is piecewise right-continuous and piecewise C^1 we take an interval $[0, \eta]$ such that (c, v) are differentiable on that interval. Fix $n' \in \mathbb{N}$ such that $1/n' < \eta$, and consider the following alternative allocation $(x_n, \bar{p}_n)$ defined as

$$\begin{aligned} x_n(t) &:= x(t) + (1 - x(0)) \text{ if } t \in [0, \tfrac{1}{n}),\ x_n(t) = x(t) \quad \text{otherwise;} \\ \bar{p}_n(t) &:= \bar{p}(t) + c\left(\tfrac{1}{n}\right)(1 - x(0)) \text{ if } t \in [0, \eta),\ \bar{p}(t) = \bar{p}(t) \quad \text{otherwise.} \end{aligned}$$

Notice that there exists $m > n'$ such that this allocation is implementable (and is also a Pareto improvement for all $n > m$). If $\pi \in \Pi_+^V$, this is a contradiction. If instead $\pi \in \Pi_-^V$ such that $\pi^S > 0$, let $k > 0$ be the supremum of the subgradients of the payoff set at π. Now

[28] Notice that the seller's types $[t^*, \bar{t})$ are made worse off, while seller's types $[0, t^*)$ are made better off. Hence, the allocation $(0, 0)$ is still optimal for types in $[\bar{t}, 1)$ when $\bar{t} < 1$.

notice that for each n the payoff of the buyer increases by $(1 - x(0)) \int_0^{\frac{1}{n}} (v(s) - c(1/n))\, ds$, while the payoff of the seller increases by $(1 - x(0)) \int_0^{\frac{1}{n}} (c(s) - c(1/n))\, ds$. Thus the ratio of the increase in the payoff of the buyer and the seller is arbitrarily large as $n \to \infty$, and for n large enough, both payoffs can be increased at a rate greater than k, a contradiction. If $\pi \in \Pi_-^V$ and $\pi^s = 0$, then either (i) c is constant in a neighborhood of 0, or (ii) $c(t) > c(0)$ for all $t > 0$. If (i) holds, then one can readily see that for some small n the alternative allocation above increases the buyer's payoff while keeping the seller's payoff constant. If (ii) holds, then since c is right-continuous we have $\pi^B = 0$ and the claim is trivially true.

Appendix B: Proof of Lemma 5 (and finite horizon)

As mentioned, we restrict ourselves to the case of extreme points that lie on the Pareto-frontier here. Considering points on the "north-west" and "south-west" of the relevant payoff set require relatively straightforward modifications.

Lemma 8 *Every extreme point (π^S, π^B) of the payoff set that can be achieved by veto-incentive compatible allocations $(x, p) \in \Pi_+^V$ for which $\pi^S(0) > v(0) - c(0)$ can be approached by a regular allocation.*

Proof. Consider an allocation (x, p) satisfying the assumptions of the lemma. This allocation maximizes the weighted sum of the buyer and the seller payoff. For future reference, let $\beta \in (0, 1)$ be the seller's weight. Taking a sufficiently close allocation if necessary, assume that $\pi^S(0) \geq v(0) - c(0)$ (implicitly we assume that $v(0) < v(1_-)$).[29]

Define $\hat{t} := \sup\{t : x(t) > 0\}$. It is straightforward to construct an allocation (x', p') such that: a) $\hat{t} := \sup\{t : x'(t) > 0\}$; b) $x'(t_-) > 0$; c) $p' \ll v$. Since the convex combination of feasible allocations is also a feasible allocation, take $\lambda \in (0, 1)$ and define $(x_\lambda, p_\lambda) := \lambda(x, p) + (1-\lambda)(x', p')$

[29] Otherwise the equilibrium outcome of the bargaining game is unique and involves the seller selling the good at $v(0)$ in the first period with probability 1.

satisfying $\|(x_\lambda, p_\lambda) - (x, p)\| < \frac{\varepsilon}{2}$ and $\pi^S_\lambda(0) > v(0) - c(0)$. Notice that for the allocation (x_λ, p_λ) we have $B_\lambda(0) > 0$ for all $t < \hat{t}$. Furthermore, notice that since (x_λ, p_λ) and (c, v) are right-continuous the assumptions a) and b) imply that there exists $t_2 < \hat{t}$ such that $p_\lambda(s) < v(t_2)$ for all $t \in (t_2, \hat{t})$. Therefore:

$$B_\lambda(t_2) = \int_{t_2}^{\hat{t}} x_\lambda(s)\left(v(s) - p_\lambda(s)\right) ds =: \vartheta > 0.$$

Next, we approach the allocation (x_λ, p_λ) with a step allocation.

- Step 1: For every $n \in \mathbb{N}$ we consider a mesh of $[0, \hat{t})$, $\left\{\mathcal{I}^n_j\right\}_{j=1}^{M_n} := \left\{[t^n_{1,1}, t^n_{2,1}), \ldots, [t^n_{1,M_n}, t^n_{2,M_n})\right\}$ such that

  i) $\sum_j \max\left|\sup_{t\in\mathcal{I}^n_j} x_n(t) - \inf_{t'\in\mathcal{I}^n_j} x_n(t')\right| < \left(\frac{1}{n}\right)$;

  ii) $x_n(t) = x_n(t')$ for all $t \in \mathcal{I}^n_j$ and $x_n\left(t^n_{2,j-}\right) = x_\lambda\left(t^n_{2,j-}\right)$ for every $t^n_{2,j}$;

  iii) $p_n(t) = p_n(t')$ for all $t \in \mathcal{I}^n_j$; $p_n\left(t^n_{2,M_n-}\right) = p\left(t^n_{2,M_n-}\right)$ and for all $j < M_n$ we define $p_n\left(t^n_{2,j-}\right)$ by

$$p_n\left(t^n_{2,j-}\right)\left(x_n\left(t^n_{2,j-}\right) - c\left(t^n_{2,j-}\right)\right) = p_n\left(t^n_{2,j+1-}\right)\left(x_n\left(t^n_{2,j+1-}\right) - c\left(t^n_{2,j-}\right)\right);$$

  iv) All discontinuity points of c belong to the boundaries of the partition.

  Notice that for every t we have

$$\bar{p}_n(t) = x_n(t)c(t) + \int_t^1 x_n(s)dc(s).$$

Notice that by construction $x_n(t) \to x(t)$ uniformly. Furthermore,

$$\begin{aligned} & |\bar{p}_n(t) - \bar{p}(t)| \\ \leq \; & |x_n(t)c(t) - x(t)c(t)| + \int_t^1 |x_n(s) - x(t)| \, dc(s). \end{aligned}$$

Hence, using iv) we conclude that $\bar{p}_n(t) \to \bar{p}(t)$ uniformly. Furthermore, there exists n_1 such that $n > n_1$ implies $B_n(t_1) \geq \frac{\vartheta}{2}$. Hence, the uniform convergence of $\bar{p}_n$ and $x_n(t)$ guarantees that there exists $n_2 > n_1$ such that $\|(x_n, p_n) - (x_\lambda, p_\lambda)\| < \frac{\varepsilon}{2}$ and $B_n(t) > 0$ for all $t < \hat{t}$.

Step 2: Notice that the allocation (x_n, p_n) is a step function allocation; hence there is a finite partition of the types $\left\{[t_{1,1}^n, t_{2,1}^n), \ldots, [t_{1,M_n}^n, t_{2,M_n}^n)\right\}$ such that all types $t \in [t_{1,j}^n, t_{2,j}^n)$ trade with the same probability. Hence, consider a fictitious game with finite types in which all types $t \in [t_{1,j}^n, t_{2,j}^n)$ have the same cost $c(t_{2,j-}^n)$, the same value $\left(\frac{\int_{[t_{1,j}^n, t_{2,j}^n)} v(s)ds}{t_{2,j}^n - t_{1,j}^n}\right)$ and trade with the same probability. Furthermore, if $t_{2,j}^n < 1$, attribute the cost $c(1)$ and the value $\left(\frac{\int_{[t_{2,j}^n, 1)} v(s)ds}{1 - t_{2,j}^n}\right)$ to all types $t \in [t_{2,j}^n, 1)$. In this finite game, consider the allocation that maximizes the weighted sum of the buyer's payoff and the seller's payoff (with weight β on the seller) such that all types $t \in [t_{1,j}^n, t_{2,j}^n)$, $j \leq M_n$ and all types $t \in [t_{2,j}^n, 1)$ trade with the same probability. This is a finite dimensional compact problem. Hence, it admits a solution. Since (x_n, p_n) is feasible, the solution leads to a weakly higher value for the objective function. It is straightforward to show that any solution to this problem is a regular allocation and, from Theorem 2.1, we know that all downward constraints are binding and the last type of the seller who trades with positive probability obtains zero payoff. This completes the proof.

■

Finite Horizon

We show below how to implement regular allocations (see Definition 1) when the horizon N ($N \geq 2$) is finite and the players do not discount the future.

- All types make non-serious offers (*e.g.*, $p_n = v(1) + 1$) in every period $n < N - 1$.

- Types $t \in [0, t_1]$ offer p_1 at period $N - 1$ which is accepted with probability 1.

- For $j = 1, \ldots, K - 3$, seller's types $t \in [t_j, t_{j+1}]$ make a non-serious offer at period $N - 1$ and offer p_{j+1} at period N. The buyer accepts this offer with probability x_{j+1}. Notice that it is rational for the buyer to randomize since the buyer breaks even by accepting any such offer (because $B(t_1) = \cdots = B(t_{K-2}) = 0$).

