Online Appendix: Omitted Proofs

Appendix D: Algorithm

Proof of Proposition 4.

First, suppose, by contradiction, that the last impasse occurs at (k, q') for some $k > 1$. We claim that $q' < \bar{q}_1$. To see this, notice that there exists $\gamma > 0$ such that for every $q \geqslant \bar{q}_1$

$$W(k, q) \geqslant \int_q^1 ((\alpha_k + \ldots + \alpha_1) v(s) - kc)\, ds > \gamma.$$

The fact that the function $W(k, \cdot)$ is strictly bounded below from zero implies that, as $\delta_n \to 1$, the time it takes for the buyer to buy the remaining k units from the types above $\bar{q}_1$ converges to zero. This result follows from well known Coasean forces.

The fact that the last impasse is at (k, q') implies that for $q \in (q', \bar{q}_1)$ we have

$$W(k, q) = \int_q^1 ((\alpha_k + \ldots + \alpha_1) v(s) - kc)\, ds <$$
$$\int_q^1 ((\alpha_k + \ldots + \alpha_2) v(s) - (k-1) c)\, ds + \int_q^{\bar{q}_1} \left(\alpha_1 v(s) - P\left(1, \bar{q}_1^-\right)\right) ds,$$

which is the payoff from buying the first $(k-1)$ units from all the remaining types at the price $(k-1)c$ and the last unit from the types in $[q, \bar{q}_1]$ at the price $P\left(1, \bar{q}_1^-\right)$. Thus, for δ_n sufficiently close to one, the buyer has a profitable deviation at (k, q).

Suppose now that no impasse occurs on the equilibrium path. Then for $q < \bar{q}_1$, we have

$$W(m, q) = \int_q^1 ((\alpha_m + \ldots + \alpha_1) v(s) - mc)\, ds <$$
$$\int_q^1 ((\alpha_m + \ldots + \alpha_2) v(s) - (m-1) c)\, ds + \int_q^{\bar{q}_1} \left(\alpha_1 v(s) - P\left(1, \bar{q}_1^-\right)\right) ds.$$

and, again, for δ_n sufficiently close to one, the buyer has a profitable deviation at (m, q). This shows that the last impasse is at $(1, \bar{q}_1)$. The fact that this impasse is of real time $1 - \left(\frac{\alpha_1 \underline{v}}{c}\right)^2$ follows from DL. ■

We now develop the necessary notation to provide a formal description of the algorithm which describes the m-limiting equilibrium outcome. We let D denote the set of the following triples:

$$D := \{(k, q, p) : k = 1, \ldots, m-1,\ q > \bar{q}_{k+1} \text{ and } p < (\alpha_k + \ldots + \alpha_1) \underline{v}\}.$$

Fix a triple $(k, q, p) \in D$. For each state (k', q'), with $k' = k+1, \ldots, m$ and $q' < q$, consider the following course of action. The buyer makes the universal offer for $(k' - k)$

units (which is accepted by all the types in $[q', 1]$) and then purchases the last k units from the types in $[q', q]$ at the price p. We let $\chi(k', k, q, p)$ denote the type q' at which the buyer breaks even. Formally, for $k' = k+1, \ldots, m$, we let $\chi(k', k, q, p)$ be implicitly defined by

$$\int_{\chi(k',k,q,p)}^{1} ((\alpha_{k'} + \ldots + \alpha_{k+1}) v(s) - (k' - k) c) \, ds + \int_{\chi(k',k,q,p)}^{q} ((\alpha_k + \ldots + \alpha_1) \underline{v} - p) \, ds = 0,$$

provided that the solution to the above equation exists and is positive. In the other cases, we set $\chi(k', k, q, p)$ equal to zero.

For each $k' = k+1, \ldots, m$, we compare $\chi(k', k, q, p)$ with zero and $\bar{q}_{k'+1}$ (recall that this is the type at which the buyer breaks even if he trades the $(m-k)$-th unit at the price c). We let $\phi_1(k, q, p)$ denote the smallest integer k' for which $\chi(k', k, q, p)$ is strictly larger than the other two quantities (see below for the case in which such an integer does not exist). Formally, suppose that $\chi(k', k, q, p) > \max\{0, \bar{q}_{k'+1}\}$ for some $k' = k+1, \ldots, m-1$. Then we let $\phi_1(k, q, p)$ be equal to

$$\phi_1(k, q, p) := \arg\min \left\{ k' = k+1, \ldots, m-1 \text{ s.t. } \chi(k', k, q, p) > \max\{0, \bar{q}_{k'+1}\} \right\}.$$

If instead $\chi(k', k, q, p) \leqslant \max\{0, \bar{q}_{k'+1}\}$ for every $k' = k+1, \ldots, m-1$, then we let $\phi_1(k, q, p)$ be equal to m.

Finally, we let

$$\phi_2(k, q, p) := \chi(\phi_1(k, q, p), k, q, p)$$

denote the critical type q' at which the buyer breaks even if he purchases $(\phi_1(k, q, p) - k)$ units at the price $(\phi_1(k, q, p) - k) c$ from the types in $[q', 1]$ and the last k units at the price p from the types in $[q', q]$.[42]

We are now ready to provide a formal description of our algorithm.

Proposition 7 *Suppose that, in the limit, the $(j-1)$-th to last impasse is at (k_{j-1}, q_{j-1}) with $k_{j-1} < m$.*

i) If $\phi_2\left(k_{j-1}, q_{j-1}, P\left(k_{j-1}, q_{j-1}^-\right)\right) = 0$, then there are no other impasses.

ii) If $\phi_2\left(k_{j-1}, q_{j-1}, P\left(k_{j-1}, q_{j-1}^-\right)\right) > 0$, then the j-th to last impasse is at

$$\left(\phi_1\left(k_{j-1}, q_{j-1}, P\left(k_{j-1}, q_{j-1}^-\right)\right), \phi_2\left(k_{j-1}, q_{j-1}, P\left(k_{j-1}, q_{j-1}^-\right)\right)\right).$$

This impasse is also the first one if $\phi_1\left(k_{j-1}, q_{j-1}, P\left(k_{j-1}, q_{j-1}^-\right)\right) = m$. Furthermore, the impasse is of real time $1 - \left(\frac{\left(\left(\alpha_{k_j} + \ldots + \alpha_1\right)\underline{v}\right)}{(k_j - k_{j-1})c + P\left(k_{j-1}, q_{j-1}^-\right)}\right)^2$, and

$$P\left(k_j, q_j^-\right) = \frac{\left(\left(\alpha_{k_j} + \ldots + \alpha_1\right)\underline{v}\right)^2}{(k_j - k_{j-1})c + P\left(k_{j-1}, q_{j-1}^-\right)} < \left(\alpha_{k_j} + \ldots + \alpha_1\right)\underline{v}$$

$$P\left(k_j, q_j^+\right) = (k_j - k_{j-1}) c + P\left(k_{j-1}, q_{j-1}^-\right).$$

[42] Recall that we set $\chi(\phi_1(k, q, p), k, q, p)$ equal to zero if the critical type q' is negative or does not exist.

