Bargaining over a Divisible Good in the Market for Lemons*

Dino Gerardi[†] Lucas Maestri[‡]

March 2016

Abstract

A seller dynamically sells a divisible good to a buyer. It is common knowledge that there are gains from trade and that the gains per unit are decreasing. Payoffs are interdependent as in Akerlof's market for lemons. The seller is informed about the quality of the good. The buyer makes an offer in every period and learns about the good's quality only through the seller's behavior. We characterize the limiting equilibrium outcome as bargaining frictions vanish and the good becomes arbitrarily divisible. We show that the gradual sale of high-quality goods arises as the main signaling device in such markets. We also show that the limiting outcome is Coasean: the competition with his future selves drives the buyer's payoff to the lowest possible level.

KEYWORDS: bargaining, gradual sale, Coase conjecture, divisible objects, interdependent valuations, market for lemons.

*We thank Carlos da Costa, Felipe Iachan, George Mailath, Michael Mandler, Tristan Tomala, Takuro Yamashita and various seminar audiences for useful comments, and Giacomo Ciapponi for superb research assistance. The financial support of the European Research Grant (Consolidator Grant, INFOASYMMARKETS) is gratefully acknowledged.

[†]Collegio Carlo Alberto, Università di Torino; dino.gerardi@carloalberto.org.

[‡]FGV/EPGE - Escola Brasileira de Economia e Finanças; maestri.lucas@gmail.com.

1 Introduction

This paper studies bargaining over a divisible good in the presence of informational asymmetries and interdependent values. Divisibility is a natural assumption in many economic applications. Consider a venture capitalist who invests in a startup initially owned by an entrepreneur. Typically, the venture capitalist's investment increases over time as he sequentially purchases different series of shares before an IPO takes place.[1] Moreover, this relationship is plagued with asymmetric information, as the entrepreneur is better informed about the project's prospects. Informational asymmetries are particularly important in the early stages of financing, when the firm develops its product and little information is made available to outsiders. Finally, there are clear gains from trade as the venture capitalist's investments relax the liquidity constraints that can severely limit the startup's expansion.[2]

Assets traded over-the-counter constitute another obvious example of bargaining with divisibility. In these markets, the parties dynamically negotiate and trade batches of securities. It is also often the case that one party (typically the seller) has superior information about the quality of the asset at hand.

Our goal is to understand how trade evolves when the parties are asymmetrically informed and negotiations are both over the quantity and the price. To this end, we extend the canonical bargaining model with incomplete information (Fudenberg, Levine and Tirole (1985), and Gul, Sonnenschein and Wilson (1986)) to account for interdependencies in values (Deneckere and Liang (2006) — DL henceforth) and divisibility. A buyer purchases a durable and divisible good from a seller who is informed about the good's quality, which may be either high or low. For each quality and quantity, there are positive gains from trade. These gains may be increasing or decreasing in the number of units already traded by the parties. However, most of our analysis focuses on the case of decreasing gains. This assumption arises naturally in situations in which the parties seek to diversify their portfolios. Furthermore, when the gains from trade are decreasing, divisibility leads to new insights into bargaining.

In our model, the good is divided into finitely many parts (or units). The buyer makes a take-it-or-leave-it offer in every period (however, as we discuss in Section 7, all our results extend to the case in which the buyer proposes menus of offers). The offer stipulates a price and a number of units to be traded. The bargaining process continues until the parties have traded all the available units.

[1] The case of RealNetworks, Inc. (discussed in chapter 23 in Berk and DeMarzo (2014)) is representative of the sequential process through which new companies raise equity capital.

[2] Davila, Foster and Gupta (2003) present conclusive evidence that startup companies' growth is delayed by their lack of financial resources.

The buyer learns about the good's quality only through the seller's behavior. In particular, owning a fraction of the good does not provide the buyer with additional information about its quality. Consider, for example, the negotiations between a venture capitalist and an entrepreneur who is developing a new product. It is difficult for the venture capitalist to obtain reliable evidence about the product's prospects until it is taken to the market. Thus, the most important channel through which the venture capitalist can learn about the profitability of the investment is the entrepreneur's willingness to accept certain terms of trade. More broadly, the game analyzed in this paper is both a benchmark model and a first step to investigate the effects of divisibility in bargaining.

Time on the market is the unique screening device when parties negotiate over the sale of an indivisible good. Therefore, the only way for a seller to convince a buyer that he has a high-quality good is by rejecting low offers in the initial phases of negotiations. Consistent with this observation, most of the theoretical literature[3] predicts similar trade patterns in dynamic markets under adverse selection: Buyers make low offers in the beginning of the relationship. These offers are accepted by some sellers with low-quality goods. However, sellers with high quality goods reject these offers, which leads to delay and often periods of market freeze. When goods are divisible, the set of possible trade outcomes becomes considerably richer. Does adverse-selection lead to the sale of low-quality goods in the beginning of the relationship, followed by a market freeze and the subsequent sale of high-quality goods, as is found in indivisible-good models? Or does the gradual sale of high-quality goods arise as the main signaling device in such markets? Our paper is the first to address and answer these questions.

Our model yields clear predictions in terms of economic behavior. We show the existence of stationary equilibrium and characterize the generically unique (stationary) equilibrium outcome. In equilibrium, the buyer employs only two types of offers: *cream-skimming* and *universal* offers. A *cream-skimming offer* is an offer to purchase all the remaining units of the good at a price that only the owner of the low-quality good would be willing to accept. A *universal offer* is an offer to purchase a fraction of the available good at a price that every type of seller will accept. Typically, the buyer starts by making *cream-skimming offers*. After several of these offers have been rejected, the buyer becomes more optimistic that the good is of high quality and decides to make a *universal offer* for a fraction of the good. Upon the acceptance of this offer, the buyer restarts the process of making *cream-skimming* offers for the remainder of the good.

We characterize the limiting equilibrium outcome (or simply limiting outcome) as bar-

[3] Among the several papers which obtained this finding, see the important contributions of DL, Daley and Green (2012), Fuchs and Skrzypacz (2013 a,b), Camargo and Lester (2014), Fuchs et. al. (2014), Kaya and Kim (2015), Kim (2015), and Moreno and Wooders (2010, 2015).

gaining frictions vanish and the good becomes arbitrarily divisible (i.e., the number of units into which the good is divided goes to infinity). Our interest in perfectly divisible goods comes from the fact that in most applications of our model, the parties can essentially trade any fraction of the good. Moreover, we get the sharpest results when the size of each part of the good shrinks to zero.

Our main result shows a gradual sale of high-quality goods. That is, the buyer purchases the good from the "high seller" (i.e., the owner of the high-quality good) smoothly over time. At each point in time, the buyer also makes an offer for the remainder of the good at a price equal to his valuation when the quality is low. The "low seller" is indifferent between the two offers (cream-skimming and universal) and gradually reveals his identity. In other words, he sells the good smoothly (pooling with the high seller) until a certain random time, and then concedes by selling the remaining fraction of the good at once.

It is well known that dynamic markets with adverse-selection may lead to impasses in bargaining (DL) and periods of illiquidity. Our model sheds light into a new economic force present in those markets — gradual trading. It shows that market mechanisms that ameliorate information asymmetries involve the slow and gradual purchase of a good at high prices, combined with substantial discounts for large purchases. Accordingly, our results bring new insights to our understanding of adverse selection in decentralized markets, in general, and over-the-counter markets, in particular. Arguably, asymmetric information in over-the-counter markets led to severe macroeconomic consequences and played a major role in the development of the recent financial crisis (Brunnermeier (2009), and Gorton (2009)). During the crisis, there was a substantial decrease in the total volume of trade of mortgage-backed securities, collateralized debt obligations, credit default swaps and several other assets traded in over-the-counter markets (Bank for International Settlements (2009), and Gorton and Metrick (2012)). Models focusing on indivisible goods predict that an increase in adverse selection in dynamic markets leads to periods of market-freezing and no trade, after which, the market clears at a high price. Our model makes a different prediction; instead of periods of complete market freezing, it predicts both the slow trade of high-quality assets and fire sales of large quantities of low-quality assets.

Our theory also makes novel and sharp predictions for venture-capital financing. Specifically, it predicts the observed gradual funding of startup companies, and is consistent with the evidence of severe liquidity constraints during the early stages of firms' life cycle.

Our second main result shows that the limiting equilibrium outcome is *Coasean*. In the limit, the buyer breaks even at any point in time, both if he makes a cream-skimming offer and if he makes a universal offer. Consequently, his limiting payoff is equal to zero. This result is reminiscent of the Coase conjecture and shows that the buyer's competition with his future selves drastically reduces his payoff. Our findings are in sharp contrast with the case in which the good is indivisible and the buyer's payoff remains positive in the limit as bargaining frictions vanish.

To provide some intuition, it is useful to compare our model with the indivisible-good version of our model analyzed in DL. When the good is indivisible, negotiations involve long impasses which severely reduce the price at which the buyer is able to purchase the good at the beginning of the bargaining process. In other words, the buyer benefits from long impasses and his initial payoff is strictly positive. Long impasses are sustained for two reasons. First, the buyer is not willing to make a generous offer unless he is sufficiently optimistic about the quality of the good. Second, the owner of the low-quality good is willing to wait and sell the good at a price much higher than his cost.

When the good is divisible, new trading opportunities are available to the buyer other than purchasing all the remaining units at a high price. In particular, when this option yields a low payoff, the buyer strictly prefers to make universal offers and purchase a certain fraction of the remaining good. Compared to DL's model, this lowers the price that the owner of the low-quality good is able to charge in the middle of the negotiations. However, lower payments in the middle of the negotiations translate into shorter impasses and higher prices paid at the beginning of the bargaining process. Consequently, the buyer's initial payoff is drastically reduced and converges to zero in the limit.

Our analysis shows that the buyer's ability to screen the seller more finely through the use of partial offers (i.e., offers for a fraction of the good) takes away the commitment power that he gains from long impasses in bargaining environments with indivisible goods.

In the final step of our analysis, we assume that the gains from trade are increasing in the amount of units purchased. This assumption arises naturally in economic situations in which different units are complementary. For example, assume that the parties negotiate stocks that yield control rights over a firm. In this case, the gains from trade may be increasing in the number of stocks purchased by the buyer, as the stocks yield him more control over the firm's decisions. We show that divisibility does not matter in this case. In fact, the limiting outcome of our game is the same regardless of whether or not the good is divisible. Therefore, the model makes very different predictions for situations in which the main reason for trade is portfolio diversification (decreasing gains of trade per unit), versus situations in which the parties trade over the control of an asset (increasing gains of trade per unit). The model predicts that high-quality goods will be gradually traded in the first case, whereas we expect large transactions after long delays in the second case.

1.1 Related Literature

Bilateral bargaining with interdependent values (and indivisibility) has received considerable attention in the literature (Samuelson (1984), Evans (1989), Vincent (1989), DL, Fuchs and Skrzypacz (2013a), and Gerardi, Hörner and Maestri (2014)). The closest papers to ours are DL and Fuchs and Skrzypacz (2013a). DL solved the one-unit version of the model in this paper. Taking their construction as a stepping stone, we build an algorithm to extend their analysis to multiple units when there are two types of sellers. In DL, the gains from trade are bounded away from zero. Fuchs and Skrzypacz (2013a) bridged the gap between the value of the good to the buyer and the cost to the seller. We find that trade happens gradually over time when the good is very divisible. This finding is reminiscent of Fuchs and Skrzypacz (2013a) who show in a model with indivisibility that, as the gains from trade from the good of highest quality vanish, the bursts of trade found in DL disappear. Like Fuchs and Skrzypacz (2013a), in our model the buyer slowly learns the seller's type. Unlike Fuchs and Skrzypacz (2013a), however, in our model the buyer makes two kinds of offers as he learns the seller's type. On the one hand, he gradually makes universally accepted offers for small pieces of the good at large per-unit prices. On the other, he makes offers for all remaining units at large discounts. Finally, another important difference between the two papers is that in our model the gains from trade are bounded away from zero.

Our paper is also related to the burgeoning body of literature that studies the effects of adverse selection in dynamic markets. An important stream of this literature focuses on markets in which one of the players is short-run. Inderst (2005) and Moreno and Wooders (2010) pioneered the study of adverse-selection in decentralized dynamic markets. Camargo and Lester (2014) and Moreno and Wooders (2015) focus on the effect of police interventions on liquidity in such markets. Hörner and Vieille (2009) show that markets in which offers are public behave considerably differently than do markets in which offers are private. Kim (2014) further analyzes the effect of a seller's past behavior on dynamic trading. Daley and Green (2012) and Kaya and Kim (2015) study the effect of exogenous learning in dynamic adverse selection. Fuchs and Skrzypacz (2013b) characterize optimal market design policies. Fuchs, Öry and Skrzypacz (2014) compare different transparency regimes and show when private offers are preferable. Beyond the issue of divisibility, our paper differs from the above studies by analyzing the strategic effects that arise when two long-run players bargain under adverse selection.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 specifies the solution concept and proves the general existence and uniqueness (for generic parameters) of the equilibrium outcome. In Section 4, we define the notion of limiting equilibrium outcome. We state the main result of the paper in Section 5 and prove it in Section 6. Section 7 analyzes a number of extensions and Section 8 concludes. Most proofs are relegated to a number of Appendices.

2 The Model

A buyer and a seller bargain over $m > 1$ parts (or units) of a divisible object. The value of each part to each trader depends on the seller's private information (i.e., his type). The seller's type is low with probability $\hat{q} \in (0,1)$ and high with probability $1-\hat{q}$. Although the seller has only two types, we find it convenient (following several papers on bargaining with incomplete information) to assume that his type q is distributed uniformly over the unit interval. For each $k = 1, \ldots, m$, the buyer's valuation of the $(m-k+1)$-th part is equal to

$$v(k,q) = \alpha_k v(q),$$

where $\alpha_m > \ldots > \alpha_1 \geqslant 1$, and the function $v(\cdot)$ is equal to:

$$v(q) = \begin{cases} \underline{v} & \text{if } q \in [0,\hat{q}] \\ \bar{v} & \text{if } q \in (\hat{q},1]. \end{cases}$$

The seller's cost of each unit is equal to

$$c(q) = \begin{cases} 0 & \text{if } q \in [0,\hat{q}] \\ c & \text{if } q \in (\hat{q},1]. \end{cases}$$

We assume that $\bar{v} > c > \underline{v} > 0$, so that it is common knowledge that for each unit there are positive gains from trade. We refer to any $q \leqslant \hat{q}$ as a low type and to any $q > \hat{q}$ as a high type.

In each period $t = 0, 1, \ldots,$ the buyer makes a proposal $\varphi_t = (k, p)$ to trade a certain number k of the remaining units in exchange for a transfer $p \geqslant 0$.[4] Then the seller decides whether to accept ($a_t = A$) or reject ($a_t = R$) the proposal φ_t. The game ends when all the m parts are traded.

The common discount factor is $\delta = e^{-r\Delta}$, where $r > 0$ is the rate at which the traders discount the future and Δ denotes the length of each period. Consider an outcome in which the seller has type q and accepts the offers $(k_1, p_1), \ldots, (k_M, p_M)$ in periods

[4] If the two players have already traded $k' = 1, \ldots, m-1$ at the end of period $t-1$, then the proposal φ_t is an element of the set $\{1, \ldots, m-k'\} \times \mathbb{R}_+$.

$t_1 < \ldots < t_{M,}$ respectively. Then the payoffs are $\sum_{i=1}^{M} \delta^{t_i} \left[p_i - k_i c(q) \right]$ for the seller and $\sum_{i=1}^{M} \delta^{t_i} \left(\left(\alpha_{\bar{k}_i+1} + \ldots + \alpha_{\bar{k}_{i-1}} \right) v(q) - p_i \right)$ for the buyer, where $\bar{k}_0 = m$ and $\bar{k}_i = \bar{k}_{1-1} - k_i$ for $i = 1, \ldots, M$. Finally, both players' payoffs are equal to zero if all the offers are rejected.

There are two different ways to interpret our model. We can either assume that the game continues until the parties trade all the m units, but the buyer does not learn the value of the object through its consumption (i.e., the consumption payoff contains noise that provides no information about the seller's type). Alternatively, we can assume that the buyer discovers the value of the part when he consumes it. However, consumption takes place only when a stochastic deadline occurs (thereby ending the game). In each period, the conditional probability that the game will continue to the next period is equal to δ.

3 Equilibrium

We let $h^0 = \emptyset$ denote the empty history, and, for each $t \geqslant 1$, we let $h^t = \left((\varphi_0, a_0), \ldots, (\varphi_{t-1}, a_{t-1}) \right)$ denote the (public) history of offers and acceptance decisions in periods $0, \ldots, t-1$. We also let (h^t, φ_t) denote the history that ends with the buyer's proposal in period t. The buyer's strategy σ_B assigns a proposal φ_t to every history h^t. The seller's strategy σ_S assigns an acceptance decision $a_t \in \{A, R\}$ to every type q and every history (h^t, φ_t). Finally, we let $\mu(h^t)$ and $\mu(h^t, \varphi_t)$ denote the buyer's beliefs over the seller's types after the histories h^t and (h^t, φ_t), respectively.[5] We use μ to denote the collection of beliefs.

Our solution concept is *stationary* perfect Bayesian equilibrium (or stationary equilibrium for brevity). In our context, the definition of perfect Bayesian equilibrium (see Fudenberg and Tirole (1991)) imposes the following two conditions on off-path beliefs:

i) $\mu(h^t, \varphi_t) = \mu(h^t)$ *for every* h^t *and* φ_t*;*

ii) Suppose that the action a_t *has positive probability after the history* (h^t, φ_t) *given the beliefs* $\mu(h^t, \varphi_t)$*. Then the beliefs* $\mu(h^t, (\varphi_t, a_t))$ *are derived from* $\mu(h^t, \varphi_t)$ *using Bayes' rule.*

The first condition captures the idea that the buyer cannot learn anything from his offer. The second condition forces the buyer to update his beliefs according to Bayes' rule when nothing "surprising" occurs.

The following observation will turn useful in the definition of the solution concept. A common feature of all perfect Bayesian equilibria of our game is that the seller will not be

[5]For any measurable subset Q of the unit interval, $\mu(Q|h^t)$ denotes the buyer's belief, at the history h^t, that the seller's type belongs to Q. The beliefs $\mu(h^t, \varphi_t)$ are defined in a similar way.

able to extract any rent once the buyer is convinced that the quality of the good is low.

Lemma 1 *Consider a perfect Bayesian equilibrium* $(\sigma_B, \sigma_S, \mu)$ *and let* h^t *be a history at which the buyer assigns probability one to low types (i.e.,* $\mu([0, \hat{q}] | h^t) = 1$*). The continuation payoff of type* $q \leqslant \hat{q}$ *at* h^t *is equal to zero.*

The proof of this lemma is standard and is relegated to Appendix A. Nonetheless, this result has important equilibrium implications. In particular, it implies that in any perfect Bayesian equilibrium, the low types are willing to accept any offer once the buyer discovers that the good is a lemon.

We extend the notion of stationary equilibrium (see Gul and Sonnenschein (1988), Ausubel and Deneckere (1992), and DL) to our setting. A natural way to define stationarity in our context is to require the acceptance decision of any seller's type to depend only on the number of units left for trade and on the buyer's proposal. However, as we already pointed out, in a perfect Bayesian equilibrium the reservation price of type $q \leqslant \hat{q}$ to sell any number of units must be equal to zero at any history h^t such that $\mu([0, \hat{q}] | h^t) = 1$. This immediately implies that in our model there do not exist perfect Bayesian equilibria that satisfy the "natural" definition of stationarity. The minimal departure from it is to let the acceptance decisions of the low types depend on whether or not the buyer is convinced that the seller's type is low. This leads us to the following definition of stationary perfect Bayesian equilibrium.

Definition 1 *A perfect Bayesian equilibrium* $(\sigma_B, \sigma_S, \mu)$ *is stationary if there exists a (measurable) function* $P(\cdot, \cdot, \cdot; \delta) : \{1, ..., m\}^2 \times [0, 1] \to \mathbb{R}_+$ *satisfying the following conditions:*

i) Suppose that k *units are left for trade. For each* $q > \hat{q}$*, an offer* (k', p)*,* $k' \leqslant k$*, is accepted by type* q *if and only if* $p \geqslant P(k, k', q; \delta)$*;*

ii) Consider a history h^t *such that* $\mu([0, \hat{q}] | h^t) < 1$ *and suppose that* k *units are left for trade. For each* $q \leqslant \hat{q}$*, an offer* (k', p)*,* $k' \leqslant k$*, is accepted by type* q *if and only if* $p \geqslant P(k, k', q; \delta)$*.*

In the next section, we establish the existence and (essentially) the uniqueness of stationary equilibria and characterize the equilibrium behavior. For brevity, when there is no ambiguity, we suppress the dependence of the reservation price function $P(k, k', \cdot; \delta)$ on δ and write $P(k, k', \cdot)$.