- Types $t \in [t_{K-2}, t_K]$ offer $v_{t_{K-2}}^{t_K}$ at period $N-1$. The buyer accepts this offer with probability $x_{K-1}\beta$ (recall that β is defined in (11)). If this offer is rejected, all types $t \in [t_{K-2}, t_{K-1}]$ offer $v_{t_{K-2}}^{t_{K-1}}$ at period N, while all types $t \in [t_{K-1}, t_K]$ offer $v_{t_{K-1}}^{t_K}$ at period N. The offer $v_{t_{K-2}}^{t_{K-1}}$ is accepted with probability ς_{K-1}, which is defined by $x_{K-1} = x_{K-1}\beta + (1 - x_{K-1}\beta)\varsigma_{K-1}$, while the offer $v_{t_{K-1}}^{t_K}$ is accepted with probability ς_K, which is defined by $x_K = x_{K-1}\beta + (1 - x_{K-1}\beta)\varsigma_K$. Again, it is rational for the buyer to randomize since he breaks even by accepting any of the offers above.

It is easy to see that the buyer has no profitable deviation. For the seller, we assume that the buyer puts probability 1 on the seller being type $t = 0$ (and never revises his belief) after an off-path offer. Therefore, the best deviation by a seller would be to imitate all types $t \in [0, t_1]$ and offer p_1 at period $N - 1$. Thus since regular allocations are incentive compatible (see (3) in Definition 1) we conclude that no type has a profitable deviation.

Appendix C: Relaxing the Co-Monotonicity of v and c

We have maintained throughout the assumption that both the seller's cost, and the buyer's value are non-decreasing. Of course, there is no loss of generality in assuming that one of these functions is non-decreasing. So let us assume that types are ordered so that only the cost function is non-decreasing, and maintain all other assumptions (besides monotonicity). In particular, gains from trade are bounded away from zero for all t, and, to avoid trivialities, the seller's highest cost exceeds the buyer's average value. Does there exist a similarly tractable characterization of the veto-incentive compatible program when the value function is not necessarily increasing? In that case, it is easy to see that $B(t) \geq 0$ for all t is no longer a necessary condition, although it remains a sufficient condition for implementability. This suggests that non-negative correlation singles out the collection of intervals $\{[t, 1] : t < 1\}$ as the relevant one for the domains of the integral constraints $B(t)$. We view it as an important next step to identify what the "right" collection of intervals is, if any, over which the expected buyer's payoff must be positive, when values are not positively correlated, before turning to more general environments with limited commitment and private information.

In the absence of such a characterization, we might still ask the question: under which conditions is the *ex ante* efficient (*i.e.*, surplus-maximizing) allocation of the commitment program also implementable in the veto-incentive program, or even in the bargaining game as frictions disappear? The answer to this question is surprisingly simple. Recall that the *ex ante* efficient mechanism under full commitment takes a very simple form, with (at most) two thresholds t_1 and t_2, with $0 < t_1 \leq t_2 \leq 1$. If $t_1 = t_2$, it is trivial to implement the allocation in the game, and, *a fortiori*, in the veto-incentive compatible program, so let us assume that $t_2 > t_1$. We have the following necessary and sufficient condition, which generalizes Proposition 1, at the cost of being stated in terms of endogenous variables (t_1, t_2).

Proposition 4 *If $t_2 > t_1$, the ex ante efficient allocation of the commitment program is imple-*

mentable in the bargaining game as $\delta \to 1$ if and only if

$$c(t_2) \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v\left(t\right).$$

Proof. Sufficiency follows closely the construction in 4.2.3 and is omitted. We focus here on necessity.

This proof makes clear that the condition is equally necessary for veto-incentive compatibility. Thus, this condition is also necessary and sufficient for implementability in the veto-incentive compatible program. Recall that, in the *ex ante* efficient allocation, the seller's expected transfers $\bar{p}\left(t\right)$ are given by

$$\bar{p}\left(t\right) = \begin{cases} \left(1-x\right)c\left(t_1\right) + xc\left(t_2\right) & t \in \left[0, t_1\right), \\ xc\left(t_2\right) & t \in \left[t_1, t_2\right], \\ 0 & t > t_2. \end{cases}$$

Define the set $\hat{T}$ as

$$\hat{T} := \left\{t \in \left[0, t_2\right] : v\left(t'\right) \leq v\left(t\right) \text{ for every } t' \in \left[0, t_2\right]\right\}.$$

Throughout we assume that the set $\hat{T}$ is nonempty (this is not guaranteed by our assumptions, and minor adjustments are necessary otherwise). To ease notation, we let $\hat{v}$ denote the value of the function v over the set $\hat{T}$.

Suppose that $c\left(t_2\right) > v_{t_1}^{t_2}$. We want to show that it is impossible to construct a collection of distributions $\left(\mu\left(\cdot \mid t\right)\right)_{t \in [0,t_2]}$ over the interval $[0, \hat{v}]$ that satisfy the following three conditions:

i) for every $t \in [0, t_2]$,

$$\int_0^{\hat{v}} d\mu\left(p \mid t\right) = x\left(t\right); \tag{15}$$

ii) for every $t \in [0, t_2]$,

$$\int_0^{\hat{v}} p d\mu\left(p \mid t\right) = \bar{p}\left(t\right); \tag{16}$$

iii) for all $p \in [0, \hat{v}]$,

$$\int_0^{t_2} (v(t) - p)\, d\mu(p \mid t) = 0.$$

(Recall that under the *ex ante* efficient mechanism the buyer's expected payoff is equal to zero.)

We approximate the function v by a sequence of step functions v^n, $n \in \mathbb{N}$. In particular, each v^n satisfies

i) for every $t \in [0, t_2]$,

$$v(t) \le v^n(t) \le \hat{v},$$

ii) for every $t \in [0, 1]$,

$$0 \le v^n(t) - v(t) \le \frac{1}{n},$$

iii) if t and t' belong to the same step of v^n, then $x(t) = x(t')$.

Finally, for each $n \in \mathbb{N}$, we let $I^n \subset [0, t_2]$ denote the union of the intervals over which the function v^n takes the value $\hat{v}$.

Fix $n \in \mathbb{N}$. For each $p < \hat{v}$ we have

$$\int_0^{t_2} (v^n(t) - p)\, d\mu(p \mid t) = \varepsilon^n(p),$$

for some $\varepsilon^n(p) \ge 0$. After dividing both sides by $\hat{v} - p$ and rearranging terms, we have

$$\int_{t \in I^n} d\mu(p \mid t) + \int_{t \in [0, t_2] \setminus I^n} \left(1 - \frac{\hat{v} - v^n(t)}{\hat{v} - p}\right) d\mu(p \mid t) = \frac{\varepsilon^n(p)}{\hat{v} - p} \ge 0.$$

We integrate the two sides of the equality over p, and get

$$z^n := \int_{t \in I^n} \int_0^{\hat{v}} d\mu(p \mid t)\, dt + \int_{t \in [0, t_2] \setminus I^n} \int_0^{\hat{v}} \left(1 - \frac{\hat{v} - v^n(t)}{\hat{v} - p}\right) d\mu(p \mid t)\, dt \ge 0.$$

For each $t \in [0, t_2] \setminus I^n$, let $\bar{\mu}(\cdot \mid t)$ denote the distribution that assigns probability $x(t)$ to the offer $\bar{p}(t) / x(t)$ (with probability $1 - x(t)$ no offer is made). Notice that the function $\frac{1}{p - \hat{v}}$ is concave in p. This, together with conditions (15) and (16), implies that, for each $n \in \mathbb{N}$,

$$\bar{z}^n := \int_{t \in I^n} \int_0^{\hat{v}} d\bar{\mu}(p \mid t)\, dt + \int_{t \in [0,1] \setminus I^n} \int_0^{\hat{v}} \left(1 - \frac{\hat{v} - v^n(t)}{\hat{v} - p}\right) d\bar{\mu}(p \mid t)\, dt \geq t z^n \geq 0. \tag{17}$$

We take the limit of $\bar{z}^n$ as n goes to infinity, so that

$$\bar{z} := \lim_{n \to \infty} \bar{z}^n = t_1 + (t_2 - t_1)\, x - \frac{\int_0^{t_1} (\hat{v} - v(t)) dt}{\hat{v} - (1-x) c(t_1) - x c(t_2)} - x \frac{\int_{t_1}^{t_2} (\hat{v} - v(t)) dt}{\hat{v} - c(t_2)} =$$

$$\frac{t_1 \left(v_0^{t_1} - (1-x) c(t_1) - x c(t_2)\right)}{\hat{v} - (1-x) c(t_1) - x c(t_2)} - \frac{x (t_2 - t_1) \left(c(t_2) - v_{t_1}^{t_2}\right)}{\hat{v} - c(t_2)} < \frac{t_1 \left(v_0^{t_1} - (1-x) c(t_1) - x c(t_2)\right) - x (t_2 - t_1) \left(c(t_2) - v_{t_1}^{t_2}\right)}{\hat{v} - (1-x) c(t_1) - x c(t_2)} = 0,$$

where the inequality follows from the fact that $c(t_2) > v_{t_1}^{t_2}$, and the last equality follows from the definition of x in equation (6). However, $\bar{z}$ being strictly negative contradicts the fact that it is the limit of a sequence of nonnegative numbers (see condition (17)). ■

Appendix D: A Sufficient Condition for the Efficient Mechanism to be Implemented in the Bargaining Game

Recall that $Y : [0,1] \to \mathbb{R}$ is defined as

$$Y(t) := \int_0^t (v(s) - c(t))\, ds = \int_0^t (v(s) - c(s) - s c'(s))\, ds.$$

Our assumptions imply that, as mentioned, $Y(0) = 0$, $Y'(0) > 0$ (if $v > c$) and $Y(1) < 0$. Let $\underline{t}$ denote the smallest local maximizer of the function Y. Also, let $\bar{t}$ denote the smallest strictly positive root of Y. For any t, let $\mu(t)$ denote the mechanism under which the types below t trade with probability one at the price $c(t)$ and the types above t do not trade. Notice that if $Y(t) \geq 0$, then the mechanism $\mu(t)$ is incentive compatible and individually rational.