Proof. The proof of Proposition 7 is split into a series of lemmata. To simplify the notation, we consider the case $j = 2$. In other words, we take as given the fact that the last impasse is at $(1, \bar{q}_1)$ and characterize the penultimate impasse. The proof for arbitrary values of j is analogous to the case analyzed here. Also, for notational convenience, we use $\hat{q}_k$, $k = 2, \ldots, m$, to denote $\chi\left(k, 1, \bar{q}_1, P\left(1, \bar{q}_1^-\right)\right)$. We remind the reader that $\hat{q}_k$ is implicitly defined by

$$\int_{\hat{q}_k}^{\bar{q}_1} \left((\alpha_k + \ldots + \alpha_1)\underline{v} - (k-1)c - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_k + \ldots + \alpha_2) v(s) - (k-1)c\right) ds = 0.$$

provided that the solution to the above equation exists and is positive. Otherwise we set $\hat{q}_k$ equal to zero.

Of course, $\hat{q}_k < \bar{q}_1$ for every $k > 1$. We also point out that for generic values of the parameters, $\hat{q}_k \neq \bar{q}_{k+1}$. In the following, we restrict our attention to generic cases.

It is easy to see that if $\hat{q}_k > 0$, then

$$(\alpha_k + \ldots + \alpha_1)\underline{v} < (k-1)c + P\left(1, \bar{q}_1^-\right).$$

Lemma 7 *Consider $k = 2, \ldots, m-1$ and suppose that $0 < \hat{q}_k < \bar{q}_{k+1}$. Then $\hat{q}_{k+1} > \hat{q}_k$.*

Proof. Notice that

$$\begin{gathered}
\textstyle\int_{\hat{q}_k}^{\bar{q}_1} \left((\alpha_{k+1} + \ldots + \alpha_1)\underline{v} - kc - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_{k+1} + \ldots + \alpha_2) v(s) - kc\right) ds = \\
\textstyle\int_{\hat{q}_k}^{\bar{q}_1} \left((\alpha_k + \ldots + \alpha_1)\underline{v} - (k-1)c - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_k + \ldots + \alpha_2) v(s) - (k-1)c\right) ds + \\
\textstyle\int_{\hat{q}_k}^{\bar{q}_{k+1}} \left(\alpha_{k+1}\underline{v} - c\right) ds + \int_{\bar{q}_{k+1}}^{1} \left(\alpha_{k+1} v(s) - c\right) ds = \\
\textstyle\int_{\hat{q}_k}^{\bar{q}_{k+1}} \left(\alpha_{k+1}\underline{v} - c\right) ds < 0.
\end{gathered}$$

Consider the function $G : \mathbb{R} \to \mathbb{R}$,

$$G(q) := \int_{q}^{\bar{q}_1} \left((\alpha_{k+1} + \ldots + \alpha_1)\underline{v} - kc - P\left(1, \bar{q}_1^-\right)\right) ds + \int_{\bar{q}_1}^{1} \left((\alpha_{k+1} + \ldots + \alpha_2) v(s) - kc\right) ds.$$

Notice that G is linear in q, $G(\bar{q}_1) > 0$ and $G(\hat{q}_k) < 0$. Therefore, the inequality above implies $\hat{q}_{k+1} \in (\hat{q}_k, \bar{q}_1)$. ■

Lemma 8 *Let $k \leqslant m$ and assume that for $j = 2, ..., k-1$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$. For $\delta_n \to 1$, consider a sequence of histories $h_n^{t_n}$ (on or off-path) associated to states (k, q_n) with $q_n \to q^* \in (\hat{q}_k, \bar{q}_1)$. Then, the state $(1, \bar{q}_1)$ is reached. Moreover, the real time required for this event converges to zero.*

Proof. By contradiction, let k be the minimal number such that the claim is violated. Without loss, assume that $q_n \in (\hat{q}_k, \bar{q}_1)$ for all n.

First, it is easy to see that the conclusion of Proposition 4 holds if the initial history is associated to a state (k, q) with $q < \bar{q}_1$. Therefore, we know that any continuation game must reach an impasse at $(1, \bar{q}_1)$. The proof will be concluded if we show that for any $\varepsilon > 0$ there exists $\gamma > 0$ such that for all $j \in \{2, ..., k\}$ and $q > \hat{q}_k + \varepsilon$ we have $W(j, q) > \gamma$.

Take $\varepsilon > 0$. For $j \in \{2, ..., k-1\}$ Lemma 7 implies $\hat{q}_k > \hat{q}_j$. Hence, for any $q \in (\hat{q}_k + \varepsilon, \bar{q}_1)$ and $\tilde{k} \in \{2, ..., k\}$ the payoff from buying $(\tilde{k} - 1)$ units in the first period and then following the equilibrium strategy for $(1, q)$ converges to

$$\int_q^{\bar{q}_1} \left((\alpha_{\tilde{k}} + \ldots + \alpha_1) \underline{v} - \left(\tilde{k} - 1\right) c - P\left(1, \bar{q}_1^-\right) \right) ds + \int_{\bar{q}_1}^1 \left((\alpha_{\tilde{k}} + \ldots + \alpha_2) v(s) - \left(\tilde{k} - 1\right) c \right) ds,$$

which is bounded away from zero. ■

Lemma 8 immediately implies the following corollary.

Corollary 2 *Let $k \leqslant m$ and assume that for $j = 2, ..., k-1$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$. Then for $q \in (\hat{q}_k, \bar{q}_1)$ we have*

$$P(k, q) = P(1, \bar{q}_1^-) + (k-1) c.$$

For $k > 1$ we define $\tilde{W}(k, \cdot)$ and $\tilde{P}(k, \cdot)$ by

$$\tilde{W}(k, q) := \textstyle\int_q^{\bar{q}_1} (\alpha_k v(s) - c)\, ds + W(k-1, q)$$
$$\tilde{P}(k, q) := c + P(k-1, q).$$

Lemma 9 *Let $k \leqslant m$ and assume that for $j = 2, ..., k-1$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$ and $\hat{q}_k > 0$. For $\delta_n \to 1$, consider a sequence of histories $h_n^{t_n}$ (on or off-path) associated to states (k, q_n) with $q_n \to q^* \in (\hat{q}_{k-1}, \hat{q}_k)$. Then:*

i) The continuation game reaches the state $(k, \hat{q}_k)$;

ii) The real time required for this event converges to zero;

iii) There is an impasse at $(k, \hat{q}_k)$ of real time $1 - \left(\frac{\sum_{j \leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)} \right)^2$.

iv) For $q \in (\hat{q}_{k-1}, \hat{q}_k)$ we have:

$$P(k, q) = \left(\frac{\sum_{j \leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)} \right)^2 P\left(k, \hat{q}_k^+\right),$$

v) For $q \in (\hat{q}_{k-1}, \hat{q}_k)$ we have:

$$W(k, q) = \int_q^{\hat{q}_k} \left(\sum_{j \leqslant k} \alpha_j \underline{v} - P\left(k, \hat{q}_k^-\right) \right) ds.$$

Proof. We begin by arguing that there must be an impasse at $(k, \hat{q}_k)$.