3.1 Existence and Uniqueness of Stationary Equilibrium

We start our analysis with a result that holds in all stationary equilibria. The high types' equilibrium behavior is rather simple. In fact, they behave myopically and accept an offer if and only if the profits generated by that offer are weakly positive.

Lemma 2 *Let $(\sigma_B, \sigma_S, \mu)$ be a stationary equilibrium. Then, $P(k, k', q) = k'c$ for every k, k' and $q > \hat{q}$.*

The statement of the lemma is an immediate consequence of the fact that at any history, the equilibrium continuation payoff of the high types is zero. The proof of this result (in Appendix A) is by induction on the number of units left for trade. The first step of the argument (i.e., the high types' continuation payoff is zero when there is only one unit left for trade) was proved by DL. We now provide a sketch of the proof of the induction argument. Suppose the high types get a payoff equal to zero when there are k or fewer units for trade. Let $\bar{u}_H$ denote the highest continuation payoff that the high types get across all histories at which there are $k+1$ units for trade.[6] By contradiction, assume that $\bar{u}_H > 0$. Our assumptions imply that, at some point, the buyer makes an offer to buy a certain number k' of the $k+1$ remaining units at the price $k'c + \bar{u}_H$. Clearly, this offer is accepted by the high types (if they reject it, they will receive at most $\delta\bar{u}_H$) as well as by the low types (if they reject the offer, their identity will be revealed and their continuation payoff will be zero). However, the same argument shows that both the high and the low types are willing to sell the k' units even at a price slightly lower than $k'c + \bar{u}_H$. We conclude that in equilibrium the buyer will never make the offer $(k', k'c + \bar{u}_H)$.

Our next result addresses the issue of the existence of stationary equilibria.

Proposition 1 *There exists a stationary equilibrium.*

The proof of Proposition 1 is in Appendix A where we construct the equilibrium strategy profile and the buyer's beliefs, and show that unilateral deviations are not profitable. Here we illustrate the equilibrium on-path behavior. In particular, the buyer employs only two types of offers. Suppose there are k units left for trade. The first type of offer is to purchase all the remaining k units at some price $p \leqslant kc$. The offer (k, p), in turn, can be accepted by all the types or only by (some of) the low types. Therefore, the rejection of (k, p) increases the buyer's confidence that the quality of the good is high. Because of this, we refer to

[6]For simplicity, here we assume that the maximum payoff does exist. The formal proof dispenses with this assumption.

an offer of the form (k, p) (when k is the number of remaining units) as a *cream-skimming offer*.

The second type of offer is to purchase some (but not all) of the remaining units, say $k' < k$, at the price $k'c$ at which the high types break even. Notice that the offer $(k', k'c)$ is accepted by all the types (recall that $P(k, k', q) = k'c$ for $q > \hat{q}$ and Lemma 1). Thus, we refer to an offer of the form $(k', k'c)$ as a *universal offer*.

The reservation price functions $P(1, 1, \cdot), P(2, 2, \cdot), \ldots, P(m, m, \cdot)$ play an important role in our analysis. To simplify notation, for the remainder of this paper we will use $P(k, \cdot)$ to indicate $P(k, k, \cdot)$.

In our equilibrium, for every $k = 1, \ldots, m$, the function $P(k, \cdot)$ is increasing and left-continuous (also recall that $P(k, q) = kc$ for $q > \hat{q}$). Therefore, at any point in time along the equilibrium path, the set of types who have not sold all their units is of the form $[q, 1]$ for some $q \in [0, 1)$. With some abuse of notation, we use (k, q) to denote an arbitrary state of the economy. In state (k, q), there are still k units left for trade, and the buyer believes that the seller's type is uniformly distributed over the set $[q, 1]$.

Let $W(k, q)$ denote the buyer's *ex-ante* expected payoff when the state is (k, q).[7] Recall that, in equilibrium, only cream-skimming and universal offers are used. Thus, for any state (k, q), $W(k, q)$ satisfies:

$$\begin{aligned} W(k, q) = \max \Big\{ & \Big(\max_{q' \in [q,1]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, q') \right] ds + \delta W(k, q') \Big), \\ & \Big\{ \int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c \right] ds + \delta W(k', q) \Big\}_{k'=1,\ldots,k-1} \Big\}. \end{aligned} \tag{1}$$

The right-hand side of the first line of equation (1) represents the buyer's expected payoff if he chooses the optimal cream-skimming offer. The second line of the equation describes the payoffs associated with the universal offers.

On the equilibrium path, the players' behavior is rather simple. Suppose that the state is (k, q). Then the buyer makes the offer (either cream-skimming or universal) that maximizes $W(k, q)$. The seller of type q' accepts any universal offer. Furthermore, he accepts a cream-skimming offer provided that the price that the buyer is willing to pay is above his reservation price $P(k, q')$.[8]

At a first glance, it may seem as if we have limited the scope of our analysis by restricting attention to the equilibrium above. Nevertheless, as Proposition 2 shows, nothing is lost

[7] Of course, the buyer's payoff $W(k, \cdot)$ depends on the discount factor. As for the reservation prices, we suppress the dependence of $W(k, \cdot)$ on δ when there is no ambiguity.

[8] Notice that along the equilibrium path the low types are indifferent between accepting and rejecting a cream-skimming offer (k, p) with $p < kc$.

inasmuch one considers stationary equilibria.[9] Two equilibria are outcome equivalent if, conditional on each quality of the good (low or high), they induce the same probability distribution over histories of accepted offers.

Proposition 2 *For generic parameters, all stationary equilibria are outcome equivalent.*

The first part of the proof (in Appendix B) shows the uniqueness of the equilibrium outcome under the assumption that the reservation price functions $P(1,\cdot),\ldots,P(m,\cdot)$ are increasing. The second part of the proof relaxes this assumption. Of course, given an equilibrium, it is also possible to construct another (outcome equivalent) equilibrium by permuting the low types. We show that, for generic values of the parameters, the equilibrium described above is unique up to a permutation of the low types.

4 Limiting Equilibrium Outcome

As it is often the case in bargaining games, it is intractable to describe the equilibrium behavior for an arbitrary value of the discount factor δ. Therefore, we focus on the equilibrium outcome when the bargaining frictions vanish. Moreover, in general, the outcome of the bargaining game depends on how the good is divided. Thus, we analyze the limiting outcome obtained when the good becomes arbitrarily more divisible.[10] This approach helps us to understand the role of divisibility and yields new insights into bargaining.

More precisely, we proceed as follows. We assume that the good has measure one. Suppose that the measure of the good on the table is $z > 0$ and that the buyer purchases a measure $z' \leqslant z$ from type q. The buyer's valuation is given by:

$$v(q)\left(\int_{z-z'}^{z}\alpha(u)\,du\right),$$

where $\alpha : [0,1] \to \mathbb{R}_{++}$ is a smooth and strictly increasing function, and $\alpha(0) = 1$. Type q's valuation of any measure z of the good is equal to zero if $q \leqslant \hat{q}$, and equal to $z\bar{c}$ if $q > \hat{q}$. Thus, $\bar{c}$ represents the cost of the whole good when the quality is high. We assume that $\bar{v} > \bar{c} > \underline{v}$.

To simplify the exposition, throughout the rest of the paper we adopt the following convention. Suppose that the buyer believes that the seller's type is uniformly distributed over the set $[\tilde{q}, 1]$ for some $\tilde{q} \in [0,1]$ (recall that along the equilibrium path, the beliefs take this form). Then we say that the buyer's belief is $\tilde{q}$.

[9]It is an open question whether there is a perfect Bayesian equilibrium that is not outcome equivalent to a stationary equilibrium.

[10]It is not possible to use our analysis to work directly with a perfectly divisible good.

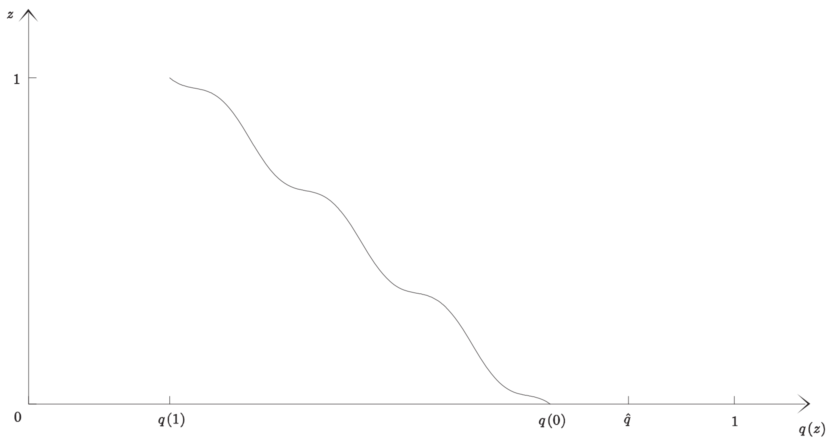

Figure 1: The function $q(z)$

For every $z \in [0,1]$, we define $q(z) \in [0,\hat{q})$ implicitly by

$$\int_{q(z)}^{1} [\alpha(z) v(s) - \bar{c}] \, ds = 0. \tag{2}$$

Suppose that the measure z of the good is still available for trade and the buyer purchases one infinitesimal unit at the price $\bar{c}$. If the buyer's belief is equal to $q(z)$, then he breaks even.

The function $q(\cdot)$ (depicted in Figure 1) is smooth and strictly decreasing since $\alpha(\cdot)$ is smooth and strictly increasing. For future reference, we let $\psi(\cdot)$ denote the inverse of $q(\cdot)$.

We take the parameters $(\underline{v}, \bar{v}, \bar{c}, \hat{q}, r)$ and the function α as given and divide the good into $m = 1, 2, \ldots$ units of measure $1/m$ each. If the seller's type is q, the buyer's valuation of the $(m-k+1)$-th unit, $k = 1, \ldots, m$, is equal to $\alpha_k v(q)$, where

$$\alpha_k := \int_{\frac{k-1}{m}}^{\frac{k}{m}} \alpha(u) \, du.$$

Type q's cost of each unit is equal to zero if $q \leqslant \hat{q}$, and equal to $\frac{\bar{c}}{m}$ if $q > \hat{q}$. Notice that since the function α is increasing, the gains from trade decrease as the parties make more and more transactions.

For every $m = 1, 2, \ldots$, we compute the limit, as the bargaining frictions vanish, of the equilibrium outcome of the game in which the good is divided into m parts. We refer to this as the m-limitimg equilibrium outcome (we simply write limiting equilibrium outcome when the number of units m is unambiguous). In particular, we fix m and consider a sequence of stationary equilibria as the length of the period Δ_n shrinks to zero (and the discount factor $\delta_n = e^{-r\Delta_n}$ converges to one). For every pair (m, Δ_n), consider the on-path history in which all the (cream-skimming) offers of the form (k, p) with $p < \frac{k\bar{c}}{m}$ are rejected, and, for every $t \geqslant 0$, let $\left(z_m^{\Delta_n}(t), q_m^{\Delta_n}(t)\right)$ denote the corresponding state in period $\left\lfloor \frac{t}{\Delta_n} \right\rfloor$.[11]

In Section 6.1, we show that the m-limiting equilibrium outcome is well defined. In fact, we establish the following result.

Fact 1 *For every $m = 1, \ldots,$ there exist two (right-continuous) functions, $z_m : \mathbb{R}_+ \to [0,1]$ and $q_m : \mathbb{R}_+ \to [0,1]$, such that for any sequence $\{\Delta_n\}_{n=1,\ldots}$ converging to zero, the sequence $\left\{\left(z_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1,\ldots}$ converges weakly to $(z_m(\cdot), q_m(\cdot))$.*

Finally, we take the limit as m goes to infinity and say that $(z_\infty(\cdot), q_\infty(\cdot))$ is the *limiting equilibrium outcome* if the sequence $\{z_m(\cdot), q_m(\cdot)\}$ converges weakly to $(z_\infty(\cdot), q_\infty(\cdot))$.

To ease notation, in what follows we focus on the extreme case of adverse selection and assume that the buyer's expected valuation of the first infinitesimal unit is smaller than the cost of the seller's high types:

$$\int_0^1 [\alpha(1) v(s) - \bar{c}]\, ds < 0. \tag{3}$$

It is easy to show that if $\int_0^1 [\alpha(0) v(s) - \bar{c}]\, ds \geqslant 0$, then for any m the m-limiting equilibrium outcome coincides with the first best and the Coase conjectures obtains.[12] Similarly, if $\int_0^1 [\alpha(z) v(s) - \bar{c}]\, ds = 0$ for some $z \in (0,1]$, and the good is infinitely divisible, then the Coase conjecture applies, but only for a measure $1 - z$ of the good.[13]

5 Limiting Equilibrium Outcome

In this section, we present the limiting equilibrium outcome of our model and show that it involves the gradual sale of the high-quality good. The outcome is determined by the following two conditions. First, it satisfies a *zero-profits* condition, which states that at any point in time the buyer breaks even both if he makes a universal offer for the first

[11] $\left\lfloor \frac{t}{\Delta_n} \right\rfloor$ denotes the largest integer not greater than $\frac{t}{\Delta_n}$.

[12] Recall that there are positive gains from trade for every unit and every type of the seller. Therefore, the first best is achieved when all the m units are traded without delay. According to the Coase conjecture, the buyer's introductory offer converges to $(m, \bar{c})$ as the length of the period between consecutive offers vanishes. It is also straightforward to check that in a model with private values and positive gains from trade ($\bar{c} \leqslant \underline{v} = \bar{v}$ in our context), the Coase conjecture continues to hold when the object is divisible.

[13] In this case, the results derived in Section 5 apply to the remaining measure z of the good.

infinitesimal unit still available, and if he makes the cream-skimming offer. In section 5.2, we explore a connection between this condition and the Coase conjecture. Second, the limiting outcome satisfies *gradual screening*, a condition that guarantees the low types' indifference (at any point in time) between revealing their identity by accepting the cream-skimming offer and pooling with the types by accepting the universal offer.

Below we derive the unique outcome that satisfies the *zero-profits* and the *gradual screening* conditions. We then show that this outcome coincides with the limiting equilibrium outcome.

Formally, let $(z^*(t), q^*(t))$ denote the state at time t in any outcome consistent with *zero-profits* and *gradual screening.* This means that the fraction of the good still on the table at time t is $z^*(t)$ and the buyer's belief is $q^*(t)$. From *zero-profits*, the universal offer for the first available (infinitesimal) unit must yield a payoff equal to zero and hence

$$\int_{q^*(t)}^{1} [\alpha(z^*(t)) v(s) - \bar{c}]\, ds = 0,$$

for every $t \geqslant 0$. Thus, for every t, we have $q^*(t) = q(z^*(t))$ and $z^*(t) = \psi(q^*(t))$ (recall the definition of functions $q(\cdot)$ and $\psi(\cdot)$ in Section 4). In particular, for $t = 0$ we have $q^*(0) = q(z^*(0)) = q(1)$.

Let $P^*(t)$ denote the price that the buyer offers at time t to purchase the remaining fraction $z^*(t)$ of the good. Also from *zero-profits*, any cream-skimming yields to the buyer a payoff equal to zero (and the offer is rejected by the high types). Thus, for every $t \geqslant 0$, we have

$$P^*(t) = \underline{v} \int_0^{z^*(t)} \alpha(u)du = \underline{v} \int_0^{\psi(q^*(t))} \alpha(u)du. \tag{4}$$

Furthermore, *gradual screening* implies that the low types are always indifferent between accepting and rejecting cream-skimming offers. In particular, they must be indifferent between accepting the cream-skimming offer at time t' and pooling with the high types (accepting, therefore, the universal offers) until time $t > t'$, at which time they finally accept the cream-skimming offer. Hence, for every $t' \geqslant 0$ and every $t > t'$ the following must hold:

$$P^*(t') = -\int_{t'}^{t} e^{-r(\tau - t')} \psi'(q^*(\tau))\, q^{*\prime}(\tau) \bar{c} d\tau + e^{-r(t-t')} P^*(t). \tag{5}$$

Using equation (4) to replace the values of $P^*(t')$ and $P^*(t)$ in equation (5), we obtain

$$\underline{v} \int_0^{\psi(q^*(t'))} \alpha(u)du = -\int_{t'}^{t} e^{-r(\tau - t')} \psi'(q^*(\tau))\, q^{*\prime}(\tau) \bar{c} d\tau + e^{-r(t-t')} \underline{v} \int_0^{\psi(q^*(t))} \alpha(u)du.$$

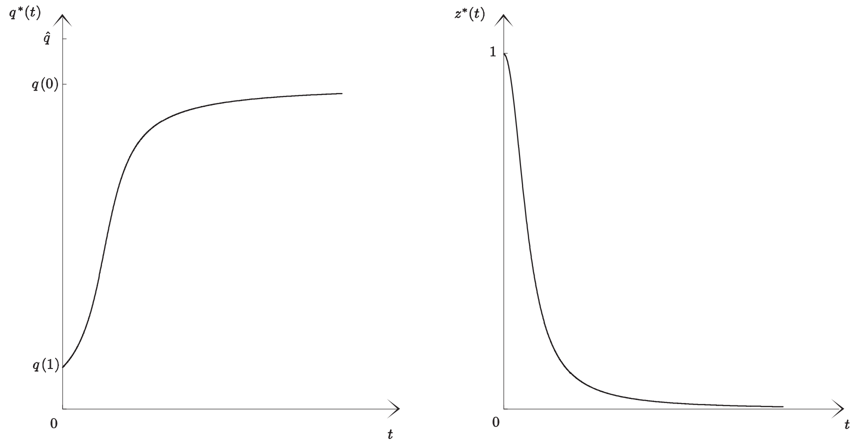

Figure 2: The Coasean outcome

Notice that the above inequality must hold for any $t > t' \geqslant 0$. Differentiating both sides with respect to t, we obtain

$$0 = -\psi'\left(q^*(t)\right) q^{*\prime}(t)\bar{c} - r\underline{v}\int_0^{\psi(q^*(t))} \alpha(u)du + \underline{v}\alpha\left(\psi\left(q^*\left(t\right)\right)\right)\psi'\left(q^*\left(t\right)\right)q^{*\prime}\left(t\right),$$

which implies

$$\begin{aligned} q^{*\prime}\left(t\right) &= \tfrac{r\underline{v}\int_0^{\psi(q^*(t))}\alpha(u)du}{\psi'(q^*(t))(\underline{v}\alpha(\psi(q^*(t)))-\bar{c})} \\ z^{*\prime}(t) = \psi'\left(q^*(t)\right)q^{*\prime}(t) &= \tfrac{r\underline{v}\int_0^{\psi(q^*(t))}\alpha(u)du}{(\underline{v}\alpha(\psi(q^*(t)))-\bar{c})}. \end{aligned} \tag{6}$$

These conditions uniquely pin down $(z^*(t), q^*(t))$. It is easy to check that $z^*(\cdot)$ is strictly decreasing and $\lim_{t\to\infty} z^*(t) = 0$ (thus, $q^*(\cdot)$ is strictly increasing and $\lim_{t\to\infty} q^*(t) = q(0)$). Figure 2 illustrates a typical trade path in the unique outcome satisfying *zero profits* and *gradual screening*.

We are now in a position to state the main result of the paper. We show that the limiting outcome (as the good becomes arbitrarily divisible and the bargaining frictions vanish) coincides with the unique outcome consistent with *zero profits* and *gradual screening*.

Theorem 1 *The limiting equilibrium outcome satisfies the zero-profits and the gradual screening conditions. That is, the sequence* $\{z_m(\cdot), q_m(\cdot)\}_{m=1,\ldots}$ *converges weakly to* $(z^*(t), q^*(t))$.

The proof of Theorem 1 is long and involved. We outline the main steps in Section 6 and leave technical arguments to the appendices.

The limiting equilibrium outcome can be interpreted as follows. At time zero, the buyer makes a cream-skimming offer for the entire good (measure one) at a price $\underline{v}\left(\int_0^1 \alpha(u)du\right)$. This offer is accepted by the low types in the interval $[0, q(1)]$. Thus, if the offer is rejected, the buyer's belief jumps to $q(1)$. At each time $t > 0$, the buyer simultaneously engages into two actions: he makes a universal offer for an infinitesimal quantity (which, of course, is accepted by all the remaining types), and a cream-skimming offer to purchase the remaining fraction $\psi(q^*(t))$ of the good at the price $\underline{v}\left(\int_0^{\psi(q^*(t))} \alpha(u)du\right)$. The low types accept the cream-skimming offer smoothly over time in such a way that the belief evolves according to $q^*(\cdot)$.

5.1 Gradual Sale of High-quality Goods

Most of the literature that studies dynamic markets with adverse selection finds that low-quality goods are sold in the beginning of the relationship, followed by periods of market freezing, after which high-quality goods are sold. Theorem 1 shows that in our model adverse selection unravels in a very different way. We predict that the gradual sale of high-quality goods arises as the main signaling device in dynamic markets with adverse selection. In times of uncertainty about fundamentals, we expect that sellers with valuable assets make small transaction over time, while sellers who hold lemons engage in fire sales of large quantities.