Consider the efficient mechanism under full commitment. We know that there exist $0 < t_1 \leq t_2 \leq 1$ such that the seller's types in $[0, t_1)$ trade with probability 1, while the types in $[t_1, t_2]$

trade with probability $x(t_1, t_2) \in [0, 1)$ (all other types of the seller do not trade). Recall that the buyer's individual rationality constraint holds with equality. Thus, we have

$$0 = \int_0^{t_1} (v(s) - c(t_1))\, ds + x\left(t_1 c(t_1) + \int_{t_1}^{t_2} v(s)\, ds - t_2 c(t_2)\right) =$$
$$Y(t_1) + x \int_{t_1}^{t_2} (v(s) - c(s) - sc'(s))\, ds = Y(t_1) + x(Y(t_2) - Y(t_1)).$$

Therefore, we can express $x(t_1, t_2)$ as

$$x(t_1, t_2) = \frac{Y(t_1)}{Y(t_1) - Y(t_2)}.$$

Consider the case in which $t_2 > t_1$, *i.e.*, there is a set of types who trade with a probability larger than zero but smaller than one. First, we must have $Y(t_2) - Y(t_1) < 0$, otherwise we may increase x and improve efficiency. This immediately implies $Y(t_1) > 0$. Second, under the optimal mechanism $Y(t_2) < 0$. In fact, if $Y(t_2) \geq 0$, it is possible to implement the mechanism $\mu(t_2)$, which is more efficient than the original one. In particular, this implies that $t_2 > \bar{t}$.

Finally, we must have $t_1 \geq \underline{t}$. Suppose that $t_1 < \underline{t}$. Fix t_2 of the original mechanism and choose $t_1' \in (t_1, \underline{t}]$. Consider the mechanism under which the types in $[0, t_1')$ trade with probability 1 while the types in $[t_1', t_2]$ trade with probability

$$x(t_1', t_2) = \frac{Y(t_1')}{Y(t_1') - Y(t_2)} > \frac{Y(t_1)}{Y(t_1) - Y(t_2)} = x(t_1, t_2),$$

where the inequality follows from $Y(t_1') > Y(t_1)$ and $Y(t_2) < 0$. Of course, the new mechanism is more efficient than the original one since the types in $[t_1, t_2]$ trade with a larger probability while the types outside this interval trade with the same probability as under the original mechanism.

We summarize our results:

Fact 5 *Let t_1 and t_2 denote the endpoints of the first two steps of the optimal mechanism. Then $t_1 \geq \underline{t}$, and $t_2 \geq \bar{t}$.*

We are now ready to provide a sufficient condition to implement the efficient mechanism in the bargaining game (when the players are sufficiently patient).

Condition 6 *For any $t \geq \bar{t}$,*

$$\int_{\underline{t}}^{t} (v(s) - c(t))\, ds \geq 0.$$

We now explain why the above condition is sufficient. Fix $0 < \tilde{t} \leq 1$, and consider the function $\varphi : \left[0, \tilde{t}\right] \to \mathbb{R}$ given by

$$\varphi(t) := \int_{t}^{\tilde{t}} \left(v(s) - c\left(\tilde{t}\right)\right) ds.$$

Under our assumptions, if $\varphi(t') \geq 0$ for some t', then $\varphi(t) > 0$ for every $t \in \left(t', \tilde{t}\right)$. Recall that the function v is increasing. Let t'' denote the value in $\left[0, \tilde{t}\right]$ such that $v(t'') = c\left(\tilde{t}\right)$ (let $t'' = \tilde{t}$ if $v\left(\tilde{t}\right) < c\left(\tilde{t}\right)$). The function φ is increasing on $[0, t'']$. By definition, φ is positive above t''.

Therefore, fix $t_2 \geq \bar{t}$. Our condition guarantees that for each $t_1 \in [\underline{t}, t_2]$,

$$\int_{t_1}^{t_2} (v(s) - c(t_2))\, ds \geq 0,$$

which implies the result, by Proposition 1.

Appendix E: Proof of Proposition 3 (Sketch)

This appendix sketches the proofs of the two harder statements in Proposition 3. We first show that the set of allocations in the buyer veto-incentive compatible program is the same whether or not one imposes *ex post* seller individual rationality. We then show that, as far as payoffs are concerned, the latter requirement can even be strengthened to seller veto-incentive compatibility. In both cases, for simplicity, we restrict attention to finite types. The extension to our set-up with a continuum of types follows by standard limiting arguments.

Lemma 9 *Assume that c and v are step functions with n steps such that $c_1 < c_2 < \cdots < c_N$, and (x, p) is an allocation that is implementable in the veto-incentive compatible program. Then there exists a measure μ that induces this allocation such that, for all $t \in T$, we have*

$$\int_{[0,c(t))} \mu(t)[1, dp] = 0.$$

Proof. Since (c, v) are step functions we can consider the model with N types in which the probability of each type is q_i. We write $\{\mu_i\}_{i=1}^{N}$ for the distribution of offers faced by type i.

Step 1: We divide the type space into 3 subsets:

$$\begin{aligned} T_1 &:= \{i \in \{1, \ldots, N\} : p_i > v_i\}, \\ T_2 &:= \{i \in \{1, \ldots, N\} : p_i < v_i\}, \\ T_3 &:= \{i \in \{1, \ldots, N\} : p_i = v_i\}. \end{aligned}$$

Step 2: For $k \leq j$, define

$$L_k^j := \sum_{i=k}^{j} q_i (x_i (v_i - p_i)).$$

Step 3: Notice that $L_0^N = B(0) \geq 0$, and let J^* be the lowest type i such that $L_0^i \geq 0$. Here we show how to construct an allocation satisfying the properties above for the special case that $J^* = N > 1$. The general proof considers a partition of the type space $\{1, \ldots, i_1\}, \{i_1 + 1, \ldots, i_2\}, \ldots, \{i_K + 1, \ldots, N\}$ and applies this procedure to each set separately.

Step 4: We will present an algorithm which delivers the desired result.

Step 4.1: Let k_1 be the smallest element in T_2.

There are 2 cases to consider:

Case 1:

$$q_1 x_1 (v_1 - p_1) + q_{k_1} x_{k_1} (v_{k_1} - p_{k_1}) < 0.$$

Case 2:

$$q_1 x_1 (v_1 - p_1) + q_{k_1} x_{k_1} (v_{k_1} - p_{k_1}) \geq 0.$$

Case 1: Notice that since $k_1 > 1$, we have $p_{k_1} \geq p_1$. From type k_1's individual rationality constraint, we have $p_{k_1} \geq c_{k_1}$. Also, there exists $\lambda \in (0,1)$ such that

$$\lambda q_1 x_1 (v_1 - p_1) + q_{k_1} x_{k_1} (v_{k_1} - p_{k_1}) = 0. \tag{18}$$

Next, notice that

$$p_1 = \alpha p_{k_1} + (1-\alpha) v_1, \tag{19}$$

for some $\alpha \in (0,1]$. Thus, applying (19) into (18) we have

$$0 = \lambda q_1 x_1 (1-\alpha)(v_1 - v_1) + \lambda q_1 x_1 \alpha (v_1 - p_{k_1}) + q_{k_1} x_{k_1} (v_{k_1} - p_{k_1}). \tag{20}$$

Next, we use (20) to show that $x = x^1 + \hat{x}^1$, where

$$x_i^1 := \begin{cases} \lambda x_1 \text{ if } i = 1, \\ x_{k_1} \text{ if } i = k_1, \\ 0 \text{ otherwise,} \end{cases}$$

and $\hat{x}^1 := x - x^1 \geq 0$. For the allocation (x^1, p), we construct a measure $\{\mu_i^1\}_{i=1}^N$ such that:

a. $\left(\int d\mu^1, \int p d\mu^1\right) = (x^1, p)$;

b. If $x_i^1 > 0$ then $\mu_i^1[0, c_i) = 0$.

For that, we define $\mu_i^1 := 0$ if $i \notin \{1, k_1\}$ and

$$\mu_1^1(\tilde{p}) := \begin{cases} \lambda x_1 \alpha \text{ if } \tilde{p} = p_{k_1} \\ \lambda x_1 (1-\alpha) \text{ if } \tilde{p} = v_1 \\ 0 \text{ otherwise} \end{cases} \quad \mu_{k_1}^1(\tilde{p}) := \begin{cases} x_{k_1} \text{ if } \tilde{p} = p_{k_1}, \\ 0 \text{ otherwise.} \end{cases}$$

Case 2: There exists $(\zeta, \gamma) \in (0,1] \times (0,1]$ such that

$$\begin{aligned} p_1 &= \zeta p_{k_1} + (1-\zeta) v_1, \\ 0 &= q_1 x_1 (1-\zeta)(v_1 - v_1) + q_1 x_1 \zeta (v_1 - p_{k_1}) + \gamma q_{k_1} x_{k_1} (v_{k_1} - p_{k_1}). \end{aligned}$$

Thus, we define

$$x_i^1 := \begin{cases} x_1 \text{ if } i = 1, \\ \gamma x_{k_1} \text{ if } i = k_1, \\ 0 \text{ otherwise,} \end{cases}$$

and $\hat{x}^1 := x - x^1 \geq 0$. For the allocation (x^1, p), we construct measures $\{\mu_i^1\}_{i=1}^N$ by setting $\mu_i^1 := 0$ if $i \notin \{1, k_1\}$ and

$$\mu_1^1(\tilde{p}) := \begin{cases} x_1 \zeta \text{ if } \tilde{p} = p_{k_1} \\ x_1 (1-\zeta) \text{ if } p = v_1 \\ 0 \text{ otherwise} \end{cases} \quad \mu_{k_1}^1(\tilde{p}) = \begin{cases} \gamma x_{k_1} \text{if } \tilde{p} = p_{k_1}, \\ 0 \text{ otherwise.} \end{cases}$$

Step 4.2: Assume that $x = \sum_{i=1}^M x^i + \hat{x}^M$. There are two possibilities:

Case i. $\big\{i \in \{1, \ldots, N\} : \hat{x}_i^M > 0\big\} \cap T_1 \neq \emptyset$.