From Lemma 8 we have $\tilde{W}(k, \hat{q}_k) = 0$ and $\tilde{W}(k, q) < 0$ for $q \in \left(\left(\frac{\hat{q}_{k-1}+\hat{q}_k}{2}\right), \hat{q}_k\right)$. First, taking a subsequence if necessary the continuation value of the low type at (k, q_n) converges to some v_L^*. We claim that $v_L^* < P(k, \hat{q}_k^+)$. Suppose this is not the case. Take a small $\varepsilon > 0$ and let $h_n^{t_1}$ the first history reached (with positive probability) by $h_n^{t_n}$ such that the state at $h_n^{t_n}$, (k_n, q_n^1) is such that either i) $k_n = k$ and $q_n^1 \geqslant \hat{q}_k + \varepsilon$ or ii) the buyer makes an offer for z units at a price zc. Using the fact that the continuation values are always positive and the reservation prices are weakly increasing, an upper bound to the buyer's payoff (for large n) is:

$$\int_{q_n}^{q_n^1} \left(\sum_{j\leqslant k} \alpha_j \underline{v} - P(k, \hat{q}_k^+) + \varepsilon\right) ds + \tilde{W}\left(k_n, q_n^1\right) + \varepsilon =$$
$$\tilde{W}(k, q_n) + \varepsilon + \varepsilon\left(q_n^1 - q_n\right)$$

which is negative for $\varepsilon < -\frac{\tilde{W}(k,q_n)}{2}$.

For some (small) $\eta > 0$, take the continuation game and define $h_n^{t_2(\eta)}$ the first history reached (with positive probability) by $h_n^{t_n}$ such that either the buyer buys z units ($z \in \{1, ..., k-1\}$) at a price cz or the continuation utility of the low type is at least $P(k, \hat{q}_k^+) - \eta$. Let (k, q_n^η) the state at $h_n^{t_2(\eta)}$. Taking a convergent subsequence, $q_n^\eta \to q^\eta$. Clearly $\lim_{\eta\to 0} q^\eta = \hat{q}_k$. The first argument of this Lemma establishes that $\liminf_{\eta\to 0} q^\eta \geqslant \hat{q}_k$, while Corollary 2 establishes $\limsup_{\eta\to 0} q^\eta \leqslant \hat{q}_k$. One can then take a small η such that:

a) q_n^η is close to $\hat{q}_k$;

b) For (k, q), such that $q \in (q_n, q_n^\eta)$ the buyer buys k units only from the low type; and

c) $P_n(k, q_n^\eta)$ is close to $P(k, \hat{q}_{k+})$.

After establishing a), b) and c), one can use an argument similar to that in DL to establish that there is an impasse at $(k, \hat{q}_k)$ corresponding to a real delay equal to $\left(\frac{\sum_{j\leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)}\right)^2$.

This implies $P(k, \hat{q}_k^-) = \left(\frac{\sum_{j\leqslant k} \alpha_j \underline{v}}{P\left(k, \hat{q}_k^+\right)}\right)^2 P(k, \hat{q}_k^+)$. Therefore, iii) and iv) are established.

Now, consider a state (k, q), with $q \in (\hat{q}_{k-1}, \hat{q}_k)$. Since the reservation price of the low type converges to $P(k, \hat{q}_k^-) < \sum_{j\leqslant k} \alpha_j \underline{v}$ and the limit payoff from buying z units ($z \in \{1, ..., k-1\}$) at a price cz converges to $\tilde{W}(k, q_n) < 0$, i), ii) and v) follow immediately. ■

Lemma 10 *Let $k < m$ and assume that for $j = 2, ..., k$ we have $\hat{q}_j \leqslant \max\{0, \bar{q}_{j+1}\}$. Then, for any initial state (k', q'), $k' = k+1, \ldots, m$, $q' \geqslant 0$, and for any $q \geqslant q'$, there is no impasse at (k, q).*

Proof. First we show an important inequality. If $c \geqslant \alpha_{k+1}\underline{v}$, then we have:

$$c+\left(\frac{\left(\left(\alpha_k+\ldots+\alpha_1\right)\underline{v}\right)^2}{(k-1)c+P(1,\bar{q}_1^-)}\right)>\left(\frac{\left(\left(\alpha_k+\ldots+\alpha_1\right)\underline{v}+\alpha_{k+1}\underline{v}\right)^2}{(k-1)c+P(1,\bar{q}_1^-)+c}\right). \tag{38}$$

To simplify notation, write V for $\left(\alpha_k+\ldots+\alpha_1\right)\underline{v}$, P for $(k-1)c+P(1,\bar{q}_1^-)$ and v_{k+1} for $\alpha_{k+1}\underline{v}$. The inequality above is equivalent to:

$$c+\left(\frac{V^2}{P}\right)>\left(\frac{\left(V+v_{k+1}\right)^2}{P+c}\right)\Leftrightarrow c\left(P^2+Pc+V^2\right)>2Vv_{k+1}P+Pv_{k+1}^2.$$

For the last inequality it suffices that

$$c\left(P^2+Pc+V^2\right)>2VPv_{k+1}+Pcv_{k+1}\Leftrightarrow c\left(P^2+P\left(c-v_{k+1}\right)+V^2\right)>2VPv_{k+1}.$$

For the last inequality, it suffices that

$$c\left(P^2+V^2\right)>2VPc\Leftrightarrow c\left(P-V\right)^2>0.$$

If $\hat{q}_k=0$, there cannot be an impasse at any state (k,q) since $W\left(k,\cdot\right)$ is bounded away from zero.

Therefore, assume that $0<\hat{q}_k<\hat{q}_{k+1}$. If the state is (k',q) and $q>\frac{\hat{q}_k+\hat{q}_{k+1}}{2}$, then there cannot be an impasse at (k,q) since $W\left(k,\cdot\right)$ is bounded away from zero.