Our model also delivers novel and testable implications for venture-capital financing. Our predictions are consistent with the observed pattern of trade between a venture capitalist and an entrepreneur in the initial stages of venture-capital financing. Typically, the venture capitalist's investment increases over time as he sequentially purchases different series of shares before an IPO takes place. Our model suggests that part of this pattern may be due to entrepreneur's way of signaling that they hold profitable prospects.

5.2 Coasean Outcome in the Market for Lemons

In his influential work, Coase (1972) conjectured that a monopolist who lacks commitment power ends up selling the good almost immediately at a price equal to the lowest valuation among all consumers. In our environment, this conjecture would be interpreted as the buyer purchasing the entire good immediately at price $\bar{c}$. However, this cannot be an equilibrium outcome of our model, as the buyer would pay a price larger than the expected valuation of the good. Thus, we conclude that, in equilibrium, there must be necessarily delay. If we wish to conjecture that the competition between the buyer's present and future selves brings his profits to the lowest possible level, we must consider an adaptation of the Coase conjecture to an environment with interdependent values.

We say that there is a *Coasean outcome in the market for lemons* (Coasean outcome for brevity) if at any time $t \geqslant 0$ the buyer breaks even both if he makes a universal offer for the first of the infinitesimal units still available and if he makes the cream-skimming offer. Theorem 1 shows that the limiting outcome of our model is Coasean.

We conclude that the combination of adverse selection and divisibility has dramatic effects on payoffs. When the good is infinitely divisible, both the buyer and the high type sellers get a payoff equal to zero. On the other hand, the low types' payoff coincides with the buyer's valuation of the low-quality good. This result is in stark contrast to the results obtained by DL in the one-unit model, where the buyer's payoff is strictly positive and the low types get less than the buyer's valuation.

When the good is indivisible, negotiations necessarily involve long impasses. In fact, the buyer is willing to pay a large price only if he is sufficiently optimistic about the quality of the good. On the other hand, the owner of the low-quality good is willing to wait for a long period until he can sell the good at a price much larger than his cost. Because of the long impasses, the price at which the buyer is able to purchase the good at the beginning of the negotiations is severely reduced. Therefore, he obtains a strictly positive payoff.

When the good is divisible, purchasing all the remaining units at a large price is only one of many options available to the buyer. When this option yields a low payoff, the buyer prefers to purchase only a fraction of the remaining good. Compared to DL's model, this lowers the price that the owner of the low-quality good is able to charge in the middle of the negotiations. However, lower payments in the middle of the negotiations correspond to shorter impasses and higher prices paid in the early stages of the negotiations. Consequently, the buyer's initial payoff is drastically reduced and converges to zero in the limit.

Our analysis shows that the possibility to make (arbitrary) partial offers is detrimental to the buyer's payoff. In the standard model with an indivisible object, the buyer would be willing to pay (ex-ante) a positive price to lose the ability to adjust the offer as he becomes more optimistic about the quality of the good. Similarly, in our model, the buyer would be willing to pay a positive price to lose the ability to make partial offers.

5.3 Comparative Statics

In this section, we investigate how the limiting equilibrium outcome depends on the primitives of the model. This will allow us to determine the effects of the economic fundamentals on the speed of trade.

For some small $\varepsilon > 0$, consider a different valuation function $\tilde{\alpha}$ such that $\tilde{\alpha}(u) \in$

$(\alpha(u), \alpha(u)+\varepsilon)$ for every $u \in [0,1]$. Define the function $\tilde{\psi}(\cdot)$ accordingly. Let $(\tilde{z}^*(\cdot), \tilde{q}^*(\cdot))$ be the functions describing the new Coasean outcome. We claim that $\tilde{z}^*(t) < z^*(t)$ for every $t > 0$. Assume towards a contradiction that this is not the case. First, notice that

$$\tilde{z}^{*\prime}(0) = \tilde{\psi}'(\tilde{q}^*(0))\, \tilde{q}^{*\prime}(0) =$$
$$\frac{r\underline{v}\int_0^1 \tilde{\alpha}(u)du}{\underline{v}\tilde{\alpha}(1)-\bar{c}} < \frac{r\underline{v}\int_0^1 \alpha(u)du}{\underline{v}\alpha(1)-\bar{c}} = z^{*\prime}(0) < 0.$$

Let $\underline{t}$ denote the smallest $t > 0$ for which $\tilde{z}^*(t) = z^*(t)$ (our contradiction hypothesis guarantees that $\underline{t}$ exists). It follows that

$$\tilde{z}^{*\prime}(\underline{t}) = \frac{r\underline{v}\int_0^{\tilde{z}^*(\underline{t})} \tilde{\alpha}(u)du}{\underline{v}\tilde{\alpha}(\tilde{z}^*(\underline{t}))-\bar{c}} < \frac{r\underline{v}\int_0^{z^*(\underline{t})} \alpha(u)du}{\underline{v}\alpha(z^*(\underline{t}))-\bar{c}} = z^{*\prime}(\underline{t}) < 0.$$

However, this and the fact that $\tilde{z}^*(t) < z^*(t)$ for every $t < \underline{t}$ imply that $\tilde{z}^*(\underline{t}) < z^*(\underline{t})$, which contradicts the definition of $\underline{t}$.

A similar argument shows that a decrease in the cost $\bar{c}$, an increase in the value $\underline{v}$ and an increase in the discount rate r all decrease $z^*(t)$ for every $t > 0$. It is also easy to check that a small change in the value $\bar{v}$ or in the probability $\hat{q}$ does not affect the speed of trade. The analysis above allows us to state the following fact.

Fact 2 *In the Coasean outcome, trade of the high-quality good happens faster when adverse selection is less severe (i.e., when α or $\underline{v}$ increases or when $\bar{c}$ decreases) or when the parties are less patient.*

Let us provide some intuition for Fact 2. First, consider an increase in α or in $\underline{v}$. In this case, the low types' payoffs from accepting cream-skimming offers increase. On the other hand, the payoff from accepting a universal offer (for an infinitesimal quantity) remains constant. Consequently, a smaller delay is necessary to induce the low types to reveal their private information. Similarly, a decrease in $\bar{c}$ makes the universal offers less attractive for the low types (in other words, the low types are less tempted to pool with the high types), and hence trade happens faster.

Fact 2 yields sharp predictions about the effect of a change of the primitives on the timing at which the high-quality good is sold. On the other hand, when the seller's type is low, the quantity of the low-quality good that is still available for trade at time $t > 0$ is a random variable. In particular, if the seller's type is smaller than $q^*(t)$, then the remaining quantity is zero. Otherwise, it is equal to $z^*(t)$. Thus, if we let $g^*(t)$ denote the expected quantity of the low-quality good that is available at time $t > 0$, we have

$$g^*(t) = \left(\frac{\hat{q} - q^*(t)}{\hat{q}}\right) z^*(t).$$

When adverse selection becomes less severe (i.e., α or $\underline{v}$ increases or $\bar{c}$ decreases) there are two opposing effects on the timing at which the low-quality good is sold. On the one hand, a decrease in adverse selection decreases $q(1)$.[14] Hence, the quantity of the low-quality good that is available right after $t = 0$, $g^*(0^+) = \left(\frac{\hat{q}-q^*(1)}{\hat{q}}\right)$, increases. On the other hand, Fact 2 establishes that a decrease in adverse selection increases the speed of trade of the high-quality good. For a given $q(1)$, this effect increases the speed of trade of the low-quality good and hence decreases the expected quantity that is available at t. It is easy to construct examples showing that the quantity of the low-quality good that is still available for trade at time t is non-monotonic in α, $\underline{v}$ or $\bar{c}$.

Finally, the effect of a change in r, $\bar{v}$, or $\hat{q}$ is unambiguous. In fact, it is simple to check that the trade of the low-quality good happens faster ($g^*(t)$ decreases for every $t > 0$) when r or $\hat{q}$ increase, or when $\bar{v}$ decreases. Clearly, there is less delay when the parties are less patient. Also, recall that a change in $\hat{q}$ or $\bar{v}$ does not affect the speed of trade of the high-quality good (the value of $z^*(t)$ remains unchanged for every t) and that the buyer must break even when he makes a universal offer. This observation, together with an increase in $\hat{q}$ (or a decrease in $\bar{v}$), immediately implies that the fraction $\frac{\hat{q}-q^*(t)}{\hat{q}}$ of the low types that the buyer still faces at time t must decrease. Therefore, the speed of trade of the low-quality good increases since the low types are quicker to accept cream-skimming offers.

6 Proof of Theorem 1

Our argument is divided into two main parts. In the first part, we fix the number of units m into which the good is split and develop an algorithm to characterize the m-limiting equilibrium outcome, i.e. the equilibrium outcome as the length of each period Δ_n shrinks to zero (and the discount factor $\delta_n = e^{-r\Delta_n}$ converges to one). In Section 6.1.1, we illustrate the limiting equilibrium outcome when the good is divided into two units. This is helpful to understand how divisibility affects the pattern of trade and the parties' payoffs. In the second part, we use the algorithm developed in the first part to characterize the limiting outcome as the good becomes more divisible.

6.1 Algorithm for the $m-$Limiting Equilibrium Outcome

As mentioned above, in this section, we fix the number of units and develop an algorithm that describes the equilibrium outcome when offers are frequent. Fix m and recall that if

[14] Remember that $q(1)$ satisfies $\int_{q(1)}^{1} [\alpha(1)v(s) - \bar{c}]\, ds = 0$ (see equation (2)).

the seller's type is q, then the buyer's valuation of the $(m-k+1)$-th unit, $k=1,\ldots,m$, is equal to $\alpha_k v(q)$, while the seller's cost is equal to zero if $q \leqslant \hat{q}$, and equal to $\frac{\bar{c}}{m}$ if $q > \hat{q}$.[15] To simplify the notation, we write c for $\frac{\bar{c}}{m}$ (there is no ambiguity since m is fixed throughout the section).

Inequality (3) implies

$$c > \alpha_k \left(\hat{q}\underline{v} + (1-\hat{q})\bar{v}\right)$$

for every $k=1,\ldots,m$. However, it should be noted that the results in this section only require that the above inequality holds for $k=1$.

Consider a sequence $\{\Delta_n\}$ converging to zero and the corresponding sequence $\{\delta_n\} = \left\{e^{-r\Delta_n}\right\}$ converging to one. For each δ_n, we consider the stationary equilibrium described in Section 3.1. DL show that the sequence of reservation prices $\{P_n(1,\cdot)\}$ (where $P_n(1,\cdot) = P(1,\cdot;\delta_n)$) converges to $P(1,\cdot)$, which satisfies

$$P(1,q) := \begin{cases} \frac{(\alpha_1 \underline{v})^2}{c} & \text{if } q < \bar{q}_1 \\ c & \text{if } q > \bar{q}_1, \end{cases}$$

where $\bar{q}_1 \in (0,\hat{q})$ is implicitly defined by

$$\int_{\bar{q}_1}^{1} (\alpha_1 v(s) - c)\, ds = 0. \tag{7}$$

Suppose that the buyer purchases the last unit at the price c from all the types above a certain threshold. The type $\bar{q}_1$ represents the threshold at which the buyer breaks even. DL also show that the sequence of expected payoff $\{W_n(1,\cdot)\}$ converges to $W(1,\cdot)$, given by

$$W(1,q) := \begin{cases} \int_q^{\bar{q}_1} \left(\alpha_1 v(s) - P\left(1,\bar{q}_1^-\right)\right) ds & \text{if } q \leqslant \bar{q}_1 \\ \int_q^1 (\alpha_1 v(s) - c)\, ds & \text{if } q > \bar{q}_1. \end{cases}$$

where $P\left(1,\bar{q}_1^-\right) = \lim_{q\uparrow\bar{q}_1} P(1,q)$. We establish similar convergence results for a finite number of units:

Lemma 3 *For every $k=2,\ldots,m$, there exist $W(k,\cdot):[0,1]\to\mathbb{R}_+$ and $P(k,\cdot):[0,1]\to\mathbb{R}_+$ such that $\{W_n(k,\cdot)\}$ has a subsequence converging uniformly to $W(k,\cdot)$, and $\{P_n(k,\cdot)\}$ has a subsequence converging pointwise to $P(k,\cdot)$.*

---

[15] Also, recall that $\alpha_k := \int_{\frac{k-1}{m}}^{\frac{k}{m}} \alpha(u)\, du$.

Proof. Notice that $W_n(k,\cdot)$ are equicontinuous functions with Lipschitz coefficient $(\alpha_k + \ldots + \alpha_1)\bar{v}$. They are also uniformly bounded by $(\alpha_k + \ldots + \alpha_1)\bar{v}$. Therefore, the conclusion follows from Arzelà-Ascoli Theorem. The functions $P_n(k,\cdot)$ are monotonic (and hence have bounded variation) and are clearly uniformly bounded. The conclusion follows from Helly Selection Theorem. ■

As will become evident in the rest of this section, a result stronger than Lemma 3 actually holds. The algorithm we construct below shows that (for generic values of the parameters) all the convergent sequences $\{W_n(k,\cdot), P_n(k,\cdot)\}$, $k = 2,\ldots,m$, have the same limit which we denote by $(W(k,\cdot), P(k,\cdot))$.

For any pair (k,q), we let

$$P(k,q^-) = \lim_{q'\uparrow q} P(k,q')$$
$$P(k,q^+) = \lim_{q'\downarrow q} P(k,q')$$

denote the limit from the left and from the right, respectively, of the function $P(k,\cdot)$ at q.

As we will see below, the limiting equilibrium outcome alternates between phases of no delay and impasses. The distinction between these two phases has to do with the discounted time that it takes for the buyer to purchase the remaining units from a set of low types with positive measure (i.e., to purchase the remaining units of the low-quality good with positive probability). We say that there is no delay between the states (k,q) and (k',q') (with $k \geqslant k'$ and $q < q'$) if the discounted time that it takes in equilibrium to transit from the state (k,q) to the state (k',q') converges to zero as bargaining frictions vanish. In contrast, we say that bargaining reaches an impasse at the state (k,q) if for every $q' > q$, the discounted time that it takes in equilibrium to transit from the state (k,q) to the state (k,q') remains bounded away from zero when the discount factor converges to one. Finally, suppose that there is an impasse at (k,q). We measure the size of the impasse in terms of the real time in which the equilibrium stays around the state (k,q). Formally, these concepts are defined in Definition 2 and Definition 3.

Fix the discount factor δ and consider the corresponding equilibrium of the game. Let h_δ denote the on-path history in which all the offers of the form (k,p) with $p < kc$ are rejected. Of course, h_δ is a finite history which ends when the buyer makes an offer for all the remaining units at the price at which the high types break even. Let T_δ denote the length of the history h_δ. For each $t < T_\delta$, we let h^t_δ denote the first $t+1$ elements of h_δ (i.e., we truncate h_δ at period t).

Definition 2 *Fix a sequence of discount factors $\{\delta_n\}_{n=1}^{\infty}$ converging to one. Consider two states, (k^1,q^1) and (k^2,q^2), with $k^1 \geqslant k^2$ and $q^1 < q^2$. We say that the real time to reach*

(k^2, q^2) from (k^1, q^1) converges to zero if the following holds. For every $\varepsilon > 0$, there exists $\bar{n}$ such that for every $n > \bar{n}$ we can find $t_1 < t_2 < T_{\delta_n}$ satisfying:

(i) for $j = 1, 2$, the state associated to the history $h_{\delta_n}^{t_j}$ is $(k^j, \tilde{q}^j)$ for some $\tilde{q}^j \in (q^j - \varepsilon, q^j + \varepsilon)$;

(ii) $\delta_n^{t_2 - t_1} > 1 - \varepsilon$.

Definition 3 *Fix a sequence of discount factors $\{\delta_n\}_{n=1}^{\infty}$ converging to one. We say that there is an impasse at the state (k, q) if there exists $\gamma \in (0, 1)$ for which the following holds. For every $\varepsilon > 0$, there exists $\bar{n}$ such that for every $n > \bar{n}$ we can find $t_1 < t_2 < T_{\delta_n}$ satisfying:*

(i) for $j = 1, 2$, the state associated to the history $h_{\delta_n}^{t_j}$ is (k, q^j) for some $q^j \in (q - \varepsilon, q + \varepsilon)$;

(ii) $\delta_n^{t_2 - t_1} < \gamma$.

We say that the impasse is of real time $1 - \xi$ if ξ is the infimum over all γ leading to an impasse at (k, q).

We are now ready to describe the limiting equilibrium behavior. Our first result shows that, similarly to the case in which the buyer and the seller trade one single unit, there is an impasse at $(1, \bar{q}_1)$. Furthermore, once the state $(1, \bar{q}_1)$ is reached, there are no other impasses.

Proposition 3 *In the limit, the last impasse is at $(1, \bar{q}_1)$ and is of real time $1 - \left(\frac{\alpha_1 v}{c}\right)^2$.*

The proof of Proposition 3 is in Appendix D. Recall that the gains per unit are decreasing and that $\bar{q}_1$ is the type at which the buyer breaks even with the last unit if he pays the price c. Thus, for any number $k \geqslant 2$ of units left for trade and for any $q \geqslant \bar{q}_1$, the buyer's expected payoff is bounded away from zero when the state is (k, q). The usual Coasean argument implies that, in the limit, there cannot be any delay once the state (k, q) is reached. This implies that if there is an impasse at (k, q') with $k \geqslant 2$, then it must necessarily be the case that $q' < \bar{q}_1$. Now, notice that once the impasse is resolved, the buyer will end up purchasing the remaining k units at the price kc. However, in the limit, the buyer can increase his payoff at (k, q) by purchasing $k - 1$ units from all the types at the price $(k - 1)c$ and then the last unit from the low types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$. This implies that for δ_n sufficiently close to one, the buyer would have a profitable deviation. A similar contradiction can be easily derived if one assumes that, in the limit, there are no impasses.

Once we have established the last impasse, we can move on to characterize the rest of the limiting equilibrium outcome. To do so, we construct an algorithm that takes as given the $(j-1)$-th to last impasse and gives us the j-th to last one. A formal description of the algorithm requires several pieces of additional information. To clarify the meaning of the new variables and help the reader follow the rest of the argument, we provide here an informal description of our procedure. In particular, we show how to identify the penultimate impasse. This is less demanding in terms of notation and allows us to focus on the intuition behind our results. In Appendix D, we provide a formal description of our algorithm.

For every $k = 2, \ldots, m$, we denote by $\bar{q}_k$ the type at which the buyer breaks even if he trades the $(m - k + 1)$-th unit at the price c. Formally, $\bar{q}_k$ is implicitly defined by

$$\int_{\bar{q}_k}^{1} (\alpha_k v(s) - c)\, ds = 0, \tag{8}$$

provided that the solution to the above equation exists and is positive. The case in which the solution is negative and the case in which the solution does not exist are not relevant for our construction. For concreteness, in these cases we set $\bar{q}_k$ equal to $-k$. Notice that $\bar{q}_m < \ldots < \bar{q}_2 < \bar{q}_1$ since $\alpha_m > \ldots > \alpha_1$.

For any $k \geqslant 2$ and $q < \bar{q}_1$, consider the strategy of acquiring $(k-1)$ units at the price $(k-1)\, c$ from all the types $[q, 1]$ and then the last unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$. Let $\hat{q}_k$ denote the type at which the buyer breaks even. To simplify the exposition, let us assume that $\hat{q}_k$ is well defined and positive.[16] Formally, $\hat{q}_k$ is the solution to the following equation:

$$\int_{\hat{q}_k}^{1} ((\alpha_k + \ldots + \alpha_2)\, v(s) - (k-1)\, c)\, ds + \int_{\hat{q}_k}^{\bar{q}_1} \left(\alpha_1 \underline{v} - P\left(1, \bar{q}_1^-\right)\right) ds = 0. \tag{9}$$

Notice that $\hat{q}_k > 0$ implies that

$$(\alpha_k + \ldots + \alpha_1)\, \underline{v} < (k-1)\, c + P\left(1, \bar{q}_1^-\right).$$

Reservation price when there are two units left for trade.

We now examine what happens when there are two units left for trade. Suppose the state $(2, q)$ is reached, either on or off the equilibrium path. If $q > \hat{q}_2$, then the buyer's expected payoff is bounded away from zero. To see this, notice that if $q \geqslant \bar{q}_1$, then the buyer can purchase the two units from all the types in $[q, 1]$ at the price $2c$. On the other hand, if $q \in (\hat{q}_2, \bar{q}_1)$, then the buyer can purchase one unit from all the types at the price c and the second unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$. In either case the buyer's payoff is positive. Thus, we conclude that for $q > \hat{q}_2$ there is no delay at the state $(2, q)$. This, in turn, implies that for $q \in (\hat{q}_2, \bar{q}_1)$, the reservation price $P(2, q)$ is equal to

$$P\left(2, \hat{q}_2^+\right) = c + P\left(1, \bar{q}_1^-\right) > (\alpha_2 + \alpha_1)\, \underline{v}.$$

[16] Of course, in the formal statement of the result and in its proof, we consider the general case.