Case ii. $\big\{i \in \{1, \ldots, N\} : \hat{x}_i^M > 0\big\} \subseteq T_2 \cup T_3$.

Assume that $\hat{x}_i^{M-1}$ is such that $\sum_{i=1}^N q_i \left(\hat{x}_i^{M-1} (v_i - p_i)\right) \geq 0$ and $\sum_{i=1}^J q_i \left(\hat{x}_i^{M-1} (v_i - p_i)\right) < 0$ if $J < N$. We claim:

Claim 7 *If **Step 4.1** is applied to* $\hat{x}_i^{M-1}$, $\hat{x}_i^{M-1} = x_i^M + \hat{x}_i^M$ *with* $\left\{i \in \{1, \ldots, N\} : \hat{x}_i^M > 0\right\} \cap T_1 \neq \emptyset$, *then* $\sum_{i=1}^N q_i \left(\hat{x}_i^M (v_i - p_i)\right) \geq 0$ *and* $\sum_{i=1}^J q_i \left(\hat{x}_i^M (v_i - p_i)\right) < 0$ *if* $J < N$.

Proof: The first conclusion follows since $\sum_{i=1}^N q_i \left(\hat{x}_i^{M-1} (v_i - p_i)\right) = \sum_{i=1}^N q_i \left(\hat{x}_i^M (v_i - p_i)\right)$. For the second, let k_{M-1} be the largest element of $\left\{i \in \{1, \ldots, N\} : \hat{x}_i^{M-1} > 0\right\} \cap T_2$. There are two possibilities:

a. $J < k_{M-1} \leq N$. In this case, the result is immediate.

b. $k_{M-1} \leq J < N$. In this case,

$$\begin{aligned}
0 &> \sum_{i \leq J} q_i \left(\hat{x}_i^{M-1} (v_i - p_i)\right) \\
&= \sum_{i \leq J} q_i \left(\hat{x}_i^{M-1} (v_i - p_i)\right) + \sum_{i \leq N} q_i \left(\left(\hat{x}_i^M - \hat{x}_i^{M-1}\right) (v_i - p_i)\right) \\
&= \sum_{i \leq J} q_i \left(\hat{x}_i^M (v_i - p_i)\right),
\end{aligned}$$

where we used the fact that $k_{M-1} \leq J$ implies

$$0 = \sum_{i \leq N} q_i \left(\left(\hat{x}_i^M - \hat{x}_i^{M-1}\right) (v_i - p_i)\right) = \sum_{i \leq J} q_i \left(\left(\hat{x}_i^M - \hat{x}_i^{M-1}\right) (v_i - p_i)\right).$$

■

From Claim 7, we can apply **Step 4.1** into $\hat{x}_i^M$ to obtain x^{M+1} and $\hat{x}^{M+1}$ and $\left\{\mu_i^{M+1}\right\}_{i=1}^N$ such that:

a'. $\left(\int d\mu^{M+1}, \int p d\mu^{M+1}\right) = (x^{M+1}, p)$;

b'. If $x_i^{M+1} > 0$ then $\mu_i^{M+1}[0, c_i) = 0$.

Notice that this procedure can take (at most) $N-1$ rounds. In order to complete the Lemma we move to **Case ii.**

Case ii: In this case, define $\left\{\mu_i^{M+1}\right\}_{i=1}^N$ by:

$$\mu_i^{M+1}(\tilde{p}) := \begin{cases} \hat{x}_i^M \text{ if } \tilde{p} = p_i \\ 0 \text{ otherwise.} \end{cases}$$

Step 5: Assume the algorithm described in **Step 4.1** and **Step 4.2** was applied to the allocation x such that $x = \sum_{j=1}^{K} x^j + \hat{x}^k$. Thus it is straightforward to verify that the measure $\{\mu_i\}_{i=1}^N$ defined by $\mu_i(\tilde{p}) := \sum_{j=1}^{K+1} \mu_i^j(\tilde{p})$ is such that $(x, p) = \left(\int d\mu, \int p d\mu\right)$ and $\mu_i[0, c_i) = 0$. This completes the proof. ■

We now turn to the other nontrivial claim: seller veto-incentive compatibility does not restrict the set of payoffs that can be achieved in the buyer veto-incentive compatible program. Here as well, attention is restricted to finite types.

Lemma 10 *Assume that the type space is finite and let $\left(\pi^B, \pi^S\right)$ be a vertex of the payoff frontier achieved in the (buyer) veto-incentive compatible program. There exists a seller veto-incentive compatible measure $\mu = \{\mu_i\}_{i=1}^N$ that achieves this payoff.*

Proof. Assume that there are N types.[30] It can be shown that if $\left(\pi^B, \pi^S\right)$ is a vertex of the payoff frontier then it is achieved by an allocation (x, p) for which there exists a partition of the type space: $\{\mathcal{P}_j\}_{j=1}^K$ with $\mathcal{P}_1 = \{1, \ldots, i_1\}$ and $\mathcal{P}_j = \{i_{j-1} + 1, \ldots, i_j\}$, with $i_K \geq 1$ such that:[31]

i. If $j < K$, then if $i, i' \in \mathcal{P}_j$ we have $p_i = p_{i'} = \mathbb{E}[v \mid \mathcal{P}_j]$.

ii. If $j = K$, then we have either a. or b. below:

  a. $(p_i, x_i) = (p_N, x_N)$ for all $i \in \mathcal{P}_K$;

[30] For simplicity of exposition we assume that all types trade with positive probability.

[31] A proof is available upon request.

b. $\mathcal{P}_K = I_1 \cup I_2$ where $I_1 = \{i_{k-1}+1, \ldots, i_l\}$ and $I_2 = \{i_l + 1, \ldots, N\}$ with $i_{k-1} \leq i_l < N$ is such that $(p_i, x_i) = (p', x')$ if $i \in I_1$ and $(p_i, x_i) = (p'', x'')$ if $i \in I_2$ with $c_{i_l} \leq \mathbb{E}[v \mid i \in I_1]$ and $p' < p''$.

Here, we prove the more challenging case b.

Step 1: Define μ_i for $i \notin \mathcal{P}_K$ by:

$$\mu_i(\tilde{p}) := \begin{cases} x_i \text{ if } \tilde{p} = p_i, \\ 0 \text{ otherwise.} \end{cases}$$

Step 2: To define μ_i for $i \in \mathcal{P}_K$, there are two cases to consider:

Case 1: $p' \leq \mathbb{E}[v \mid i \in I_1]$.

In this case we let

$$\mu_i(\tilde{p}) := \begin{cases} x' \text{ if } \tilde{p} = p' \text{ and } i \in I_1 \\ 0 \text{ if } \tilde{p} \neq p' \text{ and } i \in I_1 \end{cases} \quad \mu_i(\tilde{p}) = \begin{cases} x'' \text{ if } \tilde{p} = p'' \text{ and } i \in I_2, \\ 0 \text{ if } \tilde{p} \neq p'' \text{ and } i \in I_2. \end{cases}$$

It is straightforward to check that μ is veto-incentive compatible for the seller.

Case 2: $p' > \mathbb{E}[v \mid i \in I_1]$.

In this case, notice that since the allocation is incentive compatible we must have

$$B_{i_{k-1}+1} = \sum_{i \geq i_{k-1}+1} q_i x_i (v_i - p_i) \geq 0. \tag{21}$$

Furthermore, because $p' \in (\mathbb{E}[v \mid i \in I_1], p'')$, there exists $\alpha \in (0,1)$ such that

$$p' = \alpha \mathbb{E}[v \mid i \in I_1] + (1-\alpha) p''. \tag{22}$$

Thus, notice that from (21) and (22),

$$
\begin{aligned}
0 &\leq \sum_{i\in I_1} q_i x_i \left(v_i - p_i\right) + \sum_{i\in I_2} q_i x_i \left(v_i - p''\right) \\
&= \sum_{i\in I_1} \alpha q_i x_i \left(v_i - \mathbb{E}\left[v \mid i \in I_1\right]\right) \\
&\quad + \sum_{i\in I_1} \left(1-\alpha\right) q_i x_i \left(v_i - p''\right) + \sum_{i\in I_2} q_i x_i \left(v_i - p''\right).
\end{aligned}
$$

Thus, $\sum_{i\in I_1} \left(1-\alpha\right) q_i x_i \left(v_i - p''\right) + \sum_{i\in I_2} q_i x_i \left(v_i - p''\right) = B_{i_{k-1}+1} \geq 0$.