It remains to consider the case $q\leqslant\frac{\hat{q}_k+\hat{q}_{k+1}}{2}$. Let $\left(k',q''\right)$, $q''\in\left[q',q\right]$, denote the state at which the buyer purchases $(k'-k)$ units from all the remaining types. We now show that there exists $\gamma>0$ such that when the state is $\left(k',q''\right)$, the limit payoff from buying $(k'-k)$ units from the types in $[q'',1]$ is lower at least by γ than the payoff from buying $(k'-k-1)$ units from the types in $[q'',1]$ and then $(k+1)$ units from the types in $\left(q'',\hat{q}_{k+1}\right)$. If $q\in\left[\hat{q}_k,\left(\frac{\hat{q}_k+\hat{q}_{k+1}}{2}\right)\right]$ the result is trivial. Hence, assume $q<\hat{q}_k$.The payoff from the first strategy is:

$$\begin{gathered}\int_{q''}^{1}\left(\left(\alpha_{k'}+\ldots+\alpha_{k+1}\right)v(s)-\left(k'-k\right)c\right)ds+\int_{q''}^{\hat{q}_k}\left(\left(\alpha_k+\ldots+\alpha_1\right)\underline{v}-\left(\frac{\left(\left(\alpha_k+\ldots+\alpha_1\right)\underline{v}\right)^2}{(k-1)c+P(1,\bar{q}_1^-)}\right)\right)ds= \\ \int_{q''}^{\hat{q}_k}\left(\left(\alpha_{k'}+\ldots+\alpha_1\right)\underline{v}-\left(\frac{\left(\left(\alpha_k+\ldots+\alpha_1\right)\underline{v}\right)^2}{(k-1)c+P(1,\bar{q}_1^-)}\right)-\left(k'-k\right)c\right)ds+ \\ \int_{q''}^{1}\left(\left(\alpha_{k\prime}+\ldots+\alpha_{k+1}\right)v(s)-\left(k'-k\right)c\right)ds< \\ \int_{q''}^{\hat{q}_k}\left(\left(\alpha_{k'}+\ldots+\alpha_1\right)\underline{v}-\left(\frac{\left(\left(\alpha_k+\ldots+\alpha_1\right)\underline{v}\right)^2}{(k-1)c+P(1,\bar{q}_1^-)}\right)-c\right)ds+ \\ \int_{q''}^{1}\left(\left(\alpha_{k\prime}+\ldots+\alpha_{k+2}\right)v(s)-\left(k'-k-1\right)c\right)ds\end{gathered}$$

since $\int_{q''}^{1}(\alpha_{k+1}v(s)-c)\,ds<0$.[43] Now, from inequality (38) the expression above is strictly smaller than

$$\int_{q''}^{\hat{q}_k}\left(\left(\alpha_{\alpha_{k'}}+\ldots+\alpha_1\right)\underline{v}-\frac{((\alpha_{k+1}+\ldots+\alpha_1)\underline{v})^2}{(k-1)c+P(1,\bar{q}_1^-)+c}\right)ds+$$
$$\int_{q''}^{1}\left(\left(\alpha_{k'}+\ldots+\alpha_{k+2}\right)v(s)-\left(k'-k-1\right)c\right)ds$$

which, in turn, is strictly smaller than

$$\int_{q''}^{\hat{q}_{k+1}}\left(\left(\alpha_{k+1}+\ldots+\alpha_1\right)\underline{v}-\frac{((\alpha_{k+1}+\ldots+\alpha_1)\underline{v})^2}{(k-1)c+P(1,\bar{q}_1^-)+c}\right)ds+$$
$$\int_{q''}^{1}\left(\left(\alpha_{k'}+\ldots+\alpha_{k+2}\right)v(s)-\left(k'-k-1\right)c\right)ds$$

which is the payoff from the second strategy. ■

Recall that we set $\bar{q}_{m+1}=0$. We now need to distinguish between two cases:
(a) there exists $k=2,\ldots,m$, such that $\hat{q}_k>\max\{\bar{q}_{k+1},0\}$;
(b) for every $k=2,\ldots,m$, $\hat{q}_k\leqslant\max\{\bar{q}_{k+1},0\}$.
We start with case (a) and define $\hat{k}$ as

$$\hat{k}:=\inf\{k=2,\ldots,m-1:\ \hat{q}_k>\max\{\bar{q}_{k+1},0\}\}$$

The following Lemma follows directly from Lemma 9 and Lemma 10.

Lemma 11 *The limit functions $P\left(\hat{k},\cdot\right)$ and $W\left(\hat{k},\cdot\right)$ satisfy*

$$P\left(\hat{k},q\right):=\begin{cases}\left(\frac{(\alpha_{\hat{k}}+\ldots+\alpha_1)\underline{v}}{\left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right)}\right)^2\left(\left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right)\right) & \text{if } q<\hat{q}_{\hat{k}}\\ \left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right) & \text{if } q\in\left(\hat{q}_{\hat{k}},\bar{q}_1\right)\\ \hat{k}c & \text{if } q>\bar{q}_1\end{cases}$$

$$W\left(\hat{k},q\right):=\begin{cases}\int_q^{\hat{q}_{\hat{k}}}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}-P\left(\hat{k},q\right)\right)ds & \text{if } q\leqslant\hat{q}_{\hat{k}}\\ \begin{array}{l}\int_q^{\bar{q}_1}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}-P\left(\hat{k},q\right)\right)ds+\\ \int_{\bar{q}_1}^{1}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_2\right)v(s)-\left(\hat{k}-1\right)c\right)ds\end{array} & \text{if } q\in\left(\hat{q}_{\hat{k}},\bar{q}_1\right]\\ \int_q^1\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)v(s)-\hat{k}c\right)ds & \text{if } q>\bar{q}_1\end{cases}$$

[43] We set $\alpha_{k'}+\ldots+\alpha_{k+2}=0$ if $k'=k+1$.

Lemma 12 *The penultimate impasse occurs at* $\left(\hat{k},\hat{q}_{\hat{k}}\right)$ *and is of real time* $1-\left(\frac{\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}}{\left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right)}\right)^2$. *Furthermore, if* $\hat{k}=m$, *then the impasse at* $\left(\hat{k},\hat{q}_{\hat{k}}\right)$ *is the first impasse.*

Proof. By contradiction, suppose the claim is false. There are two possibilities:
i) the penultimate impasse occurs at (k,q') for some $k=\hat{k}+1,\ldots,m$;
ii) there is only one impasse at $(1,\bar{q}_1)$.

We consider case i) and derive a contradiction. The proof for case ii) is similar to the proof for case i); thus, we omit the details.