We conclude that as q approaches $\hat{q}_2$ from above, the buyer's expected payoff at $(2, q)$ converges to zero.

Consider now the state $(2, q)$ with $q < \hat{q}_2$. If the buyer purchases one unit from all the types at the price c and the second unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$, then his payoff will be negative. Also, for δ_n sufficiently close to one, the reservation price $P_n(2, q)$ must be bounded away from $P\left(2, \hat{q}_2^+\right)$, otherwise, the buyer's payoff would be negative. This implies that if we start from $(2, q)$ with $q < \hat{q}_2$, then we reach an impasse at $(2, \hat{q}_2)$. Using an argument similar to DL, it is easy to show that the impasse is of real time $1 - \left(\frac{(\alpha_2+\alpha_1)\underline{v}}{c+P\left(1,\bar{q}_1^-\right)}\right)^2$, and that for $q < \hat{q}_2$, the reservation price $P(2, q)$ is equal to

$$P\left(2, \hat{q}_2^-\right) = \frac{\left((\alpha_2 + \alpha_1)\underline{v}\right)^2}{c + P\left(1, \bar{q}_1^-\right)}.$$

Notice that the delay at the impasse doubles the delay necessary to decrease the seller's reservation price for the two low-quality units from $P\left(2, \hat{q}_2^+\right)$ to the buyer's valuation $(\alpha_2 + \alpha_1)\underline{v}$. Therefore, we say that there is double delay at the impasse.[17]

Let us now move to the case in which there are three units remaining. Recall that $\bar{q}_3$ is the type at which the buyer breaks even if he trades the first of these three units at the price c. We distinguish between two cases, depending on whether $\bar{q}_3$ is greater or smaller than $\hat{q}_2$. The algorithm that we construct characterizes the equilibrium outcome for a generic set of parameters. When there are three units, we restrict our attention to parameters such that $\hat{q}_2 \neq \bar{q}_3$. More generally, our algorithm restricts attention to the generic case in which $\bar{q}_{k+1} \neq \hat{q}_k$ for $k \in \{2, \ldots, m-1\}$. We consider this generic case for the remaining of this paper.

Case $\hat{q}_2 > \bar{q}_3$: there is an impasse at $(2, \hat{q}_2)$.

First, assume that $\hat{q}_2 > \bar{q}_3$. We claim that, in the limit, the penultimate impasse occurs at $(2, \hat{q}_2)$. Suppose, by contradiction, that the penultimate impasse is at (k, q) for some $k > 2$. The fact that $\hat{q}_2 > \bar{q}_3 > \bar{q}_4 > \ldots > \bar{q}_m$ implies that for $q \in (\hat{q}_2, \bar{q}_1)$ the buyer's payoff from purchasing $(k-1)$ units from all the types at the price $(k-1)c$ and the last unit from the types in $(q, \bar{q}_1)$ at the price $P\left(1, \bar{q}_1^-\right)$ is positive and bounded away from zero. Therefore, if there is an impasse at (k, q), it must be the case that $q < \hat{q}_2$.[18] Consider now the state (k, q') for some $q' \in (q, \hat{q}_2)$. In the limit, the buyer gets the same payoff as if he buys $(k-1)$ units from the types in $[q, 1]$ at the price $(k-1)c$ and the last unit from the

[17] We show that the "double delay" result holds at every impasse. This generalizes the "double delay" result found in DL to a model with divisibility.

[18] Clearly, the impasse cannot occur at (k, q) with $q \geqslant \bar{q}_1$. Recall that for any $q \geqslant \bar{q}_1$, the buyer's payoff from buying $k' > 1$ units at the price $k'c$ from the types $[q, 1]$ is strictly positive.

types in $[\underline{q}, \bar{q}_1]$ at the price $P\left(1, \bar{q}_1^-\right)$. However, we show that the payoff of this strategy is strictly smaller than the strategy under which the buyer purchases $(k-2)$ units at the price $(k-2)\,c$ and the last two units from the types in $[\underline{q}, \hat{q}_2]$ at the price $P\left(2, \hat{q}_2^-\right)$. This shows that for δ_n sufficiently close to one, the buyer would have a profitable deviation. In a similar way, we can rule out the case in which there is only one impasse at $(1, \bar{q}_1)$.

Case $\hat{q}_2 < \bar{q}_3$: there are no impasses when two units are left for trade.

Consider now the case $\hat{q}_2 < \bar{q}_3$. Notice that this implies $\hat{q}_2 < \hat{q}_3$. Using an argument similar to the one above, we show that if the state $(3, q)$ is reached (either on or off path) and $q < \hat{q}_3$, then there is an impasse at $(3, \hat{q}_3)$ of real time $1 - \left(\frac{(\alpha_3+\alpha_2+\alpha_1)\underline{v}}{2c+P\left(1,\bar{q}_1^-\right)}\right)^2$, and for $q' < \hat{q}_3$ the reservation price $P(3, q')$ is equal to

$$P\left(3, \hat{q}_3^-\right) = \frac{\left((\alpha_3 + \alpha_2 + \alpha_1)\,\underline{v}\right)^2}{2c + P\left(1, \bar{q}_1^-\right)}.$$

Finally, we use this fact to show that, on the limiting equilibrium path, there cannot be an impasse at $(2, \hat{q}_2)$. If this were the case, we could find a state (k, q) with $k > 2$ and $q < \hat{q}_2$ where the buyer purchases $(k-2)$ units from all the remaining types.[19] However, in the limit the buyer could increase his payoff by purchasing $(k-3)$ units from all the types and the last three units from the types in $(q, \hat{q}_3)$ at the price $P\left(3, \hat{q}_3^-\right)$. This implies that the buyer would have a profitable deviation for δ_n sufficiently close to one.

To sum up, we have shown that the penultimate impasse is at $(2, \hat{q}_2)$ if and only if $\hat{q}_2 > \bar{q}_3$. If, instead, $\hat{q}_2 < \bar{q}_3$, then we compare $\hat{q}_3$ and $\bar{q}_4$. Similar arguments to those developed above allow us to conclude that the penultimate impasse is at $(3, \hat{q}_3)$ if and only if $\hat{q}_3 > \bar{q}_4$. In general, we show that the penultimate impasse is at $\left(\hat{k}, \hat{q}_{\hat{k}}\right)$, where we set $\bar{q}_{m+1} = 0$ and let $\hat{k}$ denote the smallest $k = 2, \ldots, m$ for which $\hat{q}_k > \bar{q}_{k+1}$.

Once the penultimate impasse at $\left(\hat{k}, \hat{q}_{\hat{k}}\right)$ has been determined, we treat the last $\hat{k}$ units as a single unit and reapply our algorithm. This will give us the third to last impasse. In general, our algorithm takes an impasse as given and applies the procedure above to identify the impasse that comes right before. We provide a formal description of the algorithm in Appendix D.

Figure 3 illustrates a typical pattern of trade when offers are frequent. The buyer starts by making the cream-skimming offer (for all the m units). This offer is accepted by all the types smaller than q'. The rejection of the offer leads to the first impasse. After the impasse resolution, the buyer purchases (without delay) the first $m - k'$ units from all the types and the remaining k' units from the types smaller than q''. If the seller's type is larger

[19] Then the buyer purchases the last two units from the types in $(q, \hat{q}_2)$ at the price $P\left(2, \hat{q}_2^-\right)$.

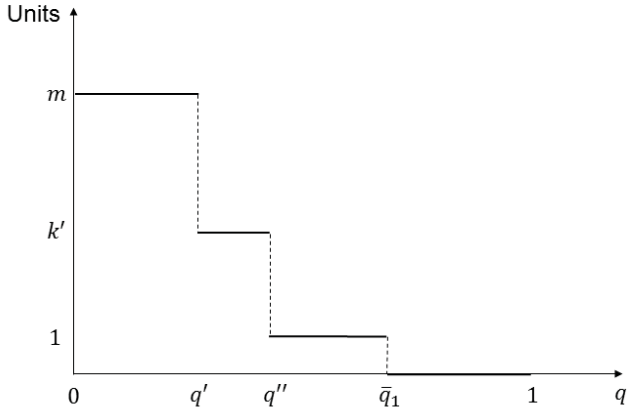

Figure 3: The m-Limiting Equilibrium Outcome

than q'' there is a second impasse. Following the impasse, the buyer makes a universal offer for $k'-1$ units and a cream-skimming offer for the last unit. The cream-skimming offer is immediately accepted by the types smaller than $\bar{q}_1$. Finally, the types larger than $\bar{q}_1$ sell their last unit when the third (and final) impasse of the game is resolved.

6.1.1 The Case $m = 2$

It is worthwhile to devote some attention to the limiting equilibrium outcome when the good is divided into two units. The case $m = 2$ provides a simple explanation of why the high-quality good is sold gradually over time and why this is disadvantageous to the buyer.

As in the previous section, we let $c = \frac{\bar{c}}{2}$ denote the seller's cost of each unit when the quality of the good is high. Recall that $\bar{q}_2$ ($\bar{q}_1$) denotes the type at which the buyer breaks even if he purchases the first (second) unit at the price c (see equations 7 and 8). Also, recall the definition of $\hat{q}_2$ from equation (9) and notice that $\hat{q}_2 < \bar{q}_2$. We assume that $\hat{q}_2 > 0$ (which implies $c + P\left(1, \bar{q}_1^-\right) > (\alpha_1 + \alpha_2)\,\underline{v}$, where $P\left(1, \bar{q}_1^-\right) = \frac{(\alpha_1 \underline{v})^2}{c}$). In section 6.1 we showed that in the limit, the seller's reservation price $P(2, \cdot)$ to sell two units of the good is equal to

$$P(2,q) = \begin{cases} \frac{((\alpha_2+\alpha_1)\underline{v})^2}{c+P\left(1,\bar{q}_1^-\right)} & \text{if } q < \hat{q}_2 \\ c + P\left(1, \bar{q}_1^-\right) & \text{if } q \in (\hat{q}_2, \bar{q}_1) \\ 2c & \text{if } q > \bar{q}_1. \end{cases}$$

When the bargaining frictions are arbitrarily small, the buyer purchases both units from the types smaller than $\hat{q}_2$ at the price $\frac{((\alpha_2+\alpha_1)\underline{v})^2}{c+P\left(1,\bar{q}_1^-\right)}$, and the bargaining process reaches the state $(2, \hat{q}_2)$ without delay. At that point a first impasse occurs (the impasse is of real time $1 - \left(\frac{((\alpha_2+\alpha_1)\underline{v})}{c+P\left(1,\bar{q}_1^-\right)}\right)^2$). After the impasse is resolved, the buyer purchases (without delay) the first unit from all the types above $\hat{q}_2$ and the second unit from the types in the interval $(\hat{q}_2, \bar{q}_1)$. The second and final impasse occurs at the state $(1, \bar{q}_1)$. The bargaining process ends as soon as the impasse is resolved, as the buyer proposes to pay c to get the second unit of the good.

Recall that the buyer's continuation payoff is equal to zero when the bargaining process reaches an impasse. Thus, the buyer's limiting equilibrium payoff is equal to

$$W(2,0) = \int_0^{\hat{q}_2} ((\alpha_2+\alpha_1)v(s) - P(2,s))\,ds = \hat{q}_2(\alpha_2+\alpha_1)\underline{v}\left(1 - \frac{((\alpha_2+\alpha_1)\underline{v})}{c+P\left(1,\bar{q}_1^-\right)}\right).$$

We now compare this case with the case in which the good is indivisible. In the context of our model, indivisibility can be simply described as a restriction on the type of admissible offers. In particular, we assume that the buyer can only make offers of the form $(2,p)$ for $p \geqslant 0$. Let $\bar{q}_{DL}$ denote the type at which the buyer breaks even if he makes the offer $(2, 2c)$. Formally, $\bar{q}_{DL}$ is the solution to the following equation

$$\int_{\bar{q}_{DL}}^{1} ((\alpha_2+\alpha_1)v(s) - 2c)\,ds = 0.$$

Notice that $\bar{q}_{DL} \in (\bar{q}_2, \bar{q}_1)$ since $\alpha_2 > \alpha_1$. It follows from DL that in the limit, the seller's reservation price (which we denote by P_{DL}) is equal to

$$P_{DL}(q) = \begin{cases} \frac{((\alpha_2+\alpha_1)\underline{v})^2}{2c} & \text{if } q < \bar{q}_{DL} \\ 2c & \text{if } q > \bar{q}_{DL}. \end{cases}$$

In the limit, the buyer purchases (without delay) the two units from the types smaller than $\bar{q}_{DL}$ at the price $\frac{((\alpha_2+\alpha_1)\underline{v})^2}{2c}$. Then, there is an impasse of real time $1 - \left(\frac{((\alpha_2+\alpha_1)\underline{v})}{2c}\right)^2$. The bargaining process ends as soon as the impasse is resolved (the buyer purchases the two units from all the remaining types at the price $2c$). Thus, the buyer's limiting equilibrium payoff is equal to

$$W_{DL}(0) = \int_0^{\bar{q}_{DL}} ((\alpha_2+\alpha_1)v(s) - P_{DL}(s))\,ds = \bar{q}_{DL}(\alpha_2+\alpha_1)\underline{v}\left(1 - \frac{((\alpha_2+\alpha_1)\underline{v})}{2c}\right).$$

It is immediate to see that the buyer is better off when the good is indivisible. In fact, $W_{DL}(0) > W(2,0)$ since $\bar{q}_{DL} > \hat{q}_2$ and $\frac{((\alpha_2+\alpha_1)\underline{v})}{2c} < \frac{((\alpha_2+\alpha_1)\underline{v})}{c+P\left(1,\bar{q}_1^-\right)}$.

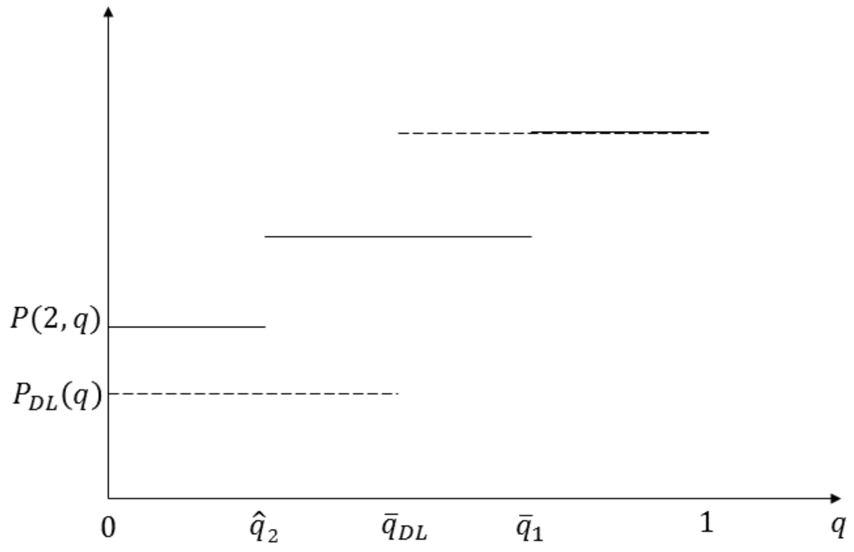

Figure 4: Reservation price for two units (benchmark model: solid line; DL: dashed line)

When the good is indivisible, the buyer pays a large price (equal to $2c$) when the belief is above $\bar{q}_{DL}$. This, however, has a positive effect on the buyer's initial payoff. A large price after the impasse implies a long delay and a severely reduced price before the impasse (see Figure 4).

Suppose now that the good is divisible and the state is $(2, q)$ with $q \in (\bar{q}_{DL}, \bar{q}_1)$. The buyer's payoff is still positive if he pays $2c$ for the two units. However, a more profitable strategy is now available. Specifically, the buyer is strictly better off by making a universal offer (for the first unit) and then proceeding with cream-skimming offers for the remaining unit. It follows that in the interval $(\bar{q}_{DL}, \bar{q}_1)$ the seller's reservation price (for two units) is lower in the model with divisibility than in the model with indivisibility (again, see Figure 4). This, in turn, has two effects. First, in the initial phase of the bargaining process (i.e., before the first impasse is reached), the buyer trades with a smaller set of types when the good is divisible ($\hat{q}_2 < \bar{q}_{DL}$).[20] Second, since there is double delay at each impasse, the buyer pays a larger initial price in the model with divisibility ($P(2,0) > P_{DL}(0)$). Clearly, both effects have a negative impact on the buyer's payoff.

[20] Notice that the strategy of making a universal offer yields a strictly positive payoff when the buyer's belief lies in the interval $(\hat{q}_2, \bar{q}_{DL})$.

6.2 Infinitely Divisible Good

In this section, we investigate the m-limiting equilibrium outcome as m goes to infinity.[21] Assume that the good is divided into $m = 1, 2, \ldots$ units. For each m, our algorithm yields a set of N_m impasses, for some $N_m \geqslant 1$. Every impasse $j = 1, \ldots, N_m$ is characterized by the number of units k_j left for trade, and by the buyer's belief q_j^m. Since m varies throughout the analysis, it is more convenient to work with the real size $\frac{k_j}{m}$ of the good that remains to be traded. We also order the impasses so that $q_1^m > q_2^m > \ldots > q_{N_m}^m$. Therefore, the first impasse in the game happens at the state $\left(\frac{k_{N_m}}{m}, q_{N_m}^m\right)$, while the last one occurs at $\left(\frac{k_1}{m}, q_1^m\right)$ (of course, $k_1 = 1$). We let

$$A_m := \left\{ \left(\frac{k_{N_m}}{m}, q_{N_m}^m \right), \ldots, \left(\frac{k_1}{m}, q_1^m \right) \right\}$$

denote the set of impasses when the good is divided into m units. Finally, for every m, we set $\left(\frac{k_0}{m}, q_0^m\right) = \left(\frac{k_1}{m}, q_1^m\right)$. The first assertion of Proposition 4 establishes that the largest probability that a cream-skimming offer is accepted converges to zero. The second result shows that the size of the largest universal offer also shrinks to zero.

Proposition 4 *The number of delays grows without bound and the size of each delay shrinks to zero:*

$$\lim_{m\to\infty} \max_{\left(q_u^m, \frac{k_u}{m}\right)\in A_m} \left| q_u^m - q_{u-1}^m \right| = 0,$$
$$\lim_{m\to\infty} \max_{\left(q_u^m, \frac{k_u}{m}\right)\in A_m} \left(\frac{k_u}{m} - \frac{k_{u-1}}{m} \right) = 0.$$

Once we have established Proposition 4, it is immediate to conclude that the limiting outcome is Coasean and trade of an infinitely divisible high-quality good will occur gradually over time. This result leads to the differential equation (6) which determines the rate at which the information is revealed and the good is sold.

To prove Proposition 4, we derive a dynamic system that links the beliefs at (the limits of) three consecutive impasses (see equation (18) below). The system follows from the following three conditions. First, the buyer's payoff from trading between two impasses is equal to zero. Second, there is double delay at every impasse. Finally, the seller's low types are indifferent between accepting and rejecting cream-skimming offers. Below is the formal derivation of equation (18). Then in Appendix C, we analyze the dynamic system

[21] This order of limits (first with respect to the length of the period, then with respect to the size of each unit) allows us to use the algorithm described in Section 6.1. It is an open question whether the same limiting outcome is obtained if the order of limits is exchanged.

and show that, in the limit, the distance between the beliefs at two consecutive impasses must shrink to zero.

Assume towards a contradiction that

$$\limsup_{m} \max_{A_m} \left|q_u^m - q_{u-1}^m\right| > \varepsilon,$$

for some $\varepsilon > 0$. By taking a subsequence if necessary, we may assume that there exists a sequence $\left\{\left(\left(z_{r_m+1}^m, q_{r_m+1}^m\right), \left(z_{r_m}^m, q_{r_m}^m\right)\right)\right\}_{m=1}^{\infty}$, with $\left(\left(z_{r_m+1}^m, q_{r_m+1}^m\right), \left(z_{r_m}^m, q_{r_m}^m\right)\right) \in A_m \times A_m$ for every m, converging to $((z_1, q_1), (z_0, q_0))$ such that

$$q_0 - q_1 = \nu > 0. \tag{10}$$

First, we claim that $z_j = \psi(q_j)$, for $j = 0, 1$. Here we provide the proof for $j = 1$ (the proof for $j = 0$ is similar and therefore omitted). It follows from the algorithm presented in Section 6.1 that $q_{r_m+1}^m$ lies between $\bar{q}_{j_m+1}$ and $\bar{q}_{j_m}$, for some $j_m = 1, \ldots, m-1$. Recall that $\bar{q}_{j_m+1}$ and $\bar{q}_{j_m}$ satisfy

$$\int_{\bar{q}_{j_m+1}}^1 \left[\left(\int_{\frac{j_m}{m}}^{\frac{j_m+1}{m}} \alpha(u)\, du\right) v(s) - \frac{\bar{c}}{m}\right] ds = 0,$$

$$\int_{\bar{q}_{j_m}}^1 \left[\left(\int_{\frac{j_m-1}{m}}^{\frac{j_m}{m}} \alpha(u)\, du\right) v(s) - \frac{\bar{c}}{m}\right] ds = 0.$$

Since α' is continuous, we conclude that $\bar{q}_{j_m+1}$ and $\bar{q}_{j_m}$ become uniformly close as m goes to infinity, which leads to the desired conclusion.