Therefore, we define μ_i by

$$
\mu_i\left(\tilde{p}\right) := \begin{cases} \alpha x' \text{ if } \tilde{p} = \mathbb{E}\left[v \mid i \in I_1\right] \text{ and } i \in I_1 \\ \left(1-\alpha\right) x' \text{ if } \tilde{p} = p'' \text{ and } i \in I_1 \\ 0 \text{ if } \tilde{p} \notin \{p', p''\} \text{ and } i \in I_1 \end{cases} \qquad \mu_i\left(\tilde{p}\right) := \begin{cases} x'' \text{ if } \tilde{p} = p'' \text{ and } i \in I_2, \\ 0 \text{ if } \tilde{p} \neq p'' \text{ and } i \in I_2. \end{cases}
$$

It is straightforward to verify that the allocation constructed is veto-incentive compatible for the seller. This completes the proof. ■

Appendix F: Proof of Proposition 2

Suppose towards a contradiction that there exists an equilibrium allocation that violates veto-incentive compatibility. Therefore, there exists $\eta > 0$ and $\tilde{t} \in (0,1)$ such that

$$
\int_{\tilde{t}}^{1} \sum_{n=0}^{\infty} \delta_n \mathbb{E}_{\sigma^*}\left[\left(v(s) - p(s)\right) \mathbf{1}_{\xi_n} \mid s\right] ds \leq -\eta, \tag{23}
$$

where ξ_n is defined as the event in which the object being sold in period n and $\mathbf{1}_{\xi_n}$ is its indicator function. Consider a typical history in which an offer is made at n, $\tilde{h}^n$. History $\tilde{h}^n$ includes: i) all previous offers (as well as the identity of the proposer) before period n; ii) the player who makes an offer at n; iii) the offer made at period n. Given an on-path history $\tilde{h}^n$, we let $\mu_{\tilde{h}^n}$ be the associated distribution of types. Since $\lim_{n\to\infty}\delta_n = 0$ there exists $N \geq 1$ such that for all $n \geq N$ we have:

$$\int_{\tilde{t}}^{1} \sum_{n=N}^{\infty} \delta_n \mathbb{E}_{\sigma^*}\left[(v(s) - p(s))\,\mathbf{1}_{\xi_n} \mid s\right] d\mu_{\tilde{h}^n} > -\eta. \tag{24}$$

Therefore, let N^* be the largest integer for which there is an on-path history $\tilde{h}^{N^*}$ such that

$$\int_{\tilde{t}}^{1} \sum_{n=N^*}^{\infty} \delta_n \mathbb{E}_{\sigma^*}\left[(v(s) - p(s))\,\mathbf{1}_{\xi_n} \mid s\right] d\mu_{\tilde{h}^{N^*}} \leq -\eta, \tag{25}$$

and consider a history $\tilde{h}^{N^*}$ satisfying (25). Let p be the offer made by the seller at $\tilde{h}^{N^*}$ and notice that from the definition of N^* we have

$$\int_{\tilde{t}}^{1} \delta_n \mathbb{E}_{\sigma^*}\left[(v(s) - p)\,\mathbf{1}_{\xi_n} \mid s\right] d\mu_{\tilde{h}^{N^*}} < 0. \tag{26}$$

There are two cases:

Case 1: *The seller is selected to make an offer in period T at $\tilde{h}^{N^*}$.*

From (26) the buyer accepts such an offer with positive probability. Since $v(\cdot)$ is an increasing function, (26) implies

$$\int_{0}^{1} \delta_n \mathbb{E}_{\sigma^*}\left[(v(s) - p)\,\mathbf{1}_{\xi_n} \mid s\right] d\mu_{\tilde{h}^{N^*}} < 0,$$

which shows that the buyer could have profitably deviated by rejecting p, offering 0 in every future period and rejecting every future offer.

Case 2: *The buyer is selected to make an offer in period n at $\tilde{h}^{N^*}$.*

Let $A \subset [0, \tilde{t}]$ be the set of types who accept this offer with probability 1. There are two possibilities.

Possibility 1: $\mu_{\tilde{h}^{N^*}}(A) = \mu_{\tilde{h}^{N^*}}([0, \tilde{t}])$.

In this case, the expected payoff of the buyer at $\tilde{h}^{N^*}$ is:

$$
\begin{aligned}
& \int_0^{\tilde{t}} \sum_{n=N^*}^{\infty} \delta_n \mathbb{E}_{\sigma^*} \left[\left(v(s) - p(s) \right) \mathbf{1}_{\xi_n} \mid s \right] d\mu_{\tilde{h}^{N^*}} + \int_{\tilde{t}}^1 \sum_{n=N^*}^{\infty} \delta_n \mathbb{E}_{\sigma^*} \left[\left(v(s) - p(s) \right) \mathbf{1}_{\xi_n} \mid s \right] d\mu_{\tilde{h}^{N^*}} \\
= \; & \int_0^{\tilde{t}} \delta_n \mathbb{E}_{\sigma^*} \left[\left(v(s) - p(s) \right) \mathbf{1}_{\xi_n} \mid s \right] d\mu_{\tilde{h}^{N^*}} + \int_{\tilde{t}}^1 \sum_{n=N^*}^{\infty} \delta_n \mathbb{E}_{\sigma^*} \left[\left(v(s) - p(s) \right) \mathbf{1}_{\xi_n} \mid s \right] d\mu_{\tilde{h}^{N^*}} \\
\leq \; & \int_{\tilde{t}}^1 \sum_{n=N^*}^{\infty} \delta_n \mathbb{E}_{\sigma^*} \left[\left(v(s) - p(s) \right) \mathbf{1}_{\xi_n} \mid s \right] d\mu_{\tilde{h}^{N^*}} < 0, \qquad (27)
\end{aligned}
$$

where we have used the fact that $v(\cdot)$ is monotonic to conclude from (25) that the first term in the second line of (27) is nonpositive. Thus the buyer obtains a negative continuation payoff at $\tilde{h}^{N^*}$, a contradiction.

Possibility 2: $\mu_{\tilde{h}^{N^*}}(A) < \mu_{\tilde{h}^{N^*}}([0,\tilde{t}])$.

In this case, the types $[0,\tilde{t}] \setminus A$ reject the offer. From (25) we know that a positive measure of seller's types in $[0,\tilde{t}]$ accepts this offer. Thus since c is monotonic we conclude that all types in $[0,\tilde{t}] \setminus A$ are indifferent between accepting this offer or not. One can easily show that if all types $[0,\tilde{t}] \setminus A$ were to accept this offer the buyer would be weakly better off. Therefore, it follows from (27) that the buyer obtains a negative continuation payoff at $\tilde{h}^{N^*}$, a contradiction.

Appendix G: Details for Samuelson's Example 1

Notice that $\mathbb{E}(v) = \frac{1}{2}k + \Delta$. If $k \geq 2$, then $E(v) \geq 1 = c(1)$ for every $\Delta \geq 0$. In words, for any $k \geq 2$ and every $\Delta \geq 0$, the first best is implementable in the veto IC program.

Similarly, if $k \in [0,2)$ and $\Delta \geq 1 - \frac{1}{2}k$, then the first best is implementable in the veto IC program.

In what follows, let us restrict attention to the set of pairs (k,Δ) with $0 \leq k < 2$ and

$$
\max\{0, 1-k\} < \Delta < 1 - \frac{1}{2}k.
$$

Consider the function

$$g(k) = \frac{4}{4-k} - k.$$

Notice that g is strictly decreasing in $[0, 2)$ and

$$\max\{0, 1-k\} < g(k) < 1 - \frac{1}{2}k$$

for every $k \in (0, 2)$ (the three quantities coincide for $k = 0$).

Claim 8 *Fix $k \in (0, 2)$. If $\Delta \in \left[g(k), 1 - \frac{1}{2}k\right)$, then condition (9) is satisfied.*

This means that if $\Delta \geq g(k)$, then the most efficient outcome of the full commitment program is implementable in the veto IC program (recall that if $\Delta \geq 1 - \frac{1}{2}k$, then the first best is implementable). When Δ belongs to the nonempty set $\left[g(k), 1 - \frac{1}{2}k\right)$, the first best is not implementable in the full commitment program. However, the second best is implementable in the veto IC program.

Proof of the Claim

The function $Y(t)$ is given by

$$Y(t) = \int_0^t (ks + \Delta - t)\, ds = \frac{1}{2}t(2\Delta - 2t + kt).$$

Let $\underline{t}$ and $\bar{t}$ denote the smallest local maximizer and the smallest strictly positive root of Y, respectively. We have

$$\underline{t} = \tfrac{\Delta}{2-k}, \quad \bar{t} = \tfrac{2\Delta}{2-k}.$$

Condition (9) becomes: For every $t \geq \frac{2\Delta}{2-k}$,

$$\begin{aligned} Z(t) &= \int_{\frac{\Delta}{2-k}}^{t} (ks + \Delta - t)\, ds \\ &= \frac{1}{2(k-2)^2} \left(k^3 t^2 - 6k^2 t^2 + 2k^2 t\Delta + 12kt^2 - 10kt\Delta + k\Delta^2 - 8t^2 + 12t\Delta - 4\Delta^2 \right) \geq 0. \end{aligned}$$

For every $k \in [0,2)$, Z is concave in t. Therefore, it is enough to check that Condition (9) holds at the extremes, $\frac{2\Delta}{2-k}$ and 1. We have

$$Z\left(\tfrac{2\Delta}{2-k}\right) = \tfrac{1}{2} k \tfrac{\Delta^2}{(k-2)^2},$$
$$Z(1) = \tfrac{1}{2(k-2)^2} \left(k^3 + 2k^2\Delta - 6k^2 + k\Delta^2 - 10k\Delta + 12k - 4\Delta^2 + 12\Delta - 8\right).$$

For every $k \in [0,2)$, $Z(1)$ is concave in Δ. Consider the expression

$$\left(k^3 + 2k^2\Delta - 6k^2 + k\Delta^2 - 10k\Delta + 12k - 4\Delta^2 + 12\Delta - 8\right).$$

The roots are

$$g(k) = \tfrac{4}{4-k} - k, \quad 2 - k.$$

Therefore, if $\Delta \in \left[g(k), 1 - \frac{1}{2}k\right)$, then $Z(1) \geq 0$ and Condition (9) is satisfied.