The fact that $\hat{q}_{\hat{k}}>\bar{q}_{\hat{k}+1}>\bar{q}_{\hat{k}+2}>\ldots\bar{q}_m$ implies that for any $k=\hat{k}+1,\ldots,m$ and any $q'\in[\hat{q}_{\hat{k}},\bar{q}_1]$

$$\int_{q'}^{\bar{q}_1}\left(\left(\alpha_k+\ldots+\alpha_1\right)v(s)-(k-1)c-P\left(1,\bar{q}_1^-\right)\right)ds+$$
$$\int_{\bar{q}_1}^{1}\left(\left(\alpha_k+\ldots+\alpha_2\right)v(s)-(k-1)c\right)ds>0.$$

Thus, if the penultimate impasse occurs at (k,q'), then $q'<\hat{q}_{\hat{k}}$. For $q\in(q',\hat{q}_{\hat{k}})$, we have

$$\begin{aligned}W(k,q)&=\int_q^{\bar{q}_1}\left(\left(\alpha_k+\ldots+\alpha_1\right)v(s)-(k-1)c-P\left(1,\bar{q}_1^-\right)\right)ds+\\&\quad\int_{\bar{q}_1}^{1}\left(\left(\alpha_k+\ldots+\alpha_2\right)v(s)-(k-1)c\right)ds\\&=\int_q^{\hat{q}_{\hat{k}}}\left(\left(\alpha_k+\ldots+\alpha_1\right)v(s)-(k-1)c-P\left(1,\bar{q}_1^-\right)\right)ds+\\&\quad\int_{\hat{q}_{\hat{k}}}^{1}\left(\left(\alpha_k+\ldots+\alpha_{\hat{k}+1}\right)v(s)-\left(k-\hat{k}\right)c\right)ds+W\left(\hat{k},\hat{q}_{\hat{k}}\right),\end{aligned}$$

where we used the expression for $W\left(\hat{k},q\right)$ for $q\geqslant\hat{q}_{\hat{k}}$. Since $W\left(\hat{k},\hat{q}_{\hat{k}}\right)=0$ the expression above is:

$$\begin{aligned}&\int_q^{\hat{q}_{\hat{k}}}\left(\left(\alpha_k+\ldots+\alpha_{\hat{k}+1}\right)v(s)-\left(k-\hat{k}\right)c\right)ds+\\&\int_q^{\hat{q}_{\hat{k}}}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)v(s)-\left(\hat{k}-1\right)c-P\left(1,\bar{q}_1^-\right)\right)ds+\\&\int_{\hat{q}_{\hat{k}}}^{1}\left(\left(\alpha_k+\ldots+\alpha_{\hat{k}+1}\right)v(s)-\left(k-\hat{k}\right)c\right)ds=\\&\int_q^{1}\left(\left(\alpha_k+\ldots+\alpha_{\hat{k}+1}\right)v(s)-\left(k-\hat{k}\right)c\right)ds+\\&\int_q^{\hat{q}_{\hat{k}}}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)v(s)-\left(\hat{k}-1\right)c-P\left(1,\bar{q}_1^-\right)\right)ds<\\&\int_q^{1}\left(\left(\alpha_k+\ldots+\alpha_{\hat{k}+1}\right)v(s)-\left(k-\hat{k}\right)c\right)ds+\\&\int_q^{\hat{q}_{\hat{k}}}\left(\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)v(s)-P\left(\hat{k},\hat{q}_{\hat{k}-}\right)\right)ds,\end{aligned}$$

which is the (limit) payoff from buying $\left(k-\hat{k}\right)$ units at a price $\left(k-\hat{k}\right)c$.

Finally, it follows immediately from Lemma 9 and Lemma 11 that the impasse at $\left(\hat{k}, \hat{q}_{\hat{k}}\right)$ is of real time $1-\left(\frac{\left(\alpha_{\hat{k}}+\ldots+\alpha_1\right)\underline{v}}{\left(\hat{k}-1\right)c+P\left(1,\bar{q}_1^-\right)}\right)^2$ and that no other impasses occur prior to it when $\hat{k}=m$.■

We now turn to case (b).

Lemma 13 *Suppose that for every* $k=2,\ldots,m$, $\hat{q}_k \leqslant \max\{\bar{q}_{k+1},0\}$. *Then in the limit there is only one impasse at* $(1,\bar{q}_1)$.

Proof. First, notice that for any $k=2,\ldots,m$, if $\hat{q}_k=0$, an impasse cannot occur at (k,q) since $W(k,\cdot)$ is bounded away from zero. Under the assumptions of the lemma, the fact an impasse cannot occur at $(k,\hat{q}_k)$ with $k=2,\ldots,m-1$ and $\hat{q}_k \in (0,\bar{q}_{k+1})$ follows from Lemma 10. ■

Appendix E: Upper Bounds

In this appendix, we prove Fact 3 and Fact 4 stated in Appendix C.

For every $i=0,-1,\ldots,$ we have $\Phi(q_{i+1},q_i,q_{i-1})=0$ (recall that the function Φ is defined in equation (32)). Using straightforward algebra and defining the function $\hat{\alpha}=\alpha\circ\psi$ we obtain

$$(q_i-q_{i+1})\frac{\int_{q_{i+1}}^{q_i}-\psi'(u)(\bar{c}-\hat{\alpha}(u)\underline{v})du}{q_i-q_{i+1}} = (q_i-q_{i+1})\left(\frac{\int_{q_{i+1}}^{q_i}\left[-\psi'(u)\left(\int_{q_i}^1[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_i-q_{i+1})^2}\right)+$$
$$\left(\frac{\int_0^{\psi(q_i)}\alpha(u)\underline{v}du}{\int_0^{\psi(q_i)}\alpha(u)\underline{v}du+\left(\frac{\int_{q_i}^{q_{i-1}}\left[-\psi'(u)\left(\int_{q_{i-1}}^1[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{q_{i-1}-q_i}\right)}\right)\left(\frac{\int_{q_i}^{q_{i-1}}\left[-\psi'(u)\left(\int_{q_{i-1}}^1[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_{i-1}-q_i)^2}\right)(q_{i-1}-q_i),$$

or equivalently

$$\frac{q_i-q_{i+1}}{q_{i-1}-q_i} = \left(\frac{\int_0^{\psi(q_i)}\alpha(u)\underline{v}du}{\int_0^{\psi(q_i)}\alpha(u)\underline{v}du+\left(\frac{\int_{q_i}^{q_{i-1}}\left[-\psi'(u)\left(\int_{q_{i-1}}^1[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{q_{i-1}-q_i}\right)}\right)\cdot$$
$$\left(\frac{\frac{\int_{q_i}^{q_{i-1}}\left[-\psi'(u)\left(\int_{q_{i-1}}^1[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_{i-1}-q_i)^2}}{\frac{\int_{q_{i+1}}^{q_i}-\psi'(u)(\bar{c}-\hat{\alpha}(u)\underline{v})du}{q_i-q_{i+1}}-\frac{\int_{q_{i+1}}^{q_i}\left[-\psi'(u)\left(\int_{q_i}^1[\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_i-q_{i+1})^2}}\right). \tag{39}$$

Consider the first term in the right-hand side of equation (39). It is bounded above by one. Hence, we obtain the following upper bound for $\frac{q_i - q_{i+1}}{q_{i-1} - q_i}$:

$$\frac{q_i - q_{i+1}}{q_{i-1} - q_i} \leqslant \frac{\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u)\left(\int_{q_{i-1}}^{1} [\hat{\alpha}(u)v(s) - \bar{c}] ds\right)\right] du}{(q_{i-1} - q_i)^2}}{\frac{\int_{q_{i+1}}^{q_i} -\psi'(u)(\bar{c} - \hat{\alpha}(u)\underline{v}) du}{q_i - q_{i+1}} - \frac{\int_{q_{i+1}}^{q_i} \left[-\psi'(u)\left(\int_{q_i}^{1} [\hat{\alpha}(u)v(s) - \bar{c}] ds\right)\right] du}{(q_i - q_{i+1})^2}}. \tag{40}$$