Second, our algorithm implies that the buyer's payoff from transiting from the state $\left(z_{r_m+1}^m, q_{r_m+1}^m\right)$ to the state $\left(z_{r_m}^m, q_{r_m}^m\right)$ must be equal to zero. The algorithm also shows that there are two different ways to express the payoff from this transition. We can either assume that the buyer first makes the universal offer to purchase $m\left(z_{r_m+1}^m - z_{r_m}^m\right)$ units from all the types above $q_{r_m+1}^m$. Then he proposes to purchase $mz_{r_m}^m$ units at the price $P\left(mz_{r_m}^m, \left(q_{r_m}^m\right)^-\right)$. This offer is accepted by all the (low) types in the interval $\left[q_{r_m+1}^m, q_{r_m}^m\right]$. Alternatively, we can assume that the buyer first makes the offer to purchase $mz_{r_m+1}^m$ units at the price $P\left(mz_{r_m+1}^m, \left(q_{r_m+1}^m\right)^+\right)$. Again, this cream-skimming offer is accepted by all the types in $\left[q_{r_m+1}^m, q_{r_m}^m\right]$. If the offer is rejected, then the buyer makes the universal offer to purchase $m\left(z_{r_m+1}^m - z_{r_m}^m\right)$ from all the types above $q_{r_m}^m$. It turns out to be more convenient to work with the second expression. Thus, we have:

$$\begin{gathered}\left(q_{r_m}^m - q_{r_m+1}^m\right)\left[\int_0^{z_{r_m+1}^m} \alpha(u)\, \underline{v} du - P\left(mz_{r_m+1}^m, \left(q_{r_m+1}^m\right)^+\right)\right] + \\ \int_{z_{r_m}^m}^{z_{r_m+1}^m} \left(\int_{q_{r_m}^m}^1 \left[\alpha(u) v(s) - \bar{c}\right] ds\right) du = 0.\end{gathered} \tag{11}$$

The expression in (11) is continuous in $\left(z^m_{r_m+1}, q^m_{r_m+1}, z^m_{r_m}, q^m_{r_m}\right)$ and $P\left(mz^m_{r_m+1}, \left(q^m_{r_m+1}\right)^+\right)$. Hence, since the sequence $\left\{\left(\left(z^m_{r_m+1}, q^m_{r_m+1}\right), \left(z^m_{r_m}, q^m_{r_m}\right)\right)\right\}_{m=1}^{\infty}$ converges to $\left((z_1, q_1), (z_0, q_0)\right)$, it follows that the sequence $\left\{P\left(mz^m_{r_m+1}, \left(q^m_{r_m+1}\right)^+\right)\right\}$ is convergent and the limit, which we denote by $\bar{P}_1$, must satisfy

$$(q_0 - q_1)\left[\int_0^{z_1} \alpha(u)\,\underline{v}du - \bar{P}_1\right] + \int_{z_0}^{z_1}\left(\int_{q_0}^{1}\left[\alpha(u)\,v(s) - \bar{c}\right]ds\right)du = 0. \tag{12}$$

The two components of the buyer's payoff from the transition between (the limits of) two consecutive impasses can be easily visualized in Figure 5.

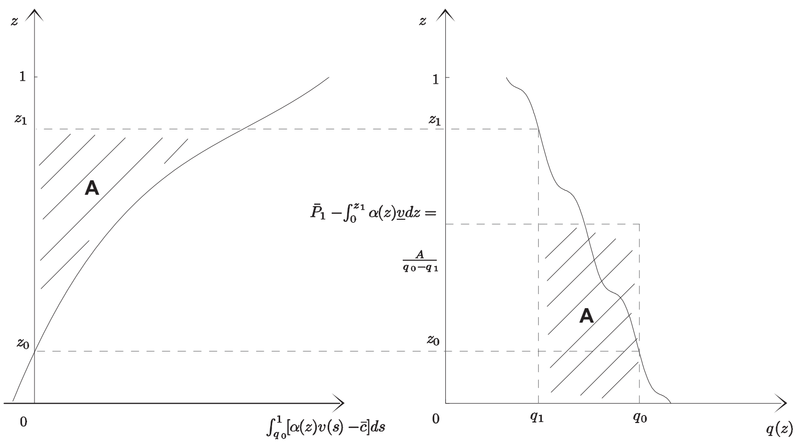

Figure 5: The buyer's payoff from transiting between two consecutive impasses

Recall that $z_j = \psi(q_j)$, for $j = 0, 1$, and $\psi'(q) < 0$ for every q. Thus, we may change variables and write equation (12) as

$$(q_0 - q_1)\left[\int_0^{\psi(q_1)} \alpha(u)\,\underline{v}du - \bar{P}_1\right] + \int_{q_1}^{q_0}\left[-\psi'(u)\left(\int_{q_0}^{1}\left[\alpha(\psi(u))\,v(s) - \bar{c}\right]ds\right)\right]du = 0. \tag{13}$$

Next, we show that $q_0 < q(0)$ (and, therefore, $z_0 > 0$). The proof is by contradiction. It follows easily from the algorithm that if $z_0 = 0$, then $\bar{P}_1 = z_1\bar{c}$. But then as m goes to infinity, the buyer's payoff from transiting from the state $\left(z^m_{r_m+1}, q^m_{r_m+1}\right)$ to the state

$\left(z_{r_m}^m, q_{r_m}^m\right)$ converges to

$$(q(0) - q_1)\left[\int_0^{z_1} \alpha(u)\underline{v}du - z_1\bar{c}\right] + \int_0^{z_1}\left(\int_{q(0)}^1 [\alpha(u)v(s) - \bar{c}]\,ds\right)du =$$
$$\int_0^{z_1}\left(\int_{q_1}^1 [\alpha(u)v(s) - \bar{c}]\,ds\right)du < 0,$$

which contradicts equation (12).

Therefore, for m large there is an impasse $\left(z_{r_m-1}^m, q_{r_m-1}^m\right)$ that comes right after $\left(z_{r_m}^m, q_{r_m}^m\right)$. Again, by taking a subsequence if necessary, assume that the sequence $\left\{\left(z_{r_m-1}^m, q_{r_m-1}^m\right)\right\}$ converges and let (z_{-1}, q_{-1}) denote its limit. We claim that $q_{-1} \in (q_0, q(0))$. The proof that $q_{-1} < q(0)$ is identical to the proof above that $q_0 < q(0)$. We now show that $q_{-1} > q_0$. Recall that $q_{r_m-1}^m > q_{r_m}^m$ for every m, and assume, by contradiction, that $q_{-1} = q_0$. Also, recall that $\left(z_{r_m+1}^m, q_{r_m+1}^m\right)$ and $\left(z_{r_m}^m, q_{r_m}^m\right)$ are two consecutive impasses. Thus, it follows from the algorithm that if $q_{-1} = q_0$, then

$$\bar{P}_1 = (z_1 - z_0)\bar{c} + \int_0^{z_0} \alpha(u)\underline{v}du.$$

This, in turn, implies that, as m goes to infinity, the buyer's payoff from transiting from $\left(z_{r_m+1}^m, q_{r_m+1}^m\right)$ to $\left(z_{r_m}^m, q_{r_m}^m\right)$ converges to

$$(q_0 - q_1)\left[\int_0^{z_1} \alpha(u)\underline{v}du - (z_1 - z_0)\bar{c} - \int_0^{z_0}\alpha(u)\underline{v}du\right] + \int_{z_0}^{z_1}\left(\int_{q_0}^1 [\alpha(u)v(s) - \bar{c}]\,ds\right)du =$$
$$\int_{z_0}^{z_1}\left(\int_{q_1}^1 [\alpha(u)v(s) - \bar{c}]\,ds\right)du < 0,$$

which, again, contradicts equation (12) (see Figure 6).

Using the same argument described above, we conclude that the sequence $\left\{P\left(mz_{r_m}^m, \left(q_{r_m}^m\right)^+\right)\right\}$ has a limit, which we denote by $\bar{P}_0$, and that

$$(q_{-1} - q_0)\left[\int_0^{\psi(q_0)} \alpha(u)\underline{v}du - \bar{P}_0\right] + \int_{q_0}^{q_{-1}}\left[-\psi'(u)\left(\int_{q_{-1}}^1 [\alpha(\psi(u))v(s) - \bar{c}]\,ds\right)\right]du = 0. \tag{14}$$

The next step is to show that we can express $\bar{P}_1$ in terms of (q_1, q_0, q_{-1}). We then substitute this expression into (13) and obtain an equation in (q_1, q_0, q_{-1}). From equation (14), we have

$$\bar{P}_0 = \int_0^{\psi(q_0)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}}\left[-\psi'(u)\left(\int_{q_{-1}}^1 [\alpha(\psi(u))v(s) - \bar{c}]\,ds\right)\right]du}{q_{-1} - q_0}\right). \tag{15}$$

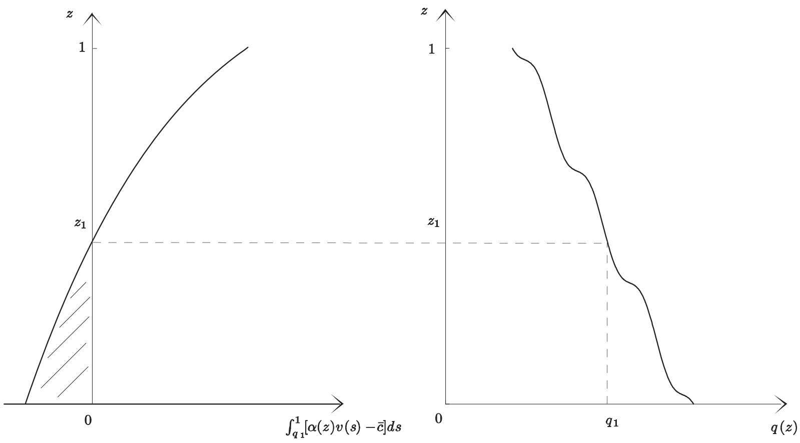

Figure 6: Profitable deviation between two impasses

It easy to show that the sequence of prices $\left\{P\left(mz^m_{r_m}, \left(q^m_{r_m}\right)^-\right)\right\}$ admits a limit which we denote by $\underline{P}_0$. Using the fact that there is double delay at every impasse and equation (15), we obtain

$$\underline{P}_0 = \left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\bar{P}_0}\right) \int_0^{\psi(q_0)} \alpha\left(u\right)\underline{v}du =$$

$$\left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}}\left[-\psi'(u)\left(\int_{q_{-1}}^{1}[\alpha(\psi(u))v(s)-\bar{c}]ds\right)\right]du}{q_{-1}-q_0}\right)}\right) \int_0^{\psi(q_0)} \alpha\left(u\right)\underline{v}du.$$

Recall that, for every m, the seller's low types are indifferent between selling the remaining $mz^m_{r_m+1}$ units at the price $P\left(mz^m_{r_m+1}, \left(q^m_{r_m+1}\right)^+\right)$ and selling the first $m\left(z^m_{r_m+1} - z^m_{r_m}\right)$ at the price $\left(z^m_{r_m+1} - z^m_{r_m}\right)\bar{c}$ and the remaining $mz^m_{r_m}$ at the price $P\left(mz^m_{r_m}, \left(q^m_{r_m}\right)^-\right)$. Thus, in the limit, as m goes to infinity, we have (recall that $z_1 - z_0 = \psi(q_1) - \psi(q_0)$)

$$\bar{P}_1 = \left(\psi(q_1) - \psi(q_0)\right)\bar{c} + \underline{P}_0 =$$

$$\left(\psi(q_1) - \psi(q_0)\right)\bar{c} + \left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}}\left[-\psi'(u)\left(\int_{q_{-1}}^{1}[\alpha(\psi(u))v(s)-\bar{c}]ds\right)\right]du}{q_{-1}-q_0}\right)}\right) \int_0^{\psi(q_0)} \alpha\left(u\right)\underline{v}du. \tag{16}$$

Finally, we substitute equation (16) into equation (13) and obtain an equation linking the beliefs at (the limits of) three consecutive impasses:

$$\Phi\left(q_1, q_0, q_{-1}\right) := \int_0^{\psi(q_1)} \alpha\left(u\right) \underline{v} du + \left(\frac{\int_{q_1}^{q_0} \left[-\psi'(u)\left(\int_{q_0}^{1}[\alpha(\psi(u))v(s)-\bar{c}]ds\right)\right]du}{q_0-q_1}\right) -$$
$$\left(\psi(q_1) - \psi(q_0)\right)\bar{c} - \left(\frac{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du}{\int_0^{\psi(q_0)} \alpha(u)\underline{v}du + \left(\frac{\int_{q_0}^{q_{-1}} \left[-\psi'(u)\left(\int_{q_{-1}}^{1}[\alpha(\psi(u))v(s)-\bar{c}]ds\right)\right]du}{q_{-1}-q_0}\right)}\right) \int_0^{\psi(q_0)} \alpha\left(u\right) \underline{v} du = 0. \tag{17}$$

Since $q_{-1} < q\left(0\right)$, for m large there is an impasse $\left(z^m_{r_m-2}, q^m_{r_m-2}\right)$ that comes right after $\left(z^m_{r_m-1}, q^m_{r_m-1}\right)$. As usual, by taking a subsequence if necessary, assume that the sequence $\left\{\left(z^m_{r_m-2}, q^m_{r_m-2}\right)\right\}$ converges and let $\left(z_{-2}, q_{-2}\right)$ denote its limit. It is also easy to show that $\Phi\left(q_0, q_{-1}, q_{-2}\right) = 0$, where the function Φ is defined in (17).

We proceed inductively and obtain a sequence of limits of impasses $\left\{\left(z_i, q_i\right)\right\}_{i=1,0,-1,\ldots}$. For every $i = 1, 0, -1, \ldots$, we have $z_i = \psi\left(q_i\right)$ and

$$\Phi\left(q_i, q_{i-1}, q_{i-2}\right) = \int_0^{\psi(q_i)} \alpha\left(u\right) \underline{v} du + \left(\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u)\left(\int_{q_{i-1}}^{1}[\alpha(\psi(u))v(s)-\bar{c}]ds\right)\right]du}{q_{i-1}-q_i}\right) -$$
$$\left(\psi(q_i) - \psi(q_{i-1})\right)\bar{c} - \left(\frac{\int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du}{\int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du + \left(\frac{\int_{q_{i-1}}^{q_{i-2}} \left[-\psi'(u)\left(\int_{q_{i-2}}^{1}[\alpha(\psi(u))v(s)-\bar{c}]ds\right)\right]du}{q_{i-2}-q_{i-1}}\right)}\right) \int_0^{\psi(q_{i-1})} \alpha\left(u\right) \underline{v} du = 0. \tag{18}$$

Also, for every $i = 1, 0, -1, \ldots$, we let $\bar{P}_i$ denote the limit, as m goes to infinity, of $P\left(m z^m_{r_m+i}, \left(q^m_{r_m+i}\right)^+\right)$.

By construction, the sequence $\{q_i\}$ is increasing and bounded above by $q\left(0\right)$. Therefore, we have

$$\lim_{i\to-\infty}\left(q_{i-1} - q_i\right) = 0. \tag{19}$$

In Appendix C, we analyze the dynamic system in (18) and show that if an impasse is small, then all the impasses that precede it are also small. Formally, we demonstrate that there exists $\varepsilon > 0$ such that if $\left(q_{i-1} - q_i\right) < \varepsilon$, for some i, then $\left(q_{i'-1} - q_{i'}\right) < \frac{\nu}{2}$ for every $i' = 1, 0, \ldots, i+1$. This contradicts equation (10), showing that $\limsup_m \max_{A_m}\left|q^m_u - q^m_{u-1}\right| = 0$ and completing the proof of Proposition 4.

7 Extensions

In this section, we consider two different extensions of the benchmark model. The first one is related to the types of trading instruments available to the buyer. In the benchmark model, the buyer can choose in every period the number of the remaining units that he wants to purchase. Theorem 1 suggests that having many trading opportunity is not beneficial to the buyer. According to this line of reasoning, one would expect that the buyer would not benefit from a larger set of trading opportunities. To assess this conjecture, in Section 7.1, we extend the contract space by allowing the buyer to offer menus of offers.

Finally, in our model we assume that the gains from trade decrease as the parties make more transactions. We investigate the importance of this assumption in Section 7.2, where we analyze the case of increasing gains from trade.

7.1 Menus of Offers

Our first extension allows the buyer to propose menus and, thus, make several offers at the same time. As in the benchmark model, an offer (k, p) specifies the number of units k that the parties trade and the transfer $p \geqslant 0$ that the seller is entitled to receive. A menu $\mathcal{M}$ is a set of offers. In particular, when the number of units on the table is $k = 1, \ldots, m$, the set of available menus is the set of all compact subsets of $\{1, ..., k\} \times \mathbb{R}_+$. Upon being offered a menu $\mathcal{M}$, the seller can either accept one offer in $\mathcal{M}$ or reject all the offers.

We continue to restrict attention to stationary (perfect Bayesian) equilibria. In our context, this means that the decision of any seller's type depends on the number of units left for trade and the menu $\mathcal{M}$ offered by the buyer.[22] Furthermore, we impose the additional requirement that all the seller's high types agree on their decisions at any point in time. Formally, we consider stationary equilibria in which the seller's strategy satisfies the following property. For every t and for every history $((\mathcal{M}_0, a_0), \ldots, (\mathcal{M}_{t-1}, a_{t-1}), \mathcal{M}_t)$ of proposed menus and seller's decisions, there exists an action that is chosen by all the types larger than $\hat{q}$. Under this refinement, the buyer's beliefs (on the equilibrium path) take a very simple form. Similarly to the benchmark model, after every on-path history, the buyer believes that the seller's type is uniformly distributed over the set $[\tilde{q}, 1]$ for some $\tilde{q} \in [0, \hat{q}]$.[23] Our refinement implies that the high types accept an offer if and only if it yields a non-negative current payoff.[24] This is in line with the high types' behavior in the

[22] As in the benchmark model, the low types' decision also depends on whether or not the buyer is convinced that the quality of the good is low (there are no stationary equilibria if we do not allow for this possibility).

[23] In contrast, if we allow the high types to make different choices, then the beliefs are not "unidimensional" (in the sense that one has to keep track of the fractions of *both* the low *and* the high types still present in the game), and the analysis is not tractable. Moreover, it is an open question whether there exist equilibria that do not satisfy our refinement.

[24] Under our refinement, it is without loss of generality (in terms of equilibrium outcomes) to restrict attention to equilibria in which the menus proposed by the buyer contains at most one offer that yields a non-negative payoff to the high types.

equilibrium of the benchmark model.

In Appendix F, we construct a stationary equilibrium of the game in which the buyer can propose menus with at most two offers. We show that, as the bargaining frictions vanish, the outcome of the equilibrium described in Appendix F converges to the m-limiting outcome of the benchmark model (where m denotes the total number of units available for trade). Furthermore, the assumption of allowing the buyer to make at most two offers is not restrictive. Consider the game in which the buyer can propose arbitrary menus (arbitrary compact subsets of the set of feasible offers). Under our refinement (and for generic values of the parameters), all stationary equilibria of this game are outcome equivalent to the equilibrium in Appendix F. This shows that the main result of this paper (Theorem 1) is robust to the introduction of menus. In particular, the analysis in this section corroborates our finding that the buyer does not benefit from additional trading instruments.

7.2 Increasing Gains

Thus far, we have considered the case in which the gains from trade are decreasing in the number of traded units. In this section, we analyze the other benchmark case, in which the gains are strictly increasing. Consider the model described in Section 2 and assume that $\alpha_m < \ldots < \alpha_1$. We refer to this as to the model with divisibility and increasing gains. The assumption of increasing gains is natural when the different units are complementary. For example, when different shareholders dispute the control of a firm, the marginal value of an additional share is often increasing in the number of shares already owned, as bigger shareholders can exert more influence on the firm's decisions. As we will see, this complementarity dramatically changes the outcome of the bargaining process.