Appendix H: Markov Equilibria

In this appendix, we show that, at least in the case of finitely (but arbitrarily) many types, restricting attention to Markov perfect equilibria does not restrict the set of limit equilibrium payoffs that we characterize in the bargaining game. A *Markov* strategy is a strategy that only depends on the (public) belief about the seller's type. An equilibrium is Markov perfect if all specified strategies are Markov.

We assume that there are N types: $T := \{t_1, \ldots, t_N\}$ and that c and v are strictly monotone in t. Let q_i be the probability of type t_i.

We claim that any regular allocation (x,p) can be approximately implemented by Markov strategies when the players are patient. In a regular allocation, there is a monotone partition of the type space $\{\mathcal{T}_1,\dots,\mathcal{T}_M\}$. We focus on the case in which all types trade with positive probability and in which $M>3$ (the other cases are analogous). For every $k\in\{1,\dots,M\}$, types $t\in\mathcal{T}_k$ trade with probability x_k at a price p_k. Notice that $1=x_1>\dots>x_M$ and $v(t_1)\le p_1<\dots<p_M$. For every $k\in\{1,\dots,M\}$, let $i(k):=\min\{i: t_i\in\mathcal{T}_k\}$ and $j(k):=\max\{i: t_i\in\mathcal{T}_k\}$. Define $\underline{t}^k=t_{i(k)}$ and $\bar{t}^k:=t_{j(k)}$. We have: $B(\underline{t}^1)\ge 0$, $B(\underline{t}^k)=0$ for every $k<M$ and $B(\underline{t}^M)>0$. Furthermore, all local incentive constraints bind.

For each $k\in\{1,\dots,M\}$ define

$$v\left(\mathcal{T}_k\right):=\frac{\sum_{t\in\mathcal{T}_k} q_i v(t_i)}{\sum_{t\in\mathcal{T}_k} q_i}.$$

Take $\varepsilon>0$. For (close enough to one) δ we specify a Markov equilibrium that implements the allocation (x^δ,p^δ). The family of allocations (x^δ,p^δ) satisfy $\lim_{\delta\uparrow 1}\left\|(x^\delta,p^\delta)-(x,p)\right\|<\varepsilon$.

Step 1: *Defining an implementable allocation (x',p') close to (x,p) which satisfies additional properties.*

Using the fact that c is strictly monotone and that all local incentive constraints bind in (x,p), it is straightforward to construct an allocation (x',p') such that:

a) Every type $t\in\mathcal{T}_k$ trades with the same probability and at the same price. The allocation is monotonic and satisfies $x'_1=1$ and $p_1\ge v(t_1)$.

b) $B(t^1)\ge 0$, $B(t^k)=0$ for every $k<M$ and $B(t^M)>0$.

c) For every $k\in\{1,\dots,M-3\}$, the type $\bar{t}^k$ strictly prefers (x'_k,p'_k) to (x'_{k+1},p'_{k+1}).

d) For every $k\in\{M-2,M-1\}$, the type $\bar{t}^k$ is indifferent between (x'_k,p'_k) to (x'_{k+1},p'_{k+1}).

e) Type t_M obtains a strictly positive payoff: $x_M(p_M-c(t_M))>0$.

f) It holds that $\|(x',p')-(x,p)\|<\frac{\varepsilon}{2}$.

With some abuse of notation we assume that the original allocation (x,p) satisfies a)–f).

Step 2: *Constructing a Markov equilibrium.*

The construction is divided into steps (1)–(7).

(1) All types $t \in \mathcal{T}_1$ make an offer p_1 at $n = 0$. The buyer accepts this offer with probability 1.

(2) Consider types $t \in \mathcal{T}_k$ for $1 < k < M-2$. All such types offer p_k in every period $n \geq 0$. The buyer randomizes and accepts this offer with probability ψ_k^δ in each period. We set ψ_k^δ such that:

$$x_k = \psi_k^\delta + \delta(1-\psi_k^\delta)\psi_k^\delta + \cdots = \left(\frac{\psi_k^\delta}{1-\delta(1-\psi_k^\delta)}\right). \tag{28}$$

(3) Consider types $t \in \mathcal{T}_{M-2}$. For each small $\eta > 0$, let

$$p^{M-2}(\eta) := \frac{\sum_{t_i \in \mathcal{T}_{M-2} \setminus \{\bar{t}^{M-2}\}} q_i v(t_i) + q_{j(M-2)} v(\bar{t}^{M-2})(1-\eta)}{\sum_{t_i \in \mathcal{T}_{M-2} \setminus \{\bar{t}^{M-2}\}} q_i + q_{j(M-2)}(1-\eta)}. \tag{29}$$

That is, assume that a measure $\eta q_{j(M-2)}$ of the type $\bar{t}^{M-2}$ "leaves" the partition $\mathcal{T}_{M-2}$. Thus, $p^{M-2}(\eta)$ is the expected value to the buyer from this new set. Notice that since $M > 3$, we have $B(\underline{t}^{M-2}) = B(\underline{t}^{M-1}) = 0$ and hence $p_{M-2} = v(\mathcal{T}_{M-2})$. Consequently, we have $\lim_{\eta \to 0} p^{M-2}(\eta) = p^{M-2}$.

All types $t_i \in \mathcal{T}_{M-2} \setminus \{\bar{t}^{M-2}\}$ offer $p^{M-2}(\eta)$ in every period $n \geq 0$. The type $\bar{t}^{M-2}$ randomizes. With probability $(1-\eta)$, he "joins" this partition and offers $p^{M-2}(\eta)$ in every period $n \geq 0$. With complementary probability, his behavior is determined by point (4) below. The probability that the offer $p^{M-2}(\eta)$ is accepted in each period, ψ_{M-2}^δ, is set to yield the payoff $x_{M-2}(p_{M-2} - c(\bar{t}^{M-2}))$ to type $\bar{t}^{M-2}$:

$$x_{M-2}(p_{M-2} - c(\bar{t}^{M-2})) = \left(\frac{\psi_{M-2}^\delta}{1-\delta(1-\psi_{M-2}^\delta)}\right)\left(p^{M-2}(\eta) - c(\bar{t}^{M-2})\right). \tag{30}$$

For future reference, let $\bar{\psi}^{\delta}_{M-2}$ solve:

$$x_{M-2}(p_{M-2} - c(\bar{t}^{M-2})) = \left(\frac{\bar{\psi}^{\delta}_{M-2}}{1-\delta(1-\bar{\psi}^{\delta}_{M-2})}\right)\left(v(\bar{t}^{M-2}) - c(\bar{t}^{M-2})\right), \tag{31}$$

and notice that $\bar{\psi}^{\delta}_{M-2} < \psi^{\delta}_{M-2}$ if and only if $\mathcal{T}_{M-2} \setminus \{\bar{t}^{M-2}\} \neq \emptyset$.

(4) Next, we specify the behavior of the remaining types. Remember that $B(\underline{t}^{M-1}) = 0$ and $B(\underline{t}^{M}) > 0$ and hence $p_{M-1} > v(\mathcal{T}_{M-1})$. Therefore, there is a unique $\beta \in (0,1)$ such that:

$$x_{M-1}p_{M-1} = \beta x_{M-1}v(\mathcal{T}_{M-1}) + (1-\beta)x_{M-1}v(\mathcal{T}_{M-1} \cup \mathcal{T}_{M}). \tag{32}$$

In period $n = 0$, all types $t \in \mathcal{T}_{M-2} \cup \mathcal{T}_{M-1}$ together with a measure $\left(\frac{\eta}{2}\right)$ of type $\bar{t}^{M-2}$ offer

$$p_0^{M-1,M-2} := \frac{\sum_{t_i \in \mathcal{T}_{M-2}\cup\mathcal{T}_{M-1}} q_i v(t_i) + q_{j(M-2)}v(\bar{t}^{M-2})\left(\frac{\eta}{2}\right)}{\sum_{t_i \in \mathcal{T}_{M-2}\cup\mathcal{T}_{M-1}} q_i + q_{j(M-2)}\left(\frac{\eta}{2}\right)}. \tag{33}$$

The buyer accepts this offer with probability ψ^{δ}_0. This probability is set such that the type $\bar{t}^{M-2}$ is indifferent between:

a) Offering $v(\bar{t}^{M-2})$ in every future period. In this case, the buyer randomizes and accepts this offer with probability $\bar{\psi}^{\delta}_{M-2}$ (see (31)) the type $\bar{t}^{M-2}$ obtains a payoff $x_{M-2}(p_{M-2} - c(\bar{t}^{M-2}))$.

b) Offering $p_0^{M-1,M-2}$ for one period. If the buyer rejects it (which happens with probability $(1-\psi^{\delta}_0)$) the seller reverts to strategy a) in the next period.

A measure $\left(\frac{\eta}{2}\right)$ of types $\bar{t}^{M-2}$ chooses b).