Using the fact that $\int_{q_i}^{1} [\hat{\alpha}(q_i) v(s) - \bar{c}] \, ds = 0$ and the mean value theorem, we obtain

$$\begin{gathered} \int_{q_{i+1}}^{q_i} \left[-\psi'(u) \left(\int_{q_i}^{1} [\hat{\alpha}(u) v(s) - \bar{c}] \, ds\right)\right] du = \\ \int_{q_i}^{1} v(s) \, ds \int_{q_{i+1}}^{q_i} \left(\psi'(u) \int_u^{q_i} \hat{\alpha}'(s) \, ds\right) du = \left(\frac{\psi'(q_i')\hat{\alpha}'\left(q_i''\right) \int_{q_i}^{1} v(s) ds}{2}\right) (q_i - q_{i+1})^2 \end{gathered} \tag{41}$$

for some $(q_i', q_i'') \in [q_{i+1}, q_i]^2$.

In a similar way we obtain

$$\int_{q_i}^{q_{i-1}} \left[-\psi'(u) \left(\int_{q_{i-1}}^{1} [\hat{\alpha}(u) v(s) - \bar{c}] \, ds\right)\right] du = \left(\frac{\psi'(q_{i-1}')\hat{\alpha}'\left(q_{i-1}''\right) \int_{q_{i-1}}^{1} v(s) \, ds}{2}\right) (q_{i-1} - q_i)^2 \tag{42}$$

for some $\left(q_{i-1}', q_{i-1}''\right) \in [q_i, q_{i-1}]^2$.

Also, it follows from the mean value theorem that

$$\frac{\int_{q_{i+1}}^{q_i} -\psi'(u) (\bar{c} - \hat{\alpha}(u) \underline{v}) \, du}{q_i - q_{i+1}} = -\psi'(q_i''') (\bar{c} - \hat{\alpha}(q_i''') \underline{v}), \tag{43}$$

for some $q_i''' \in [q_{i+1}, q_i]$.

Hence, from equations (40)-(43) we conclude that the function $\Upsilon(q_{i-1}, q_i, q_{i+1})$ defined by

$$\Upsilon(q_{i-1}, q_i, q_{i+1}) := \frac{\left(\frac{\psi'(q_{i-1}')\hat{\alpha}'\left(q_{i-1}''\right) \int_{q_{i-1}}^{1} v(s) ds}{2}\right)}{-\psi'(q_i''') (\bar{c} - \hat{\alpha}(q_i''') \underline{v}) - \left(\frac{\psi'(q_i')\hat{\alpha}'\left(q_i''\right) \int_{q_i}^{1} v(s) ds}{2}\right)},$$

is an upper bound to $\frac{q_i - q_{i+1}}{q_{i-1} - q_i}$.[44]

We let $h = q_{i-1} - q_i$ and express q_i as $q_{i-1} - h$. It easily follows from equation (32) that q_{i+1} is a (locally) well defined function of q_{i-1} and h and we write $q_{i+1}(q_{i-1}, h)$ for it.[45]

[44] To simplify the notation, in the definition of the function Υ we suppress the dependence of q_i', q_i'' and q_i''' on q_{i+1} and q_i. Similarly, we suppress the dependence of q_{i-1}' and q_{i-1}'' on q_i and q_{i-1}.

[45] Notice that $\lim_{h \to 0} q_{i+1}(q_{i-1}, h) = q_{i-1}$.

Next, we define $\hat{\Upsilon}(q_{i-1},h) := \Upsilon(q_{i-1}, q_{i-1}-h, q_{i+1}(q_{i-1},h))$. We now compute the limit of $\hat{\Upsilon}(q_{i-1},h)$ as h shrinks to zero. To do this, we first need to evaluate $\hat{\alpha}'(q)$. Recall that, by definition, $\int_q^1 (\hat{\alpha}(q)v(s)-\bar{c})\,ds = 0$. This immediately implies $\hat{\alpha}'(q) = \left(\frac{\hat{\alpha}(q)\underline{v}-\bar{c}}{\int_q^1 v(s)ds}\right)$ for every $q \leqslant \hat{q}$. Hence, for every q_{i-1} we have

$$\lim_{h\to 0}\hat{\Upsilon}(q_{i-1},h) = \frac{\frac{\left(\frac{\hat{\alpha}(q_{i-1})\underline{v}-\bar{c}}{\int_{q_{i-1}}^1 v(s)ds}\right)\int_{q_{i-1}}^1 v(s)ds}{2}}{(\hat{\alpha}(q_{i-1})\underline{v}-\bar{c}) - \left(\frac{\left(\frac{\hat{\alpha}(q_{i-1})\underline{v}-\bar{c}}{\int_{q_{i-1}}^1 v(s)ds}\right)\int_{q_{i-1}}^1 v(s)ds}{2}\right)} = 1. \tag{44}$$

Next, we state some simple mathematical facts (the proofs are standard and, therefore, omitted).

Claim 6 *Let f and $\tilde{f}$ be two continuous functions from $[0,1]\times[a,b]$ into $\mathbb{R}$. Assume that f and $\tilde{f}$ are four times continuously differentiable in the first argument. Assume that these derivatives are continuous in the second argument. There exists $M>0$ such that for every $y\in[a,b]$ and $x\in(0,1)$*

i) $\left|\frac{d}{dx}\left(\frac{\int_0^x f(s,y)ds}{x}\right)\right| < M;$

ii) $\left|\frac{d}{dx}\left(\frac{\int_0^x f(s,y)\int_0^s \tilde{f}(u,y)duds}{x^2}\right)\right| < M.$

Claim 6 is used to prove the next result.

Claim 7 *$\hat{\Upsilon}(q_{i-1},h)$ is differentiable in h. Furthermore, there exist $M>0$ and $h_1>0$ such that $h<h_1$ implies $\left|\frac{\partial\hat{\Upsilon}(q_{i-1},h)}{\partial h}\right| < M$ for all $q_{i-1}\in(0,q(0))$.*

We use the implicit function theorem to evaluate $\frac{\partial\hat{\Upsilon}(q_{i-1},h)}{\partial h}$. It is straightforward to bound this derivative for every q_{i-1} bounded away from $q(0)$. We use Claim 6 to bound the derivative for q_{i-1} close to $q(0)$. Straightforward but tedious algebra yields the Lipschitz constant.

Claim 7 and equation (44) imply the following result.