Proposition 1 and Proposition 2, which guarantee the existence and the uniqueness of stationary equilibrium outcomes, respectively, continue to hold in the model with divisibility and increasing gains. In this section, we investigate the limiting equilibrium outcome as the length of each period Δ_n shrinks to zero. Our main result asserts that divisibility does not play a role when the gains from trade are increasing. To state our result, we need to consider the model (analyzed by DL) in which the m units are indivisible and the buyer can only purchase them all at once. We refer to this as to the model with indivisibility.

Proposition 5 *For any $m \geqslant 2$, the limiting equilibrium outcome of the model with divisibility and increasing gains coincides with the limiting equilibrium outcome of the model with indivisibility.*

We outline the proof of Proposition 5 in Appendix G.

8 Concluding Remarks

This paper studies bargaining with interdependent values and divisible goods. We show that a new pattern of trade, gradual trading, arises in these markets and that the possibility of purchasing fractions of the good is detrimental to the uninformed buyer. In the limit, when offers are frequent and the good is arbitrarily divisible, the high-quality good is sold smoothly over time and the buyer's payoff converges to the lowest possible level. When adverse selection is particularly severe, the buyer's limiting equilibrium payoff is equal to zero.

Throughout the paper we have made a few simplifying assumptions which make the analysis tractable. In particular, we have assumed that the quality of the good can take only two values (i.e., the seller has only two types). This assumption guarantees the existence of stationary equilibria. We are unable to show that stationary equilibria exist in a model with more than two types. However, under some conditions on the primitives of the model, the main result of the paper extends to the case of finitely many types.

Consider the model described in Section 4 but assume that $c(q)$ and $v(q)$ are two (non-decreasing) step functions with finitely many discontinuity points. Without loss, we assume that $c(0) = 0$ and let $\hat{q} = \sup_{\{q:c(q)=0\}} q$ denote the probability that the good is of the worst quality. Also, suppose that for $z = 0, 1$, there exists $q(z) \in (0, \hat{q})$

$$\int_{q(z)}^{1} [\alpha(z) v(s) - c(1)] \, ds = 0.$$

In words, the belief at which the buyer breaks even when he purchases any unit of the good at the price $c(1)$ corresponds to a type with the lowest quality of the good. It is not difficult to show that under these assumptions Theorem 1 continues to hold.

In our model, the buyer learns about the quality of the good only through the seller's behavior. This assumption is reasonable in a number of important applications and our model constitutes a useful theoretical benchmark to study bargaining over divisible objects. However, it would be interesting to extend the model to allow for additional forms of learning. In particular, learning could be endogenous (i.e., coming from the consumption of parts of the good) or exogenous. We leave the study of bargaining with learning for future work.

Finally, our results show that the limiting equilibrium outcome depends on whether the gains from trade are decreasing or increasing in the number of traded units. The techniques developed in this paper do not extend to the case of constant gains from trade. This is another interesting avenue for future research.

Appendix A: Existence of Stationary Equilibria

Proof of Lemma 1.

Given a perfect Bayesian equilibrium $(\sigma_B, \sigma_S, \mu)$, we let $H_k(\sigma_B, \sigma_S, \mu)$, $k = 1, \ldots, m$, denote the set of histories h^t after which there are k units left for trade and such that $\mu([0, \hat{q}] | h^t) = 1$.

First, we show that the claim is true if $h^t \in H_1(\sigma_B, \sigma_S, \mu)$, that is, there is only one unit left for trade. Define $\bar{u}_L$ as the supremum, over all the perfect Bayesian equilibria, of the continuation payoffs of the low types $q \leqslant \hat{q}$ at histories $h^t \in H_1(\sigma_B, \sigma_S, \mu)$. Assume towards a contradiction that $\bar{u}_L > 0$. Take $\varepsilon = \left(\frac{1-\delta}{2}\right)\bar{u}_L$ and notice that there exists a perfect Bayesian equilibrium $(\sigma_B, \sigma_S, \mu)$ and a history $\bar{h}^t \in H_1(\sigma_B, \sigma_S, \mu)$ at which the buyer makes the proposal $\varphi_t = (1, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$. We claim that the conditional probability, given the beliefs $\mu\left(\bar{h}^t\right)$, that the proposal φ_t is accepted must be one. To see this, notice that if φ_t is accepted with a probability of less than one (given $\mu\left(\bar{h}^t\right)$), then $\left(\bar{h}^t, (\varphi_t, R)\right) \in H_1(\sigma_B, \sigma_S, \mu)$ (i.e., $\mu\left([0, \hat{q}] | \bar{h}^t, (\varphi_t, R)\right) = 1$) and the continuation payoff of the low types is at most $\bar{u}_L$. But then it is not optimal for a low type to reject φ_t since $\bar{u}_L - \varepsilon > \delta\bar{u}_L$. However, using a similar argument it is easy to show that the conditional probability, given the belief $\mu\left(\bar{h}^t\right)$, that the proposal $\varphi'_t = \left(1, \bar{u}_L - \left(\frac{3}{2}\right)\varepsilon\right)$ is accepted is also one (since $\bar{u}_L - \left(\frac{3}{2}\right)\varepsilon > \delta\bar{u}_L$). Thus, the buyer has a profitable deviation at $\bar{h}^t$ since he strictly prefers the proposal φ'_t to φ_t.

Next, assume that the claim is true for any $h^t \in H_1(\sigma_B, \sigma_S, \mu) \cup \ldots \cup H_{k-1}(\sigma_B, \sigma_S, \mu)$, $k = 2, \ldots, m$. We show that the claim is also true for any $h^t \in H_k(\sigma_B, \sigma_S, \mu)$. Again, towards a contradiction, let $\bar{u}_L > 0$ be the supremum, over all the perfect Bayesian equilibria $(\sigma_B, \sigma_S, \mu)$, of the continuation payoffs of the low types $q \leqslant \hat{q}$ at histories $h^t \in H_k(\sigma_B, \sigma_S, \mu)$ .Take $\varepsilon = \left(\frac{1-\delta}{2}\right)\bar{u}_L$ and notice that there exist a perfect Bayesian equilibrium $(\sigma_B, \sigma_S, \mu)$ and a history $\bar{h}^t \in H_k(\sigma_B, \sigma_S, \mu)$ at which the buyer makes the proposal $\varphi_t = (k', p)$ for some $k' = 1, \ldots, k$ and $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$. Using the induction hypothesis and an argument similar to the one presented in the previous paragraph, we conclude that the conditional probability, given $\mu\left(\bar{h}^t\right)$, that the proposal $\varphi_t = (k', p)$ is accepted is equal to one. However, the same is true for the proposal $\varphi'_t = \left(k', \bar{u}_L - \left(\frac{3}{2}\right)\varepsilon\right)$ which is, therefore, strictly preferred to φ_t. Again, this shows that the buyer has a profitable deviation at $\bar{h}^t$ and concludes our proof. ■

Proof of Lemma 2.

The first step of the proof is to show that, at any history, the equilibrium continuation payoff of the high types is zero. From the characterization in DL, we know that this is true at every history h^t at which there is only one unit left for trade. For $k = 1, \ldots, m - 1$, we assume that the continuation payoff (in any stationary equilibrium) of the high types $q > \hat{q}$ is zero when k or fewer units are left for trade. We show that the continuation payoff of the high types is also zero when the number of remaining units is $k+1$.

Define $\bar{u}_H$ as the supremum, over all stationary equilibria, of the continuation payoffs of the high types $q > \hat{q}$ at histories in which the remaining units are $k+1$. Trivially, we have $\bar{u}_H \in [0, (\alpha_1 + \ldots + \alpha_{k+1}) \bar{v}]$. We claim that $\bar{u}_H = 0$. By contradiction, assume that $\bar{u}_H > 0$. Take $\varepsilon = \left(\frac{1-\delta}{2}\right) \bar{u}_H$ and notice that there exist a stationary equilibrium $(\sigma_B, \sigma_S, \mu)$ and a history $\bar{h}^t$ at which the high types $q > \hat{q}$ obtain the continuation payoff $u_H \in [\bar{u}_H - \varepsilon, \bar{u}_H]$ by accepting a certain offer $\varphi_t = (k', k'c + u_H)$ for some $k' \in \{1, \ldots, k+1\}$. We claim that the conditional probability, given the beliefs $\mu\left(\bar{h}^t\right)$, that the proposal φ_t is accepted is one. First, every type $q > \hat{q}$ must accept φ_t at $\bar{h}^t$. This is because every high type gets at least $\bar{u}_H - \varepsilon$ by accepting φ_t, whereas he gets at most $\delta \bar{u}_H < \bar{u}_H - \varepsilon$ by rejecting φ_t. Thus, our claim is true if $\mu\left([0, \hat{q}] \,|\bar{h}^t\right) = 0$. If, on the other hand, $\mu\left([0, \hat{q}] \,|\bar{h}^t\right) > 0$ and the conditional probability that φ_t is accepted is not one, then $\mu\left([0, \hat{q}] \,|\bar{h}^t, (\varphi_t, R)\right) = 1$ and the continuation payoff of the low types $q \leqslant \hat{q}$ will be zero (recall Lemma 1 above). But then it is not optimal for the low types to reject φ_t.

Consider now the proposal $\varphi_t' = \left(k', k'c + u_H - \frac{\varepsilon}{2}\right)$. Using an argument similar to the one above, it is easy to see that the conditional probability, given $\mu\left(\bar{h}^t\right)$, that the proposal φ_t' is accepted is also one. Since in a stationary equilibrium the continuation payoff of the buyer depends only on the number of units left and his beliefs about the seller's type, φ_t' is a profitable deviation at $\bar{h}^t$. Thus, we conclude that $\bar{u}_H$ is equal to zero.

Fix a stationary equilibrium $(\sigma_B, \sigma_S, \mu)$ and let h^t denote an arbitrary history. Suppose that the buyer makes the offer $\varphi_t = (k, p)$ at h^t and consider type $q > \hat{q}$. The fact that q's continuation payoff is zero immediately implies that he must accept the offer φ_t if $p > kc$ and must reject it if $p < kc$. Recall that the definition of stationarity requires that q accepts the offer if and only if $p \geqslant P(k, k', q)$. It thus follows that, in any stationary equilibrium, $P(k, k', q) = k'c$ for every k, k' and $q > \hat{q}$. ■

Proof of Proposition 1.

Recall that $W(k, q)$ denotes the buyer's expected payoff when the state is (k, p) and satisfies equation (1). Let $Y(k, q)$ denote the arg max correspondence in (1). As mentioned in Section 3.1, the solution can either be of the form (k, q'), for some $q' \in [q, \hat{q}] \cup \{1\}$, or of the form (k', q) for some $k' = 1, \ldots, k-1$. In the first case, the buyer makes the cream-skimming offer $(k, P(k, q'))$ and purchases all the remaining k units from the types in the interval $[q, q']$.[25] In the second case, the buyer makes a universal offer and purchases $(k - k')$ units from the types in $[q, 1]$.

[25] Below, we show that if (k, q') is a solution of (1), then $P(k, q'') > P(k, q')$ for every $q'' > q'$.

Of course, for $q > \hat{q}$ equation (1) implies

$$W(k,q) = (1-q)\left[(\alpha_k + \ldots + \alpha_1)\bar{v} - kc\right].$$

Although we have a continuum of low types, it is obvious that, at every history, all of them must get the same continuation payoff (otherwise, some of these types would have an incentive to deviate and mimic the behavior of other low types). We refer to this payoff as the payoff of the low type. We let $Z_L(k,q)$ denote the low type's payoff when the state is (k,q). When the set $Y(k,q)$ is a singleton, the buyer's optimal behavior is uniquely determined. When $Y(k,q)$ contains more than one element, we select the solution in $Y(k,q)$ that yields the lowest payoff to the low types. Therefore, for any $k = 1,\ldots,m$ and any $q \in [0,\hat{q}]$, $Z_L(k,q)$ satisfies:

$$Z_L(k,q) = \min\left\{\left\{\min_{(k,q')\in Y(k,q)} P(k,q')\right\}, \left\{\min_{(k',q)\in Y(k,q)} \left[(k-k')c + \delta Z_L(k',q)\right]\right\}\right\}. \quad (20)$$

We let $t(k,q)$ denote the solution to (20). (If there are multiple solutions, then there exists at least one solution of the form (k',q) and we pick the one with the lowest k'.)

Finally, for every $k = 1,\ldots,m$, we define $P(k,\cdot) : [0,1] \to \mathbb{R}_+$ to be the largest increasing function that is (weakly) below the function $\delta Z_L(k,\cdot)$. Formally, $P(k,\cdot)$ satisfies the following condition.

Condition 2 *For every q, $P(k,q)$ satisfies:*
(i) $P(k,q) \leqslant \delta Z_L(k,q)$;
(ii) $P(k,q) \leqslant P(k,q')$ for every $q' > q$;
(iii) For every $\eta \in (P(k,q), \delta Z_L(k,q))$ there exists $q' > q$ such that $\delta Z_L(k,q') < \eta$.

Next, we show that it is never optimal for the buyer to choose a point in the interior of a flat segment of $P(k,\cdot)$. That is, we show that if $t(k,q) = (k,q')$, then $P(k,q'') > P(k,q')$ for every $q'' > q'$.

By contradiction, suppose $t(k,q) = (k,q')$ and $P(k,q'') = P(k,q')$ for $q'' > q'$. Suppose also that $t(k,q') = (k',q')$ and $t(k',q') = (k',q''')$ for some $q''' > q''$. Notice that we have

$$P(k,q'') = P(k,q') \leqslant \delta(k-k')c + \delta^2 P(k',q''').$$

Given the state (k,q), consider the following strategy. The buyer buys k parts from the types in $[q,q'']$ at the price $P(k,q'') = P(k,q')$. Then if the seller rejects the offer, the buyer purchases $(k-k')$ units from all the types. Finally the buyer purchases k' parts from the types in $(q'', q''']$. The difference between the buyer's payoff from following this strategy and $W(k,q)$ is equal to:

$$\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_1)\,v(s)-P(k,q'')\right]ds-\delta\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_{k'+1})\,v(s)-(k-k')\,c\right]ds-$$

$$\delta^2\int_{q'}^{q''}\left[(\alpha_{k'}+\ldots+\alpha_1)\,v(s)-P(k',q''')\right]ds\geqslant\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_1)\,v(s)\right]ds-$$

$$\delta\int_{q'}^{q''}\left[(\alpha_k+\ldots+\alpha_{k'+1})\,v(s)\right]ds-\delta^2\int_{q'}^{q''}\left[(\alpha_{k'}+\ldots+\alpha_1)\,v(s)\right]ds>0.$$

Here we considered a specific continuation strategy following the state (k,q'). In a similar way, it is easy to show that if the buyer does not choose the endpoint of a flat segment, then there is a strictly profitable deviation for any possible continuation strategy. We omit the details.

A quadruplet of functions $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ satisfying equations (1) and (20), and Condition 2 defines the following stationary equilibrium. First, at any history of the game, type $q>\hat{q}$ accepts an offer to sell k units if and only if the transfer is at least kc. Consider now type $q<\hat{q}$. Suppose that there are k units left for trade and type q has never accepted a transfer smaller than $k'c$ to sell $k'\leqslant m-k$ units. In this case, type q accepts the proposal (k'', p), $k''\leqslant k$, if and only if $p\geqslant\min\{P(k,q), k''c\}$.[26] On the other hand, if type q has already accepted a transfer smaller than $k'c$ to sell $k'\leqslant m-k$ units, then he accepts any offer.

We now turn to the buyer. In the first period, the buyer makes an offer consistent with $t(m,0)$.[27] Suppose that the seller has never accepted an offer less than $k'c$ for k' units and that the state is (k,q). Then the buyer makes an offer consistent with $t(k,q)$.[28] Finally, suppose that in the past the seller has accepted a transfer less than $k'c$ for k' units. Then the buyer offers zero to purchase all the remaining units.

As far as the buyer's beliefs are concerned, they are pinned down by Bayes' rule except for the following two cases. First, suppose that the state is (k,q), the buyer makes the offer (k', p), with $k'<k$ and $p<P(k,k',q)$, and the offer is accepted. In this case, we assume that the buyer assigns probability one to the fact that the good is of low quality. Second, assume that the state is (k,q), the buyer makes the offer (k', p), with $k'\leqslant k$ and

[26] This defines the functions $P(k,k',\cdot)$ for $k>k'$ and $q\in[0,\hat{q}]$.

[27] This means that if $t(m,0)=(m,q)$ for some $q>0$, then the buyer proposes to purchase all the m units at the price $P(m,q)$. If instead $t(m,0)=(k,0)$ for some $k<m$, then the buyer purchases $(m-k)$ units at the price $(m-k)\,c$.

[28] If the previous offer p for k units was not in the range of $P(k,\cdot)$, then the buyer randomizes among the offers consistent with the elements of $Y(k,q)$ so as to rationalize the low types' acceptance decision of the offer p.

$p \geqslant k'c$, and the offer is rejected. In this case, the buyer does not update his belief and the state remains (k, q).

Given how we construct the quadruplet $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ and how we define the buyer's strategy, it is immediate to see that the seller cannot profitably deviate. It is also straightforward to check that the buyer's behavior is optimal if we restrict the buyer to choosing between cream-skimming and universal offers. To conclude that we indeed have an equilibrium, it remains to show that it is not optimal for the buyer to purchase a fraction of the remaining units only from the low types. Suppose the state is (k, q), for some $q \leqslant \hat{q}$, and consider the offer (k', p), with $k' < k$ and $P(k, k', q) \leqslant p < k'c$.[29] Let $[q, q']$ denote the set of types who accept this offer. Given the seller's strategy, we have $p \geqslant P(k, q'')$ for $q'' \leqslant q'$ and $p < P(k, q'')$ for $q'' > q'$. Consider now the offer (k, p). This offer is also accepted by the types in $[q, q']$. Clearly, the buyer strictly prefers the offer (k, p) to the offer (k', p) (while the two offers specify the same payment, the discounted consumption is larger under the former offer than under the latter).

To conclude the proof of Proposition 1 it remains to show that there exists a quadruplet $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ satisfying equations (1) and (20), and Condition 2.

Claim 1 *For every $k = 1, \ldots, m$, there exists $\bar{q} < \hat{q}$ such that for $q \in [\bar{q}, \hat{q}]$*

$$
\begin{aligned}
W(k, q) &= \textstyle\int_q^1 [(\alpha_k + \ldots + \alpha_1) v(s) - kc]\, ds > 0 \\
Z_L(k, q) &= kc \\
P(k, q) &= \delta kc \\
t(k, q) &= (k, 1).
\end{aligned}
$$

Proof. If the buyer proposes to buy all the k units at a price smaller than kc, his expected payoff is bounded above by

$$(\hat{q} - q)(\alpha_k + \ldots + \alpha_1)\underline{v} + (1 - \hat{q})\delta[(\alpha_k + \ldots + \alpha_1)\bar{v} - kc],$$

which is smaller than

$$\int_q^1 [(\alpha_k + \ldots + \alpha_1) v(s) - kc]\, ds$$

[29]Suppose that the state is (k, q) and consider the offer (k', p), with $k' < k$ and $p < P(k, k', q)$. Below we show that $W(k'', q'') > 0$ for all k'' and $q'' < 1$. This immediately implies that the offer (k', p) is not optimal. This is because the offer is rejected with probability one and the buyer will get at most $\delta W(k, q) < W(k, q)$.

for q sufficiently close to $\hat{q}$. Using an inductive argument, it is also easy to check that buying $k' < k$ units from all the types (i.e., at the price $k'c$) is not optimal when q is close to $\hat{q}$. ■

For $k = 1$, the quadruplet $(W(1,\cdot), P(1,\cdot), Z_L(1,\cdot), t(1,\cdot))$ is as in DL. For $k = 2, \ldots, m$, we assume that $(W(\cdot), P(\cdot), Z_L(\cdot), t(\cdot))$ are defined for $k' < k$ and extend the construction to k. Claim 1 also allows us to assume that $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$ are defined over the interval $[q_n, 1]$ for some $q_n < \hat{q}$ and $W(k,q) > 0$ for every $q \in [q_n, 1]$. We now extend the quadruplet $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$ to the interval $[q_{n+1}, 1]$ for some $q_{n+1} < q_n$.

For $q \in [0, q_n]$, define $\tilde{W}(k,q)$ and $\tilde{X}(k,q)$ as follows:

$$\tilde{W}(k,q) = \max_{q' \geqslant q_n} \int_q^{q'} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k,q')] \, ds + \delta W(k,q'), \tag{21}$$

$$\tilde{X}(k,q) = \arg\max_{q' \geqslant q_n} \int_q^{q'} [(\alpha_k + \ldots + \alpha_1) v(s) - P(k,q')] \, ds + \delta W(k,q').$$

It is easy to check that if $q' \in \tilde{X}(k,q)$, then $P(k,q'') > P(k,q')$ for every $q'' > q'$ (i.e., the maximum is never achieved on the interior of a flat segment of P). The objective function in (21) has strictly increasing differences in q at all maximizers q'. Thus, $\tilde{X}$ is a nondecreasing correspondence: if $q > q'$, then $\tilde{q} \geqslant \tilde{q}'$ for any pair $\tilde{q} \in \tilde{X}(k,q)$, $\tilde{q}' \in \tilde{X}(k,q')$. Also, from the theorem of the maximum, $\tilde{X}$ is upper hemicontinuous.