Inductively, we define

$$p_n^{M-1,M-2} := \frac{\sum_{t_i \in \mathcal{T}_{M-2}\cup\mathcal{T}_{M-1}} q_i v(t_i) + q_{j(M-2)}v(\bar{t}^{M-2})\left(\frac{\eta}{2^{n+1}}\right)}{\sum_{t_i \in \mathcal{T}_{M-2}\cup\mathcal{T}_{M-1}} q_i + q_{j(M-2)}\left(\frac{\eta}{2^{n+1}}\right)}, \tag{34}$$

and define ψ^{δ}_n analogously for all $n \leq n^*$ (n^* is defined below). Notice that the total discounted probability that the buyer purchases the good in periods $n = 0, \dots, n^*$ is:

$$X^\delta_{n^*} := \psi^\delta_0 + \delta(1-\psi^\delta_0)\psi^\delta_1 + \cdots + \delta^{n^*}\left[\prod_{n=0}^{n^*-1}(1-\psi^\delta_n)\right]\psi^\delta_{n^*}. \tag{35}$$

Next, define n^* as the minimum integer n such that $X^\delta_n \geq (1-\beta)x_{M-1}$ (see (32)). It is straightforward to show that the difference $\left|X^\delta_{n^*} - (1-\beta)x_{M-1}\right| \to 0$ as $\delta \uparrow 1$ and $\eta \to 0$. Below we define the continuation behavior after n^*.

The type $\bar{t}^{M-2}$ offers $v(\bar{t}^{M-2})$ in every future period (and this offer is accepted with probability $\bar{\psi}^\delta_{M-2}$).

All types $t \in \mathcal{T}_{M-1}$ offer $v(\mathcal{T}_{M-1})$. This offer is accepted with constant probability ψ^δ_{M-1} in every future period. ψ^δ_{M-1} is set such that type $\bar{t}^{M-2}$ is indifferent:

$$\left(\frac{\bar{\psi}^\delta_{M-2}}{1-\delta(1-\bar{\psi}^\delta_{M-2})}\right)\left(v(\bar{t}^{M-2}) - c(\bar{t}^{M-2})\right) = \left(\frac{\psi^\delta_{M-1}}{1-\delta(1-\psi^\delta_{M-1})}\right)\left(v(\mathcal{T}_{M-1}) - c(\bar{t}^{M-2})\right). \tag{36}$$

All types $t \in \mathcal{T}_M$ offer $v(\mathcal{T}_M)$. This offer is accepted with constant probability ψ^δ_M in every future period. ψ^δ_M is set such that type $\bar{t}^{M-1}$ is indifferent:

$$\left(\frac{\psi^\delta_{M-1}}{1-\delta(1-\psi^\delta_{M-1})}\right)\left(v(\mathcal{T}_{M-1}) - c(\bar{t}^{M-1})\right) = \left(\frac{\psi^\delta_{M}}{1-\delta(1-\psi^\delta_{M})}\right)\left(v(\mathcal{T}_{M}) - c(\bar{t}^{M-2})\right). \tag{37}$$

Next, we explain in (5) why the induced allocation (x^δ, p^δ) is close to (x, p) when η is small and δ large. Then we verify in (6) that the induced allocation is indeed an equilibrium when when η is small and δ large. In (7) we show that the equilibrium is Markov.

(5) It is clear that for all $k < M-1$ all $t \in \mathcal{T}_k$ trade with probability $x^\delta(t)$ close to $x(t)$ and at a price $p^\delta(t)$ close to $p(t)$ (when η is small and δ large). Next, we consider types $t \in \mathcal{T}_{M-1}$. From (36) the type $\bar{t}^{M-2}$ is indifferent between the allocation $(x^\delta(t), p^\delta(t))$ and always imitating

types $t \in \mathcal{T}_{M-1}$. Therefore, using (30) and (36) we obtain:

$$\begin{aligned} & x_{M-2}(p_{M-2} - c(\bar{t}^{M-2})) \\ = \;& \psi_0^\delta \left(p_0^{M-1,M-2} - c(\bar{t}^{M-2})\right) + \cdots + \delta^{n^*} \left[\prod_{n=0}^{n^*-1}(1-\psi_n^\delta)\right] \psi_{n^*}^\delta \left(p_{n^*}^{M-1,M-2} - c(\bar{t}^{M-2})\right) \\ & + \delta^{n^*+1}\left(\prod_{n=0}^{n^*}(1-\psi_n^\delta)\right)\left(\frac{\psi_{M-1}^\delta}{1-\delta(1-\psi_{M-1}^\delta)}\right)\left(v(\mathcal{T}_{M-1}) - c(\bar{t}^{M-2})\right). \end{aligned} \tag{38}$$

Next, notice that as $\eta \to 0$ we have $p_{n^*}^{M-1,M-2} \to v(\mathcal{T}_{M-1} \cup \mathcal{T}_M)$ and as $\delta \uparrow 1$ we have $\left|X_{n^*}^\delta - (1-\beta)x_{M-1}\right| \to 0$. Therefore, for any $\kappa > 0$ we can find $\eta_1 > 0$ and $\delta_1 \in (0,1)$ such that whenever $\eta < \eta_1$and $\delta > \delta_1$ we have:

$$\left| \begin{array}{c} \psi_0^\delta \left(p_0^{M-1,M-2} - c(\bar{t}^{M-2})\right) + \cdots + \delta^{n^*} \left[\prod_{n=0}^{n^*-1}(1-\psi_n^\delta)\right] \psi_{n^*}^\delta \left(p_{n^*}^{M-1,M-2} - c(\bar{t}^{M-2})\right) \\ -(1-\beta)x_{M-1}v(\mathcal{T}_{M-1}\cup\mathcal{T}_M) \end{array} \right| < \kappa.$$

Thus, for such parameters we have

$$\begin{aligned} & x_{M-2}(p_{M-2} - c(\bar{t}^{M-2})) \\ = \;& (1-\beta)x_{M-1}(v(\mathcal{T}_{M-1}\cup\mathcal{T}_M) - c(\bar{t}^{M-2})) \\ & + \delta^{n^*+1}\left(\prod_{n=0}^{n^*}(1-\psi_n^\delta)\right)\left(\frac{\psi_{M-1}^\delta}{1-\delta(1-\psi_{M-1}^\delta)}\right)\left(v(\mathcal{T}_{M-1}) - c(\bar{t}^{M-2})\right) + z^\delta, \end{aligned} \tag{39}$$

where $z^\delta \leq |\kappa|$. Next, using (32) and the fact that type $\bar{t}^{M-2}$ is indifferent between the allocations (x_{M-2}, p_{M-2}) and (x_{M-1}, p_{M-1}), we have

$$\begin{aligned} & x_{M-2}(p_{M-2} - c(\bar{t}^{M-2})) \\ = \;& \beta x_{M-1}(v(\mathcal{T}_{M-1}) - c(\bar{t}^{M-2})) + (1-\beta)\, x_{M-1}(v(\mathcal{T}_{M-1}\cup\mathcal{T}_M) - c(\bar{t}^{M-2})). \end{aligned} \tag{40}$$

From (39) and (40) we immediately have:

$$\begin{aligned}
&\beta x_{M-1}(v(\mathcal{T}_{M-1}) - c(\bar{t}^{M-2})) \\
= \;& \delta^{n^*+1} \left(\prod_{n=0}^{n^*} (1-\psi_n^\delta) \right) \left(\frac{\psi_{M-1}^\delta}{1-\delta(1-\psi_{M-1}^\delta)} \right) \left(v(\mathcal{T}_{M-1}) - c(\bar{t}^{M-2}) \right) + z^\delta,
\end{aligned}$$

which implies that $\delta^{n^*+1} \left(\prod_{n=0}^{n^*} (1-\psi_n^\delta) \right) \left(\frac{\psi_{M-1}^\delta}{1-\delta(1-\psi_{M-1}^\delta)} \right) \to \beta x_{M-1}$. Using a similar argument we conclude that for all $t \in \mathcal{T}_M$ the allocation $\left(x^\delta(t), p^\delta(t) \right)$ can be made as close as we want to $(x(t), p(t))$ by taking η is small and δ close to 1.

(6) We show that the induced allocation is indeed an equilibrium.

We start defining the off-path behavior. First, consider a deviation by the buyer. The only off-path action is to reject the offer p_1 at $n = 0$ (as the buyer should randomize over all other offers). If the buyer rejects p_1 at $n = 0$, the equilibrium specifies that the buyer does not update his belief and accepts the same offer with probability 1 in the next period. Following this deviation, the equilibrium prescribes that the seller makes the same offer in the subsequent period. If the seller deviates and does not offer p_1, the continuation equilibrium is the same as the one triggered by an off-path offer made by the seller (see below).

Now, consider an off-path deviation by the seller. First, assume that the seller makes an off-path offer. In this case, we impose that the buyer puts probability 1 on the seller being type t_1 and never revises his belief again. The buyer accepts any future offer p if and only if $p \leq v(t_1)$. Type t_i offers $v(t_1)$ if $(v(t_1) - c(t_i)) \geq 0$ and $v(t_M) + 1$ otherwise. Therefore, he guarantees a payoff $[v(t_1) - c(t_i)]^+$. We postpone the description of the seller's continuation strategy after a deviation to an offer which is made on the equilibrium path to the end of (6).

Now, we show that the buyer does not have a profitable deviation. Notice that since $B(\underline{t}^1) \geq 0$ and $B(\underline{t}^2) = 0$ we have $v(\mathcal{T}_1) \geq p_1$ and hence the buyer would never profit by rejecting the offer p_1. Next, notice that the buyer obtains zero payoffs from all other (on-path) offers and hence he cannot profitably deviate by accepting or rejecting any such offer with probability 1. Finally, consider the buyer off-path behavior induced by a seller deviation. In this case, the buyer puts probability 1 on the seller having type t_1. Since no offer lower than $v(t_1)$ will ever be made, it is evident that the strategy of accepting an offer p if and only if $p \leq v(t_1)$ is optimal.

Let us now show that the seller has no profitable deviation.

First, consider a seller with a type $t < \bar{t}^{M-2}$ and assume that $t \in \mathcal{T}_k$. Let us first contemplate the deviation to some offer $p_j \in \{p_1, \ldots, p_{M-3}\} \setminus \{p_k\}$ in the first period. Notice that since the buyer's acceptance rate is constant we may (w.l.o.g.) assume that the seller offers p_j in every subsequent period. Since the allocation (x, p) is monotonic and incentive compatible, (28) implies that there is no profitable deviation.