Claim 8 *For every $\varepsilon>0$ there exists $h_1>0$ such that $h<h_1$ implies $\Upsilon(q_{i-1},h)<1+\varepsilon$ for every $q_{i-1}\in(0,q(0))$.*

The next result is an immediate corollary of Claim 8 and allows us to prove Fact 3 (in Appendix C).

Corollary 3 *For every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $i = 0, -1, \ldots,$ if $q_{i-1} - q_i < \eta$, then $q_i - q_{i+1} < (1+\varepsilon)(q_{i-1} - q_i)$.*

We now turn to the proof of Fact 4. It follows from equation (34) that for every $i = 1, 0, -1, \ldots,$ we have

$$\frac{\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du\right)}{q_{i-1}-q_i} =$$

$$\frac{\int_{q_i}^{q_{i-1}} \left[-\psi'(u)\left(\int_{q_{i-1}}^1 [\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_{i-1}-q_i)^2} - \tag{45}$$

$$\frac{\int_{q_{i-1}}^{q_{i-2}} \left[-\psi'(u)\left(\int_{q_{i-2}}^1 [\hat{\alpha}(u)v(s)-\bar{c}]ds\right)\right]du}{(q_{i-2}-q_{i-1})^2} \left(\frac{q_{i-2}-q_{i-1}}{q_{i-1}-q_i}\right).$$

Similarly to what we did above, we let $h = q_{i-2} - q_{i-1}$ and express q_{i-1} as $q_{i-2} - h$ and compute q_i as function of q_{i-2} and h (and write $q_i(q_{i-2}, h)$ for it). It follows from equation (41) that as h goes to zero the right-hand side of (45) converges to

$$\left(\frac{\psi'(q_{i-2})\hat{\alpha}'(q_{i-2}) \int_{q_i}^1 v(s)\,ds}{2}\right)\left(1 - \frac{h}{q_{i-2} - h - q_i(q_{i-2}, h)}\right).$$

It follows from Corollary 3 that for every $\varepsilon > 0$ there exists $h_1 > 0$ such that $h < h_1$ implies $\frac{h}{q_{i-2}-h-q_i(q_{i-2},h)} > \frac{1}{1+\varepsilon}$ for every $q_{i-2} \in (0, q(0))$. This allows us to establish our next result.

Claim 9 *As h goes to zero the function* $\max\left\{\left(\frac{\psi'(q_{i-2})\hat{\alpha}'(q_{i-2})\int_{q_i}^1 v(s)ds}{2}\right)\left(1 - \frac{h}{q_{i-2}-h-q_i(q_{i-2},h)}\right), 0\right\}$ *converges uniformly (in q_{i-2}) to zero.*

Finally, using straightforward algebra one can show that there exists an upper bound for the function above, which implies the first lemma of this section.

Lemma 14 *There exists $b_1 > 0$ such that for every $i = 1, 0, -1, \ldots,$*

$$\frac{\left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du\right) - \left(\bar{P}_{i-1} - \int_0^{\psi(q_{i-1})} \alpha(u)\underline{v}du\right)}{q_{i-1} - q_i} \leqslant b_1(q_{i-1} - q_i).$$

Next, using equations (34) and (42) we obtain

$$\left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u)\underline{v}du\right) = \left(\frac{\psi'(q'_{i-1})\hat{\alpha}'(q''_{i-1})\int_{q_{i-1}}^1 v(s)\,ds}{2}\right)(q_{i-1} - q_i), \tag{46}$$

for some $\left(q'_{i-1}, q''_{i-1}\right) \in [q_i, q_{i-1}]^2$. Letting b_2 be a uniform bound for the reciprocal of the term in parenthesis in the right-hand side of (46), we obtain our second important lemma.

Lemma 15 *There exists $b_2 > 0$ such that for every $i = 1, 0, -1, \ldots,$*

$$q_{i-1} - q_i \leqslant b_2 \left(\bar{P}_i - \int_0^{\psi(q_i)} \alpha(u) \underline{v} du \right).$$

Appendix F: Increasing Gains

Proof of Proposition 6.

For every $k = 1, \ldots, m$, consider the DL's model in which the parties trade k *indivisible* units and let $W(k, \cdot)$ and $P(k, \cdot)$ denote the buyer's continuation payoff and the seller's reservation price, respectively. Also, for every $q \in [0, 1)$, let $t_k(q) > q$ be such that

$$W(k, q) = \int_q^{t_k(q)} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k, t_k(q)) \right] ds + \delta W(k, t_k(q)).$$

The proof of Proposition 6 is complete if we show that it is suboptimal for the buyer to make universal offers when the number of remaining units is $k = 1, \ldots, m$. By definition, our claim holds when there is only one unit left for trade. We now assume that the claim holds when the number of remaining units is at most $k-1$, $k = 2, \ldots, m$, and show that it also holds when there are k units left for trade.

By contradiction, suppose that there exist $k' < k$ and q such that

$$\begin{gathered} W(k, q) \leqslant \int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c \right] ds + \delta W(k', q) < \\ \int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c \right] ds + W(k', q), \end{gathered} \tag{47}$$

where the second inequality follows from the fact that $W(k', q)$ is strictly positive for every q (see DL). This, together with the fact that $t_k(q) \leqslant 1$, imply

$$\begin{gathered} \int_q^1 \left[(\alpha_k + \ldots + \alpha_1) v(s) - kc \right] ds \leqslant W(k, q) < \\ \int_q^1 \left[(\alpha_k + \ldots + \alpha_{k'+1}) v(s) - (k - k') c \right] ds + W(k', q), \end{gathered}$$

and, thus,

$$W(k', q) > \int_q^1 \left[(\alpha_{k'} + \ldots + \alpha_1) v(s) - k'c \right] ds. \tag{48}$$

Recall that $W(k', q)$ is the buyer's continuation payoff when he makes cream-skimming offers (which are, by assumption, optimal when there are k' units left for trade). Thus, there exists $T \in \mathbb{Z}_+$ such that

$$W(k', q) = \sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_{k'} + \ldots + \alpha_1) v(s) - P(k', t_{k'}^{\tau}(q)) \right] ds,$$

where we define $t_{k'}^{0}(q) = q$, and for $\tau = 1, \ldots, T$, $t_{k'}^{\tau}(q) = t_{k'}\left(t_{k'}^{\tau-1}(q)\right)$ (of course, $t_{k'}^{T}(q) = 1$).

A lower bound to the continuation payoff $W(k, q)$ can be computed by assuming that the buyer purchases after $\tau - 1$ periods, $\tau = 1, \ldots, T$, the k units from the types between $t_{k'}^{\tau-1}(q)$ and $t_{k'}^{\tau}(q)$ at the price $P(k, t_{k'}^{\tau}(q))$. Thus, we have

$$W(k, q) \geqslant \sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_{k} + \ldots + \alpha_1) v(s) - P(k, t_{k'}^{\tau}(q)) \right] ds. \tag{49}$$

Using equations (5)-(12) in DL (pages 1318-1319), it is easy to check that

$$\frac{P(k, q')}{k} \leqslant \frac{P(k', q')}{k'} \tag{50}$$

for every $q' \in [0, 1]$.