We let $\tilde{x}(k,q) \in [q_n, 1]$ denote the smallest element in $\tilde{X}(k,q)$. We also define for $q \in [0, q_n]$ the acceptance function $\tilde{P}(k,q) = \delta P(k, \tilde{x}(k,q))$.

Next, for $q \in [0, q_n]$, define $\hat{W}(k,q)$ as follows:

$$\hat{W}(k,q) = \max_{q' \in [q, q_n]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - \tilde{P}(k,q')\right] ds + \delta \tilde{W}(k,q').$$

We let $\hat{x}(k,q) \in [q, q_n]$ denote the smallest element of the arg max correspondence in the above expression. Finally, define $q_{n+1} = \max\left\{q \in [0, q_n] : \tilde{W}(k,q) \leqslant \hat{W}(k,q)\right\}$ whenever the set is nonempty and $q_{n+1} = 0$ otherwise. Notice that $\tilde{W}(k,q_n) > 0$ ($\tilde{W}(k,q_n)$ is equal to $\delta W(k,q_n) > 0$ if we choose $q' = q_n$ in (21)), and $\hat{W}(k,q_n) = \delta \tilde{W}(k,q_n) < \tilde{W}(k,q_n)$. By the theorem of the maximum, $\tilde{W}$ and $\hat{W}$ are continuous and, therefore, $q_{n+1} < q_n$. By definition of q_{n+1}, we have $\tilde{W}(k,q) > \hat{W}(k,q)$ for $q > q_{n+1}$. It is easy to show that $\tilde{W}(k,q) \leqslant \hat{W}(k,q)$ for $q \leqslant q_{n+1}$ (the function $\tilde{W}(k,q) - \hat{W}(k,q)$ is increasing in $[0, q_{n+1}]$ and is equal to zero at q_{n+1}).

Next, we claim that $\tilde{W}(k,q) > 0$ for all $q \in [q_{n+1}, q_n]$. For q_n we have $\tilde{W}(k,q_n) \geqslant \delta W(k,q_n) > 0$. For $q \in [q_{n+1}, q_n)$, let $x(k,q) \in [q_n, 1]$ be the largest element in $\tilde{X}(k,q)$

and let ε be such that $q+\varepsilon \in (q, q_n)$. Since $x(k,q)$ is feasible at $q+\varepsilon$, we have

$$\tilde{W}(k, q+\varepsilon) \geqslant \int_{q+\varepsilon}^{x(k,q)} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, x(k,q))\right] ds + \delta W(k, x(k,q)).$$

It then follows from

$$\tilde{W}(k,q) = \int_{q}^{x(k,q)} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, x(k,q))\right] ds + \delta W(k, x(k,q))$$

that

$$\tilde{W}(k, q+\varepsilon) \geqslant \tilde{W}(k,q) - \int_{q}^{q+\varepsilon} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, x(k,q))\right] ds. \tag{22}$$

Since $q+\varepsilon < q_n$ and $\tilde{W}(k,q) > \hat{W}(k,q)$, we have

$$\tilde{W}(k,q) > \int_{q}^{q+\varepsilon} \left[(\alpha_k + \ldots + \alpha_1) v(s) - \tilde{P}(k, q+\varepsilon)\right] ds + \delta \tilde{W}(k, q+\varepsilon). \tag{23}$$

Substituting (22) into (23) yields

$$(1-\delta)\tilde{W}(k,q) >$$
$$\int_{q}^{q+\varepsilon} \left\{(1-\delta)(\alpha_k + \ldots + \alpha_1) v(s) - \delta\left[(P(k, \tilde{x}(k, q+\varepsilon)) - P(k, x(k,q)))\right]\right\} ds.$$

The proof is complete if we show that

$$\lim_{\varepsilon \downarrow 0} \left[(P(k, \tilde{x}(k, q+\varepsilon)) - P(k, x(k,q)))\right] = 0. \tag{24}$$

Because $\tilde{X}(k,q)$ is a nondecreasing upper hemicontinuous correspondence, we have $\lim_{\varepsilon \downarrow 0} \tilde{x}(k, q+\varepsilon) = x(k,q)$. Therefore, equality (24) can fail only if $P(k,\cdot)$ has a discontinuity at $x(k,q)$. But then we must have $\tilde{x}(k, q+\varepsilon) = x(k,q)$ for a sufficiently small ε.

Next, we define the quadruplet $(W^1(k,\cdot), P^1(k,\cdot), Z_L^1(k,\cdot), t^1(k,\cdot))$. For $q > q_n$, we let $(W^1(k,\cdot), P^1(k,\cdot), Z_L^1(k,\cdot), t^1(k,\cdot))$ be equal to $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$. For $q \in [q_{n+1}, q_n]$ define

$$W^1(k,q) = \max\left\{\tilde{W}(k,q), \left\{\int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k-k')c\right] ds + \delta W(k', q)\right\}_{k'=1,\ldots,k-1}\right\},$$

and let $t^1(k,q)$ be the solution with the lowest continuation payoff to the low type. (If there are multiple solutions with the same continuation payoff, then there exists at least one solution of the form (k', q), and we pick the one with the lowest k'.)

If $t^1(k,q) = (k, \tilde{x}(k,q))$, then let $Z_L^1(k,q) = \delta Z_L(k, \tilde{x}(k,q))$. If $t^1(k,q) = (k',q)$, then let $Z_L^1(k,q) = (k-k')c + \delta Z_L(k',q)$. Finally, for $q \in [q_{n+1}, q_n]$, we define $P^1(k,q)$ to be the largest increasing function that is (weakly) below the function $\delta Z_L^1(k,\cdot)$.

We now inductively define a sequence of quadruplets $\left\{\left(W^\ell(k,\cdot), P^\ell(k,\cdot), Z_L^\ell(k,\cdot), t^\ell(k,\cdot)\right)\right\}_{\ell=1,2,\ldots,}$. Given $\left(W^\ell(k,\cdot), P^\ell(k,\cdot), Z_L^\ell(k,\cdot), t^\ell(k,\cdot)\right)$, we define the next element of the sequence as follows:

$$W^{\ell+1}(k,q) = \max\left\{\left(\max_{q' \in [q,1]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P^\ell(k,q')\right] ds + \delta W^\ell(k,q')\right),\right.$$
$$\left.\left\{\int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k-k')c\right] ds + \delta W(k',q)\right\}_{k'=1,\ldots,k-1}\right\}.$$

We let $t^{\ell+1}(k,q)$ be the solution to the above problem with the lowest continuation payoff to the low type which we denote by $Z^{\ell+1}(k,q)$. Finally, $P^{\ell+1}(k,\cdot)$ is the largest increasing function below $\delta Z^{\ell+1}(k,\cdot)$.

Claim 2 *There exists ℓ^* such that*

$$\left(W^{\ell^*}(k,\cdot), P^{\ell^*}(k,\cdot), Z_L^{\ell^*}(k,\cdot), t^{\ell^*}(k,\cdot)\right) = \left(W^{\ell^*+1}(k,\cdot), P^{\ell^*+1}(k,\cdot), Z_L^{\ell^*+1}(k,\cdot), t^{\ell^*+1}(k,\cdot)\right).$$

Proof. Since $W^1(k,\cdot)$ is strictly positive on $[q_{n+1}, 1]$, there exists $\Delta > 0$, such that $W^1(k,q) > \Delta$ for every $q \in [q_{n+1}, 1]$. For each ℓ, $W^\ell(k,\cdot)$ is uniformly continuous and $W^\ell(k,q) \geqslant W^1(k,q)$. This implies that there exists $\varepsilon_\ell > 0$ such that for every $q \in [q_{n+1}, 1]$, if $t^\ell(k,q) = (k,q')$, then $q' > q + \varepsilon_\ell$.

Recall that Δ is a lower bound to $W^\ell(k,\cdot)$ for each ℓ. Let T be an integer such that

$$\Delta > \left(\frac{1}{T} + \delta^T\right)(\alpha_k + \ldots + \alpha_1)\bar{v}.$$

If the claim fails, then we can find ℓ, $q \in [q_{n+1}, q_n)$, and a sequence $q = q^0 < q^1 < \ldots < q^T < q + \frac{1}{T}$ such that for every $\tau = 1, \ldots, T$, $t^\ell(k, q^{\tau-1}) = (k, q^\tau)$. But then we have the following contradiction:

$$\Delta < W^\ell(k,q) < \left(\frac{1}{T} + \delta^T\right)(\alpha_k + \ldots + \alpha_1)\bar{v} < \Delta.$$

■

At this point we are ready to extend $(W(k,\cdot), P(k,\cdot), Z_L(k,\cdot), t(k,\cdot))$ to the interval $[q_{n+1}, q_n]$ by setting them equal to $\left(W^{\ell^*}(k,\cdot), P^{\ell^*}(k,\cdot), Z_L^{\ell^*}(k,\cdot), t^{\ell^*}(k,\cdot)\right)$.

Finally, we show that it takes finitely many steps to extend the function $W(k,\cdot)$ to the whole unit interval. By contradiction, suppose that $\lim_{n\to\infty} q_n = q^* > 0$. We distinguish between the following two cases.

Case 1. First, consider the case in which there exists a sequence of points $\{q_j\}_{j=1}^{\infty}$ converging (from above) to q^* such that for each q_j

$$W(k, q_j) = \int_{q_j}^{q'_j} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P\left(k, q'_j\right) \right] ds + \delta W\left(k, q'_j\right) \tag{25}$$

for some $q'_j > q_j$.

From this it follows that $\lim_{q \to q^*} P(k, q) = 0$ and, therefore, $\inf_{(q^*,1]} W(k, q) > 0$. We now choose $\varepsilon > 0$ to satisfy

$$[(\alpha_k + \ldots + \alpha_1) \bar{v} + \delta] \varepsilon < (1 - \delta) \inf_{(q^*,1]} W(k, q). \tag{26}$$

Because of the uniform continuity of $W(k, \cdot)$, given ε we can find $\eta \in (0, \varepsilon)$, such that for every $(q, q') \in (q^*, 1]^2$, if $|q - q'| < \eta$, then $|W(k, q) - W(k, q')| < \varepsilon$.

The fact that the sequence $\{q_j\}_{j=1}^{\infty}$ converges to q^* and satisfies (25) implies that there exists $\hat{\jmath} = 1, 2, \ldots,$ such that $|q_{\hat{\jmath}} - q'_{\hat{\jmath}}| < \eta < \varepsilon$. If we let $\underline{w}$ denote the minimum between $W(k, q_{\hat{\jmath}})$ and $W\left(k, q'_{\hat{\jmath}}\right)$ we have the following contradiction:

$$\begin{gathered} \underline{w} \leqslant W(k, q_{\hat{\jmath}}) \leqslant (\alpha_k + \ldots + \alpha_1) \bar{v}\varepsilon + \delta W\left(k, q'_{\hat{\jmath}}\right) \leqslant \\ (\alpha_k + \ldots + \alpha_1) \bar{v}\varepsilon + \delta (\underline{w} + \varepsilon) < \underline{w} \end{gathered}$$

where the last inequality follows from (26) and $\underline{w} \geqslant \inf_{(q^*,1]} W(k, q)$.

Case 2. Consider now the case in which there exists an interval $(q^*, \breve{q})$ and $k' < k$ such that for each $q \in (q^*, \breve{q})$, $t(k, q) = (k', q)$.

In this case, we can find $\hat{n}$ such that for every $n \geqslant \hat{n}$, $t(k, q_n) = (k', q_n)$ and $t(k', q_n) = (k', \tilde{q})$ for some $\tilde{q} > q_{\hat{n}}$. Furthermore, the sequence $\{q_n\}_{n=\hat{n}}^{\infty}$ converges (from above) to q^*. First, we show that $\lim_{n \to \infty} W(k, q_n) = 0$. To see this, recall our definition of $\tilde{W}$, $\tilde{P}$ and $\hat{W}$. The function $\tilde{W}$ is uniformly continuous. Fix $\varepsilon > 0$, and let $\eta \in (0, \varepsilon)$ be such $\left|\tilde{W}(k, q) - \tilde{W}(k, q')\right| < \varepsilon$ for every (q, q') with $|q - q'| < \eta$. Furthermore, there exists n' such that $q_n - q_{n+1} < \eta$ for every $n > n'$. Therefore, if $n > n'$ we have

$$\begin{gathered} \tilde{W}(k, q_{n+1}) = \hat{W}(k, q_{n+1}) = \max_{q' \in [q_{n+1}, q_n]} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - \tilde{P}(k, q') \right] ds + \delta \tilde{W}(k, q') \leqslant \\ \varepsilon (\alpha_k + \ldots + \alpha_1) \bar{v} + \delta \sup_{q' \in [q_{n+1}, q_n]} \tilde{W}(k, q') \leqslant \varepsilon [(\alpha_k + \ldots + \alpha_1) \bar{v} + \delta] + \delta \tilde{W}(k, q_{n+1}) \end{gathered}$$

and, therefore,

$$\tilde{W}(k, q_{n+1}) \leqslant \frac{\varepsilon [(\alpha_k + \ldots + \alpha_1) \bar{v} + \delta]}{(1 - \delta)}.$$

This implies $\lim_{n\to\infty}\tilde{W}(k,q_n) = 0$. On the other hand, for $n > \hat{n}$, $\tilde{W}(k,q_{n+1}) \geqslant -\varepsilon kc + \delta W(k,q_n)$. Thus, we have $\lim_{n\to\infty} W(k,q_n) = 0$.

Next, we claim that

$$\delta\left[(k-k')\,c + \delta P(k',\tilde{q})\right] \geqslant (\alpha_k + \ldots + \alpha_1)\,\underline{v}.$$

The left hand side is an upper bound to the reservation price $P(k,q_n)$ for k units of any type q_n with $n \geqslant \hat{n}$. If the inequality is violated, then $W(k,q_n)$ is bounded below by

$$(q_{\hat{n}} - q_n)\left[(\alpha_k + \ldots + \alpha_1)\,\underline{v} - \delta\left[(k-k')\,c + \delta P(k',\tilde{q})\right]\right] > 0$$

contradicting the fact that $\lim_{n\to\infty} W(k,q_n) = 0$.

For $q < q_{\hat{n}}$, suppose the buyer adopts the following strategy. First, he purchases k units at the price $P(k,q_{\hat{n}}) \leqslant \delta\left[(k-k')\,c + \delta P(k',\tilde{q})\right]$ from the types in $[q,q_{\hat{n}}]$. Then he purchases $(k-k')$ units from all the types in $[q_{\hat{n}},1]$. Finally, he follows the optimal strategy given $(k',q_{\hat{n}})$. The buyer's payoff from adopting such a strategy is weakly larger than $R(q)$ defined by

$$R(q) = \int_q^{q_{\hat{n}}} \left[(\alpha_k + \ldots + \alpha_1)\,\underline{v} - \delta\left[(k-k')\,c + \delta P(k',\tilde{q})\right]\right] ds +$$
$$\delta \int_{q_{\hat{n}}}^1 \left[(\alpha_k + \ldots + \alpha_{k'+1})\,v(s) - (k-k')\,c\right] ds + \delta^2 \int_{q_{\hat{n}}}^{\tilde{q}} \left[(\alpha_{k'} + \ldots + \alpha_1)\,\underline{v} - P(k',\tilde{q})\right] ds + \delta^3 W(k',\tilde{q}).$$

Notice that $R(q)$ is increasing in q.

For $q < q_{\hat{n}}$, let $V(q)$ be equal to

$$V(q) = \int_q^{q_{\hat{n}}} \left[(\alpha_k + \ldots + \alpha_{k'+1})\,\underline{v} - (k-k')\,c\right] ds + \int_{q_{\hat{n}}}^1 \left[(\alpha_k + \ldots + \alpha_{k'+1})\,v(s) - (k-k')\,c\right] ds +$$

$$\delta \int_q^{q_{\hat{n}}} \left[(\alpha_{k'} + \ldots + \alpha_1)\,\underline{v} - P(k',\tilde{q})\right] ds + \delta \int_{q_{\hat{n}}}^{\tilde{q}} \left[(\alpha_{k'} + \ldots + \alpha_1)\,\underline{v} - P(k',\tilde{q})\right] ds + \delta^2 W(k',\tilde{q})$$

and notice that $W(k,q) = V(q)$ for $q \in (q^*, q_{\hat{n}}]$. Let $\bar{q}$ be such that $R(\bar{q}) = V(\bar{q})$. Such $\bar{q}$ is well defined since it is the solution to the following equation:

$$\left[\int_{q_{\hat{n}}}^1 \left[(\alpha_k + \ldots + \alpha_{k'+1})\,v(s) - (k-k')\,c\right] ds + \delta \int_{q_{\hat{n}}}^{\tilde{q}} \left[(\alpha_{k'} + \ldots + \alpha_1)\,\underline{v} - P(k',\tilde{q})\right] ds + \delta^2 W(k',\tilde{q})\right] =$$
$$\int_{\bar{q}}^{q_{\hat{n}}} \left[(\alpha_{k'} + \ldots + \alpha_1)\,\underline{v} + (k-k')\,c + \delta P(k',\tilde{q})\right] ds. \tag{27}$$

For $q \in (q^*, q_{\hat{n}}]$, the left hand side is larger than the right hand side in (27) since $V(q)$ is the value of the optimal strategy. Of course, the inequality is reversed as $\bar{q}$ goes to $-\infty$. Thus $\bar{q}$ exists and is weakly smaller than q^*.

Using (27), we obtain

$$R(\bar{q}) = V(\bar{q}) \geqslant (q_{\hat{n}} - \bar{q})(\alpha_k + \ldots + \alpha_1)\underline{v} > 0.$$

Recall that $\bar{q} \leqslant q^*$ and $R(\cdot)$ is increasing in q. For every $n \geqslant \hat{n}$, we have

$$W(k, q_n) = V(q_n) \geqslant R(q_n) \geqslant R(\bar{q}) > 0$$

which contradicts the fact that $\lim_{n\to\infty} W(k, q_n) = 0$. ■

Appendix B: Uniqueness

Proof of Proposition 2.

First, we prove the result under the assumption that $P(k, \cdot)$ is weakly increasing and, without loss of generality, left-continuous for all k. Second, we give a general argument.

Step 1: *For generic parameters all stationary equilibria with increasing reservation price functions are outcome equivalent.*

DL establish that when the parties trade a single unit there is (for a generic set of parameters) a unique left-continuous reservation price function $P(1, \cdot)$, which is equal to the one constructed in Appendix A. First, we consider equilibria satisfying Condition 1 below.

Condition 1*: At any history in which there are k units remaining, the buyer either makes a cream-skimming offer or makes a universal offer for $k' < k$ units.*

Assume that for some $k = 2, \ldots, m$, there is a generic set of parameters for which for all $k' < k$, $P(k', \cdot)$ constructed in Appendix A is the unique reservation price function in all stationary equilibria. It is straightforward to show that there is $\bar{q} < 1$ such that if the state is (k, q') with $q' \geqslant \bar{q}$, then the buyer makes the offer (k, kc) and this offer is accepted with probability one in all stationary equilibria. Define

$$q^* := \inf \left\{ \begin{array}{c} q' : \text{there exists a generic set of parameters such that for all } q > q' \\ P(k, q) \text{ constructed in Appendix A is the unique} \\ \text{reservation price in all stationary equilibria} \end{array} \right\}.$$

We show that there is $\varepsilon > 0$ and a generic set of parameters such that for all $q > q^* - \varepsilon$, our $P(k, q)$ is the unique reservation price in all stationary equilibria.

Consider a state (k, q) with $q < q^*$. First, impose that the buyer is obliged to make a cream-skimming offer $(k, P(k, \tilde{q}))$ for some $\tilde{q} \geqslant q$. Using the same argument as in Appendix A we conclude that there exists $\varepsilon > 0$ such that if $q > q^* - \varepsilon$ then in any stationary equilibria, the buyer's optimal offer is $(k, P(k, q'))$ with $q' > q^* + \varepsilon$. Let us now restrict our attention to the states (k, q) with $q \in (q^* - \varepsilon, q^*]$.

Consider the choice between a cream-skimming offer and a universal offer for $k' < k$ units. Observe that, by assumption, for every (k', q) with $k' < k$ and $q \in [0, 1]$ or $k' = k$ and $q > q^*$, $P(k', q)$ is unique in all stationary equilibria. Hence, the low type's continuation payoff at the state (k', q) is the same in all stationary equilibria. Thus, for a generic set of parameters, the buyer has a unique best response at (k, q). This implies that the unique (left-continuous) extension of P to (k, q) with $q > q^* - \varepsilon_1$ is the reservation price function that we constructed in Appendix A. This, in turn, shows the uniqueness of the reservation price functions under the assumption that the functions are weakly increasing and the buyer makes only cream-skimming or universal offers.