Next, let us consider a deviation to the offer $p_{M-2}(\eta)$ at $n = 0$ (thus assume implicitly that $k < M-2$). Notice that we have assumed in (c) in Step 1 that type $\bar{t}^{M-3}$ strictly prefers (x_{M-3}, p_{M-3}) to (x_{M-2}, p_{M-2}). Notice that the allocation $(x^{\delta}_{M-2}, p_{M-2}(\eta))$ approaches (x_{M-2}, p_{M-2}) as $\eta \downarrow 0$. Therefore, offering $p_{M-2}(\eta)$ at every $n \geq 0$ is strictly dominated by following the equilibrium strategy.

Finally, let us contemplate the deviation to some offer $p \in \left\{v(\bar{t}^{M-2}), p_0^{M-1,M-2}\right\}$ in the first period. Assume that type t seller deviates by pooling with types $t \in \mathcal{T}_{M-1} \cup \mathcal{T}_M$ until period $n \leq n^*$.

First, assume that $n = n^*$. At time $n^* + 1$ one of the following 4 options is a best-response for type t : i) Offer $v(t_1)$ in every $n \geq n^* + 1$ with the buyer accepting this offer with constant probability 1 in each future period; ii) Offer $v(\bar{t}^{M-2})$ in every $n \geq n^* + 1$ with the buyer accepting this offer with constant probability $\bar{\psi}^{\delta}_{M-2}$ in each future period; iii) Offer $v(\mathcal{T}_{M-1})$ in every $n \geq n^* + 1$ with the buyer accepting this offer with constant probability ψ^{δ}_{M-1} in each future period; iv) Offer $v(\mathcal{T}_M)$ in every $n \geq n^* + 1$ with the buyer accepting this offer with constant probability ψ^{δ}_M in each future period. By construction type $\bar{t}^{M-1}$ is indifferent between iii) and iv), hence by single-crossing type $\bar{t}^{M-2}$ prefers iii) to iv). By construction type $\bar{t}^{M-2}$ is indifferent between iii) and ii). Furthermore, since $x_{M-2}(p_{M-2} - c(\bar{t}^{M-2})) \geq (v(t_1) - c(\bar{t}^{M-2}))$ (because we have a regular allocation) we conclude that option ii) is a best-response to type $\bar{t}^{M-2}$. Thus, from single-crossing any best-response for type t is either i) or ii).

Next, let us analyze the incentives of type t in period n^*. Consider the fictional environment in which the possibilities of type t are enriched in period n^* : He has options i), ii) (above) and v): Offer $p_{n^*}^{M-1,M-2}$ in every period $n \geq n^*$ with the buyer accepting this offer with constant probability $\psi_{n^*}^{\delta}$ in each future period. Type $\bar{t}^{M-2}$ is indifferent between ii) and v) and hence type t would never choose v) at n^*. Thus we conclude that type t would never pool with types $t \in \mathcal{T}_{M-1} \cup \mathcal{T}_M$ until period n^*. The (essentially) same argument implies that type t would never pool with types $t \in \mathcal{T}_{M-1} \cup \mathcal{T}_M$ until period $n = n^* - 1$.

By induction, we conclude that type t never pools with types $\mathcal{T}_{M-1} \cup \mathcal{T}_M$ and hence he has a best-response in which he offers $p \in \left\{p_1, \ldots, p_{M-2}(\eta), v(\bar{t}^{M-2})\right\}$ in every $n \geq 0$. However, type $\bar{t}^{M-2}$ is indifferent between offering $\left\{p_{M-2}(\eta), v(\bar{t}^{M-2})\right\}$ in every $n \geq 0$. Thus, from single-crossing we conclude that type t has a best-response in the set $\{p_1, \ldots, p_{M-2}(\eta)\}$. Therefore, from the analysis in the previous two paragraphs we conclude that type t does not have a profitable deviation.

One can use an analogous argument to show that no type $t > \bar{t}^{M-2}$ has a profitable deviation.

Finally, we specify the seller's strategy after a deviation to an offer which is made on the equilibrium path. We consider a type $t < \bar{t}^{M-2}$. A similar construction holds for every type $t \geq \bar{t}^{M-2}$ (omitted for brevity). First, assume that type $t \in \mathcal{T}_k$ has deviated in every period $n \leq \tilde{n}$ and offered $p \in \left\{p_1, \ldots, p_{M-2}(\eta), v(\bar{t}^{M-2})\right\}$ and let $\psi^{\delta}(p)$ denote the (constant) probability that the buyer accepts this offer in each future period. The best-response of the seller depends on the maximizer of

$$A := \left\{ \left(\frac{\psi^{\delta}(p)}{1 - \delta(1 - \psi^{\delta}(p))} \right) (p - c(t)), (v(t_1) - c(t)), 0 \right\}.$$

If $0 \in \arg\max A$, then the seller offers $v(t_M) + 1$ in every future period. Otherwise, if

$\left(\frac{\psi^{\delta}(p)}{1-\delta(1-\psi^{\delta}(p))}\right)(p-c(t)) \in \arg\max A$ the seller offers p in each future period.

Finally, if $(v(t_1)-c(t)) = \arg\max A$, the seller offers $v(t_1)$ in each future period.

Next, assume that type t has offered $p_n^{M-1,M-2}$ in every period $n \leq \tilde{n}$ ($\tilde{n} \leq n^*$). We have to compare $\left(\frac{\bar{\psi}^{\delta}_{M-2}}{1-\delta(1-\bar{\psi}^{\delta}_{M-2})}\right)(v(\bar{t}_{M-2})-c(t))$ and $(v(t_1)-c(t))$. If $\left(\frac{\bar{\psi}^{\delta}_{M-2}}{1-\delta(1-\bar{\psi}^{\delta}_{M-2})}\right)(v(\bar{t}_{M-2})-c(t)) \geq (v(t_1)-c(t))$ (resp. $\left(\frac{\bar{\psi}^{\delta}_{M-2}}{1-\delta(1-\bar{\psi}^{\delta}_{M-2})}\right)(v(\bar{t}_{M-2})-c(t)) < (v(t_1)-c(t))$) the seller offers $v(\bar{t}_{M-2})$ (resp. $v(t_1)$) in every future period.

Now, assume that type t has offered $p_n^{M-1,M-2}$ in every period $n \leq n^*$ and offered $p \in \left\{v(\bar{t}^{M-2}), v(\mathcal{T}_{M-1}), v(\mathcal{T}_M)\right\}$ in every period $n \in \{n^*+1, \ldots, n^*+k\}$. Let $\psi^{\delta}(p)$ be the (constant) probability that the buyer accepts the offer p in each period. As above, the seller's offer in every future period is determined by the maximizer of A.

The case in which the seller has offered $p_n^{M-1,M-2}$ in every period $n \leq \tilde{n}$ ($\tilde{n} \leq n^*$) and has offered $v(\bar{t}^{M-2})$ in every period $n \in \{\tilde{n}+1, \ldots, \tilde{n}+k\}$ is clearly analogous to the cases above (omitted for brevity).

(7) Now we establish that the strategies are Markov.

Let

$$\mathbf{P} :=\{p_1, \ldots, p_{M-3}, p_{M-2}(\eta), v(\bar{t}^{M-2}), p_0^{M-1,M-2}, \ldots, p_{n^*}^{M-1,M-2}, v(\mathcal{T}_{M-1}), v(\mathcal{T}_M)\}$$

be the set of on-path offers (assume that $p_{M-2}(\eta) \neq v(\bar{t}^{M-2})$, otherwise eliminate $v(\bar{t}^{M-2})$ from $\mathbf{P}$).

Notice that in the equilibrium that we constructed, the behavioral strategy of type t at a history h depends on which partition the history h belongs to.

i) Partition 1: $h = \emptyset$, the initial history;

ii) Partition p ($p \in \mathbf{P}$, thus there are $|P|$ of such partitions): $h \neq \emptyset$ and "no deviation" has been detected by the buyer. In this case, the behavioral strategy of the type t seller depends only on the offer p that was made in the last period.

iii) Partition D : The buyer detected a deviation in h. (Remember that in this case the offer made by type t is determined by $\arg\max\left\{\left(v(t_1)-c(t)\right),0\right\}$.)

Notice that the history i) is associated with the initial belief $\mathbf{q}_0 \in \mathbf{\Delta}(T)$. Notice also that if the buyer has not detected any deviation then each offer $p \in \mathbf{P}$ made in the last period leads to a different posterior which we call $\mathbf{q}(p)$. Finally, notice that if an offer belongs to the partition D then the buyer puts probability 1 on the seller being type t_1. We write $\mathbf{q}\left(\{t_1\}\right)$ for this posterior.[32] Therefore, the seller's strategy depends only on the prior $\mathbf{q}$ $\left(\mathbf{q} \in \left\{\mathbf{q}\left(\{t_1\}\right), \mathbf{q}_0\right\} \cup \left\{\mathbf{q}(p) : p \in \mathbf{P}\right\}\right)$.

Finally, since the buyer's behavioral strategy precludes that he accepts any on-path $p \in \mathbf{P}$ with probability $\psi^\delta(p)$ and accepts any off-path offer if and only if it is no greater than $v(t_1)$ we conclude that the buyer plays a Markov strategy.

References

Billingsley, P. (1968). *Convergence of Probability Measures*, John Wiley & Sons.

[32] We remark that if $\mathcal{T}_1 = \{t_1\}$ then the partitions D and p_1 (notice that in this case $p_1 = v(t_1)$) are the same. This causes no difficulty.