Combining equations (47), (49), and (50), we obtain

$$\begin{gathered}
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_{k} + \ldots + \alpha_1) v(s) - \tfrac{k}{k'} P(k', t_{k'}^{\tau}(q)) \right] ds \leqslant \\
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_{k} + \ldots + \alpha_1) v(s) - P(k, t_{k'}^{\tau}(q)) \right] ds \leqslant W(k, q) < \\
\int_{q}^{1} \left[(\alpha_{k} + \ldots + \alpha_{k'+1}) v(s) - (k - k') c \right] ds + W(k', q) = \\
\int_{q}^{1} \left[(\alpha_{k} + \ldots + \alpha_{k'+1}) v(s) - (k - k') c \right] ds + \\
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[(\alpha_{k'} + \ldots + \alpha_1) v(s) - P(k', t_{k'}^{\tau}(q)) \right] ds,
\end{gathered}$$

which implies (we compare the first and the last term after multiplying both of them by $\frac{k'}{k-k'}$)

$$\begin{gathered}
\sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[\tfrac{k'}{k-k'} (\alpha_{k} + \ldots + \alpha_{k'+1}) v(s) - P(k', t_{k'}^{\tau}(q)) \right] ds < \\
\int_{q}^{1} \left[\tfrac{k'}{k-k'} (\alpha_{k} + \ldots + \alpha_{k'+1}) v(s) - k' c \right] ds.
\end{gathered}$$

Recall that the gains from trade are increasing ($\alpha_m < \ldots < \alpha_1$). Thus, we have

$$\frac{k'}{k-k'}\left(\alpha_k + \ldots + \alpha_{k'+1}\right) < \alpha_{k'} + \ldots + \alpha_1.$$

It follows immediately from the last two inequalities that

$$W\left(k',q\right) = \sum_{\tau=1}^{T} \delta^{t-1} \int_{t_{k'}^{\tau-1}(q)}^{t_{k'}^{\tau}(q)} \left[\left(\alpha_{k'} + \ldots + \alpha_1\right) v\left(s\right) - P\left(k', t_{k'}^{\tau}\left(q\right)\right)\right] ds <$$
$$\int_q^1 \left[\left(\alpha_k + \ldots + \alpha_{k'+1}\right) v\left(s\right) - k'c\right] ds,$$

which contradicts inequality (48) and concludes our proof.

Appendix G: Menus of Offers

As anticipated in Section 9, in this appendix we construct a stationary equilibrium of the game in which the buyer can propose menus with at most two offers. In particular, we focus on the equilibrium on-path behavior. The equilibrium off-path behavior and the buyer's beliefs are derived similarly to those in Appendix A (we omit the details).

In equilibrium, the buyer proposes two types of menus. Let $k = 1, \ldots, m$ denote the number of units on the table. A menu $\mathcal{M}$ of the first type takes the form $\mathcal{M} = \{(k,p), (k', k'c)\}$ for some $k' = 1, \ldots, k$ and $p < kc$. Thus, the first offer (k,p) is for all the remaining units and can be accepted only by the low types. The second offer $(k', k'c)$ is for a fraction of the remaining units and the price is such that the high types break even.

The second type of menus contains only one offer of the form (k,p) with $p \leqslant kc$ (i.e., the buyer proposes to purchase all the remaining units). Although a menu $\mathcal{M}$ of the second type contains only one offer, we find it convenient to denote it as $\mathcal{M} = \{(k,p), (0,0)\}$. In this case, we also say that the seller accepts the offer $(0,0)$ if he rejects the offer (k,p).

For every $k = 1, \ldots, m$ and every $k' = 0, \ldots, k-1$, we define the function $P_{\mathfrak{m}}(k, k', \cdot) : [0,1] \to \mathbb{R}_+$. The function $P_{\mathfrak{m}}(k, k', \cdot)$ is weakly increasing, left-continuous, and satisfies $P_{\mathfrak{m}}(k, k', q) = kc$ for every $q > \hat{q}$. Suppose that the buyer offers the menu $\mathcal{M} = \{(k,p), (k', k'c)\}$. In equilibrium, type q accepts the first offer if $p \geqslant P_{\mathfrak{m}}(k, k', q)$, and the second offer if $p < P_{\mathfrak{m}}(k, k', q)$.

As in the equilibrium of the benchmark model, the set of types who have not sold all their units is of the form $[q, 1]$ for some $q \in [0, 1)$. Thus, we continue to use (k, q) to denote an arbitrary state of the economy (where k denotes the number of units on the table). We also let $W_{\mathfrak{m}}(k, q)$ denote the buyer's expected payoff when the state is (k, q). Clearly, for every (k,q) , $W_{\mathfrak{m}}(k,q)$ satisfies

$$W_{\mathfrak{m}}(k,q) = \max_{q' \geqslant q, k'=0,\ldots,k-1} \int_q^{q'} \left[(\alpha_k + \ldots + \alpha_1) v(s) - P(k,k',q')\right] ds + \int_{q'}^{1} \left[(\alpha_k + \ldots + \alpha_{k-k'+1}) v(s) - k'c\right] ds + \delta W(k-k',q'), \tag{51}$$

where we set $\alpha_k + \ldots + \alpha_{k-k'+1} = 0$ if $k' = 0$.

The existence of the functions $P_{\mathfrak{m}}(k,k',\cdot)$ and $W_{\mathfrak{m}}(k,\cdot)$ satisfying equation (51) and the other equilibrium conditions is established along the lines of Appendix A.

Also, we can repeat the analysis in Section 6 and Appendix D to develop an algorithm that pins down the limiting equilibrium outcome as the bargaining frictions vanish (and the discount factor converges to one). It follows from equation (51) that the algorithm for the model with menus shares several similarities with the algorithm of the benchmark model and delivers the same limiting equilibrium outcome (again, we omit the details).

Finally, we proceed as in Appendix B to show that generically all stationary equilibria (satisfying our refinement that all the high types always agree on their decisions) of the model with arbitrary menus are outcome equivalent to the equilibrium above (with two offers). In particular, we first establish outcome equivalence for equilibria in which the seller's reservation price functions are increasing. Clearly, there is a unique equilibrium outcome when the state is (k,q) and q is sufficiently close to $\hat{q}$. Then we assume that there is a unique outcome when the state is (k,q) and then establish the same result for some state (k,q') with $q' < q$ until we reach the state $(k,0)$. We then relax the monotonicity assumption. An argument similar to the final part of the proof of Proposition 2 (Appendix B) shows that it is possible to relabel the types in such a way that for any stationary equilibrium there exists an outcome equivalent equilibrium with increasing reservation price functions.