Next, we show that Condition 1 holds generically (this guarantees the uniqueness of stationary equilibrium outcomes when the reservation price functions are increasing). It follows from DL that Condition 1 holds for $k = 1$. Assume that for some $k = 2, \ldots, m$, Condition 1 holds for all $k' < k$. Notice that there exists $\tilde{q} \in (0, 1)$ such that whenever the state is (k, q) and $q > \tilde{q}$ the buyer makes the offer (k, kc). Define

$$q^* := \inf \left\{ \begin{array}{c} q' : \text{there exists a generic set of parameters such that for all } q > q' \\ \text{Condition 1 holds at the state } (k, q) \end{array} \right\}.$$

We now show that there exists $\varepsilon > 0$ such that generically Condition 1 holds at every state (k, q) with $q > q^* - \varepsilon$. By contradiction, assume that this is not true. It is straightforward to show that there exist $\varepsilon > 0$, a stationary equilibrium and a state (k, q), with $q > q^* - \varepsilon$, such that the following two conditions are satisfied. First, the buyer makes the offer (k', p) with $k' < k$ and $p < k'c$. Second, if the offer is rejected, then the state becomes (k, q') with $q' > q^* + \varepsilon$.

From our analysis above, we may assume that for all $q > q^*$ the reservation prices coincide with those constructed in Appendix A. This and the fact that $q' > q^* + \varepsilon$ imply that the low types' continuation payoff from rejecting the offer (k', p) is at least $\delta Z_L (k, q')$ (recall that in Appendix A the reservation price functions are constructed to give the lowest continuation payoff to the seller's low types). Now it follows from Lemma 1 that the continuation payoff of the (low) types who accept the offer (k', p) is equal to zero. Thus, $p \geqslant \delta Z_L (k, q')$ since some low types accept the offer (k', p). But then the buyer has a profitable deviation at the state (k, q) by making the cream-skimming offer $(k, \delta Z_L (k, q'))$ (notice that this offer is weakly more likely to be accepted than the equilibrium offer (k', p)). This concludes the proof of Step 1.

Step 2: *General result*

Next, we claim that there is a generic set of parameters for which, for every stationary equilibrium, there is an outcome equivalent stationary equilibrium in which all the reservation price functions are increasing. This, together with Step 1, proves Proposition 2.

From the results in DL we know that the claim above is true when $m = 1$. Consider the game in which there are $m > 1$ parts and take a stationary equilibrium. Let $P(1, \cdot), \ldots, P(m, \cdot)$ denote the corresponding reservation price functions. We consider the following relabeling process, which does not change the outcome of the game.

i) *Relabeling the types for the first unit*

Since $P(1, \cdot) : [0, 1] \to \mathbb{R}$ is measurable, there exists a one-to-one (measurable) function $g : [0, 1] \to [0, 1]$ such that $\hat{P}(1, \cdot)$ defined by

$$\hat{P}(1, q) = P(1, g(q))$$

is increasing (almost everywhere). Without loss we assume that $\hat{P}(1, \cdot)$ is increasing everywhere and left-continuous.[30] For every $k = 2, ..., m$ define $\hat{P}(k, \cdot) : [0, 1] \to \mathbb{R}$ by:

$$\hat{P}(k, q) = P(k, g(q)).$$

From DL we know that $\hat{P}(1, \cdot)$ is a step function. We claim that for every $k \in \{2, ..., m\}$ the regions in which $\hat{P}(k, \cdot)$ fails to be monotonic lie in flat regions of $\hat{P}(1, \cdot)$. Using m for the Lebesgue measure, the formal statement is as follows.

Claim 3 *Let Ω be the generic set of parameters under which Step 1 holds. Let $q_1 < q_2 < q_3 < q_4$ be such that $\hat{P}(1, \cdot)$ is constant both in the interval $[q_1, q_2]$ and in $[q_3, q_4]$, and $\hat{P}(1, q_1) < \hat{P}(1, q_3)$. Take $k \in \{2, ..., m\}$, $p \in \mathbb{R}$ and assume that $\hat{P}(k, q) \geqslant p$ for some subset of $[q_1, q_2]$ with positive measure. Finally, let $A := \{q \in [q_3, q_4] : \hat{P}(k, q) < p\}$. Then $m(A) = 0$.*

Proof. The proof is by contradiction. Define $B := \{q \in [q_1, q_2] : \hat{P}(k, q) \geqslant p\}$ and assume that $m(A) > 0$ and $m(B) > 0$.

Consider now the following history. At the beginning of the game the buyer purchases $m - k$ units from all the types at the price $(m - k) c$. Then he makes an offer for the k remaining units which is accepted by all the types in A and is rejected by all the types in B. Following this rejection the buyer purchases $k - 1$ units from all the remaining types at the price $(k - 1) c$. Let $C \subset [0, 1]$ denote the set of types who still have one unit to trade.

[30] This is possible because the set of discontinuities of $\hat{P}(1, \cdot)$ is countable.

Notice that C contains "gaps". We would like to relabel C in such a way that it contains no "gaps". To do so, define a mapping $h : C \to [0,1]$ implicitly by:

$$m(q' : q' \in C \cap [q,1]) = 1 - h(q).$$

Let $\underline{q} = \inf\{h(q) : q \in C\}$, and notice that $m(C) = 1 - \underline{q}$ and for any $\tilde{q} \in (\underline{q}, 1)$ we have

$$m(q' : q' \in C, h(q') \geqslant \tilde{q}) = \int_{q \in C \cap [\inf\{q' : h(q') \geqslant \tilde{q}\},1]} dm(q) = 1 - \tilde{q}.$$

Next, we define a function $\tilde{P}(1,\cdot) : [1-\underline{q}, 1] \to \mathbb{R}$ by:

$$\tilde{P}(1,q) := \sup\{\hat{P}(1, h(\tilde{q})) : \; h(\tilde{q}) < q\}.$$

The function $\tilde{P}(1,\cdot)$ is increasing since both $\hat{P}(1,\cdot)$ and $h(\cdot)$ are increasing. By changing $\tilde{P}(1,\cdot)$ in a set of measure zero if necessary, we may assume that $\tilde{P}$ is left-continuous. Therefore, it follows from the analysis in Step 1 that the functions $\tilde{P}(1,\cdot)$ and $\hat{P}(1,\cdot)$ must coincide on the interval $[1-\underline{q}, 1]$. However, we now show that they differ. Indeed, define q^* and q^{**} by

$$q^* := \sup\{q : \hat{P}(1,q) \leqslant \hat{P}(1,q_3)\}, \quad q^{**} := \sup\{q : \tilde{P}(1,q) \leqslant \hat{P}(1,q_3)\}.$$

The fact that $m(A) > 0$ and $m(B) > 0$ implies $q^* < q^{**}$ and, therefore, $m(\{q \geqslant 1 - \underline{q} : \hat{P}(1,q) > \tilde{P}(1,q)\}) > 0$. Thus, we conclude that the parameters of the model do not belong to Ω.■

ii) *Inductively relabeling the types for the next unit*

Recall that $P(1,\cdot), \ldots, P(m,\cdot)$ denote the equilibrium reservation price functions. Assume, as an induction hypothesis, that for some $\tilde{k} = 2, \ldots, m-1$ there exists a one-to-one (measurable) function $g : [0,1] \to [0,1]$ such that $\hat{P}(k,\cdot)$, $k = 1, \ldots, \tilde{k}$, defined by

$$\hat{P}(k,q) = P(k, g(q))$$

is an increasing and left-continuous step function. For every $k' = \tilde{k}+1, \ldots, m$ define $\hat{P}(k',\cdot) : [0,1] \to \mathbb{R}$ by:

$$\hat{P}(k',q) = P(k', g(q)).$$

The proof is complete if we show that for every $k' > \tilde{k}$ the regions in which $\hat{P}(k',\cdot)$ fails to be monotonic lie in flat regions of $\hat{P}(1,\cdot), \ldots, \hat{P}\left(\tilde{k},\cdot\right)$.

Claim 4 *Let Ω be the generic set of parameters under which Step 1 holds. Fix $k = 1, \ldots, \tilde{k}$ and let $q_1 < q_2 < q_3 < q_4$ be such that $\hat{P}(k, \cdot)$ is constant both in the interval $[q_1, q_2]$ and in $[q_3, q_4]$, and $\hat{P}(k, q_1) < \hat{P}(k, q_3)$. Take $k' \in \tilde{k}+1, \ldots, m$, $p \in \mathbb{R}$ and assume that $\hat{P}(k', q) \geqslant p$ for some subset of $[q_1, q_2]$ with positive measure. Finally, let $A := \{q \in [q_3, q_4] : \hat{P}(k', q) < p\}$. Then $m(A) = 0$.*

Proof Define $B := \{q \in [q_1, q_2] : \hat{P}(k', q) \geqslant p\}$ and assume towards a contradiction that $m(A) > 0$ and $m(B) > 0$. Consider now the following history. At the beginning of the game the buyer purchases $m - k'$ units from all the types at the price $(m - k')\, c$. Then he makes an offer for the k' remaining units, which is accepted by all the types in A and is rejected by all the types in B. Following this rejection, the buyer purchases $k' - k$ units from all the remaining types at the price $(k' - k)\, c$. Proceeding analogously to the proof of Claim 3 we conclude that in the model with k units the outcome of stationary equilibria with increasing reservation price functions is not unique. Then, it follows from Step 1 that the parameters of the model do not belong to Ω and. This concludes the proof of Proposition 2. ■

Appendix C: Small Units

Proof of Proposition 4

Consider the sequence $\left\{\left(q_i, \bar{P}_i\right)\right\}_{i=1,0,-1,\ldots}$ constructed in Section 6.2 and recall that for every $i = 1, 0, -1, \ldots$ we have $\Phi\left(q_i, q_{i-1}, q_{i-2}\right) = 0$ (see equation (18)) and

$$\bar{P}_i = \int_0^{\psi(q_i)} \alpha\left(u\right) \underline{v} du + \left(\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} \left[\alpha\left(\psi\left(u\right)\right) v(s) - \bar{c}\right] ds \right) \right] du}{q_{i-1} - q_i} \right) \tag{28}$$

Also recall that the sequence $\{q_1, q_0, q_{-1}, \ldots\}$ is increasing and bounded above by $q\left(0\right)$. Therefore, the sequence is convergent, and we denote its limit by $q_{-\infty}$.

In Appendix E, we establish the following two facts.

Fact 3 *There exists $\eta^* > 0$ such that for every $i = 0, -1, \ldots$, if $q_{i-1} - q_i < \eta^*$, then $q_i - q_{i+1} < \frac{4}{3}\left(q_{i-1} - q_i\right)$.*

Fact 4 *There exist two constants $b_1 > 0$ and $b_2 > 0$ such that for every $i = 1, 0, -1, \ldots$, we have*

i)

$$\frac{\left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha\left(u\right) \underline{v} du\right) - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha\left(u\right) \underline{v} du\right)}{q_{i-1} - q_i} \leqslant b_1(q_{i-1} - q_i), \tag{29}$$

and ii)

$$q_{i-1} - q_i \leqslant b_2 \left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u) \underline{v} du \right). \tag{30}$$

Proposition 4 follows from Lemmas 4-6 below.

Lemma 4 *Let j and j' be two integers satisfying $1 \geqslant j' > j$, and consider the beliefs $q_{j'} < q_{j'-1} < \ldots < q_j$. Let ε and M be two positive numbers such that $q_{i-1} - q_i < \varepsilon$ for every $i = j+1, \ldots, j'$, and $q_j - q_{j'} < M^{-1}$. Then for every $i = j+1, \ldots, j'$, we have*

$$\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u) \underline{v} du < \bar{P}_j - \int_0^{\psi(q_j)} \alpha(u) \underline{v} du + \varepsilon b_1 M^{-1}.$$

Proof. Notice that $(q_{i-i} - q_i) < \varepsilon$ implies

$$\bar{P}_i - \textstyle\int_0^{\psi(q_i)} \alpha(u) \underline{v} du =$$

$$\bar{P}_{i-1} - \textstyle\int_0^{\psi(q_{i-1})} \alpha(u) \underline{v} du + \left(\frac{\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du \right)}{q_{i-i} - q_i} \right) (q_{i-i} - q_i) <$$

$$\bar{P}_{i-1} - \textstyle\int_0^{\psi(q_{i-1})} \alpha(u) \underline{v} du + \varepsilon b_1 (q_{i-i} - q_i),$$

where the inequality follows from equation (29). Therefore for every $i = j+1, \ldots, j'$, we have

$$\bar{P}_i - \textstyle\int_0^{\psi(q_i)} \alpha(u) \underline{v} du < \bar{P}_j - \int_0^{\psi(q_j)} \alpha(u) \underline{v} du + \varepsilon b_1 \displaystyle\sum_{i'=j+1}^{j'} (q_{i'-i} - q_{i'}) <$$

$$\bar{P}_j - \textstyle\int_0^{\psi(q_j)} \alpha(u) \underline{v} du + \varepsilon b_1 M^{-1}.$$

■

Lemma 5 *Let j and j' be two integers satisfying $0 \geqslant j' > j$, and let ε and M be two positive numbers such that $\varepsilon < \eta^*$, $q_{i-1} - q_i < \varepsilon$ for every $i = j+1, \ldots, j'$, $\bar{P}_j - \int_0^{\psi(q_j)} \alpha(u) \underline{v} du < \left(\frac{\varepsilon}{3b_2} \right)$, and $M > 3b_1 b_2$. If $q_j - q_{j'} < M^{-1}$, then $q_{j'} - q_{j'+1} < \varepsilon$.*

Proof. We have

$$q_{j'-1} - q_{j'} \leqslant b_2 \left(\bar{P}_{j'} - \textstyle\int_0^{\psi(q_{j'})} \alpha(u) \underline{v} du \right) <$$

$$b_2 \left(\bar{P}_j - \textstyle\int_0^{\psi(q_j)} \alpha(u) \underline{v} du + \varepsilon b_1 M^{-1} \right) <$$

$$b_2 \left(\left(\tfrac{\varepsilon}{3b_2} \right) + \varepsilon b_1 M^{-1} \right) < \tfrac{2}{3} \varepsilon,$$

where the first inequality follows from Fact 4 (equation (30)), the second follows from Lemma 4, and the last two inequalities are an immediate consequence of the assumptions in Lemma 5. Finally, the inequality above together with Fact 3 and the assumption $\varepsilon < \eta^*$ implies

$$q_{j'} - q_{j'+1} < \frac{4}{3}(q_{j'-1} - q_{j'}) = \frac{4}{3}\frac{2}{3}\varepsilon < \varepsilon.$$

■

Lemma 6 *For every $\varepsilon > 0$ there exists $\kappa > 0$ such that for every $i = 0, -1, \ldots,$ the following is true. If max $\left\{q_{i-1} - q_i, \bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du\right\} < \kappa$, then $q_{j-1} - q_j < \varepsilon$ for every $j = 1, 0, \ldots, i+1$.*

Proof. Take $\bar{M} > 3b_1b_2$ and an integer N such that $\left(\frac{q_{-\infty} - q_1}{N}\right) < \left(\frac{1}{2\bar{M}}\right)$, and $q_0 > q_1 + \left(\frac{q_{-\infty} - q_1}{N}\right)$. Consider the partition of $[q_1, q_{-\infty}]$ into N intervals. Let the integer m^* be such

$$q_1 + m^*\left(\frac{q_{-\infty} - q_1}{N}\right) \leqslant q_0 < q_1 + (m^* + 1)\left(\frac{q_{-\infty} - q_1}{N}\right),$$

and for $m = m^* + 1, \ldots, N-1$ define

$$j_m := \max\left\{j : q_j \geqslant q_1 + m\left(\frac{q_{-\infty} - q_1}{N}\right)\right\}.$$

Also define $j_N = -\infty$. Consider $m = m^* + 1, \ldots, N$. It follows from Lemma 4 and Lemma 5 that for every $\varepsilon_m > 0$, there exists $\kappa_m > 0$ such that max $\left\{q_{i-1} - q_i, \bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du\right\} < \kappa_m$, for some $i = j_{m-1} - 1, j_{m-1} - 2, \ldots, j_m$, implies max $\left\{q_{i'-1} - q_{i'}, \bar{P}_{i'} - \int_0^{\psi(q_{i'})} \alpha(u)\underline{v}du\right\}$ $< \varepsilon_m$ for every $i' = j_{m-1}, \ldots, i+1$.

This, together with Fact 3, immediately implies Lemma 6. ■

Recall that $\lim_{i\to-\infty} q_i - q_{i-1} = 0$ (see equation (19)). Using this fact and equation (28), it is easy to check that

$$\lim_{i\to-\infty} \bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du = 0. \tag{31}$$

Combining Lemma 6 with equation (19) and equation (31) we obtain a contradiction to equation (10). This concludes the proof of Proposition 4. ■

References

[1] Ausubel, L. M., and R. J. Deneckere (1992): "Bargaining and the Right to Remain Silent," *Econometrica*, 60, 597-625.

[2] Bank for International Settlements (2009). OTC derivatives market activity in the second half of 2008. Monetary and Economics Department.

[3] Berk, J., and P. DeMarzo (2014): *Corporate Finance (3rd edition)*, Pearson Prentice Hall.

[4] Brunnermeier, M. (2009). "Deciphering the Liquidity and Credit Crunch 2007-2008," *Journal of Economic Perspectives,* 23, 77–100.

[5] Camargo, B. and B. Lester (2014): "Trading Dynamics in Decentralized Markets with Adverse Selection," *Journal of Economic Theory*, 153, 534-568.

[6] Coase, R. H. (1972): "Durability and Monopoly," *Journal of Law and Economics*, 15, 143-149.

[7] Daley, D., and B. Green (2012): "Waiting for News in the Market for Lemons," *Econometrica*, 80, 1433-1504.

[8] Davila, A., G. Foster, and M. Gupta (2003): "Venture Capital Financing and the Growth of Startup Firms," *Journal of Business Venturing*, 18, 689-708.

[9] Deneckere, R. J., and M.-Y. Liang (2006): "Bargaining with Interdependent Values," *Econometrica*, 74, 1309-1364.

[10] Evans, R. (1989): "Sequential Bargaining with Correlated Values," *Review of Economic Studies*, 56, 499-510.

[11] Fuchs, W., A. Öry and A. Skrzypacz (2014): "Transparency and Distressed Sales under Asymmetric Information," mimeo.

[12] Fuchs, W., and A. Skrzypacz (2013a): "Bridging the Gap: Bargaining with Interdependent Values," *Journal of Economic Theory*, 148, 1226-1236.

[13] Fuchs, W., and A. Skrzypacz (2013b): "Costs and Benefits of Dynamic Trading in a Lemons Market," mimeo.

[14] Fudenberg, D., D. K. Levine, and J. Tirole (1985): "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information," in *Game Theoretic Models of Bargaining*, ed. by A. Roth. Cambridge University Press.

[15] Fudenberg, D., and J. Tirole (1991): *Game Theory*, The MIT Press.

[16] Gerardi, D., J. Hörner and L. Maestri (2014): "The Role of Commitment in Bilateral Trade," *Journal of Economic Theory*, 154, 578-603.

[17] Gorton, G. (2009): "The Panic of 2007," in Maintaining Stability in a Changing Financial System, Proceedings of the 2008 Jackson Hole Conference, Federal Reserve Bank of Kansas City.

[18] Gorton, G., and A. Metrick (2012): "Securitized Banking and the Run on Repo," *Journal of Financial Economics*, 104, 425-451.

[19] Gul, F., and H. Sonnenschein (1988): "On Delay in Bargaining with One-Sided Uncertainty," *Econometrica*, 56, 601-611.

[20] Gul, F., H. Sonnenschein, and R. Wilson (1986): "Foundations of Dynamic Monopoly and the Coase Conjecture," *Journal of Economic Theory*, 39, 155-190.

[21] Hörner, J. and N. Vieille (2009): "Public vs. Private Offers in the Market for Lemons," *Econometrica*, 77, 29-70.

[22] Inderst, R. (2005): "Matching Markets with Adverse Selection," *Journal of Economic Theory*, 121, 145-166.

[23] Kaya, A. and K. Kim (2015): "Trading Dynamics with Private Buyer Signals in the Market for Lemons," mimeo.

[24] Kim, K. (2015): "Information about Sellers' Past Behavior in the Market for Lemons," mimeo.

[25] Moreno, D. and M. Wooders (2010): "Decentralized Trade Mitigates the Lemons Problem," *International Economic Review,* 121, 51, 383-399.

[26] Moreno, D. and M. Wooders (2015): " Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention," *Theoretical Economics*, forthcoming.

[27] Samuelson, W. (1984): "Bargaining under Asymmetric Information," *Econometrica*, 52, 995-1005.

[28] Vincent, D. R. (1989): "Bargaining with Common Values," *Journal of Economic Theory*, 48, 47-62.