Competitive on-the-job search*

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January 14, 2015

Abstract

The paper proposes a model of on-the-job search and industry dynamics in which search is directed. Firms permanently differ in productivity levels, their production function features constant returns to scale, and search costs are convex in search intensity. Wages are determined in a competitive manner, as firms advertise wage contracts (expected discounted incomes) so as to balance wage costs and search costs (queue length). An important assumption is that a firm is able to sort out its coordination problems with its employees in such a way that the on-the-job search behavior of workers maximizes the match surplus. Our model has several novel features. First, it is close in spirit to the competitive model, with a tractable and unique equilibrium, and is therefore useful for empirical testing. Second, the resulting equilibrium gives rise to an efficient allocation of resources. Third, the equilibrium is characterized by a job ladder, where unemployed workers apply to low-productivity firms offering low wages, and then gradually move on to more productive, higher-paying firms. Finally, the equilibrium offers different implications for the dynamics of job-to-job transitions than existing models of random search.

Key words: Directed search, competitive search equilibrium, firm dynamics, efficiency.
JEL codes: J60, C62

*A simpler version of the model with preliminary results was published in the note "Job-to-job movements in a simple search model", *AER papers and proceedings* 2010.

1 Introduction

There is broad evidence that job-to-job transitions play an important role in the labour market. The last decade has witnessed a growing literature modeling and estimating firm dynamics, job-to-job and unemployment-to-employment transitions, as well as wage distributions. This literature is broadly coherent with the the DMP (Diamond, Mortensen, Pissarides) search paradigm. Most of these contributions assume that search is random, and firms can not use wages as an instrument for fast recruiting.

In the present paper we set up and analyze a model of directed search with identical workers and large and (ex post) heterogeneous firms. In competitive search equilibrium, firms advertise wages and workers choose optimally which firms to apply to. On-the-job search is efficient, in the sense that workers search behaviour maximizes the joint income of workers and firms. We obtain a tractable model of on-the-job search, which delivers an efficient allocation of resources, and in which on-the-job search is an optimal response to search frictions and heterogeneous firms.

The equilibrium leans towards a job ladder, where unemployed workers search for low-productivity firms offering low wages, and then gradually advances to higher paid jobs. Productive firms pay higher wages and grow faster than less productive firms. The economic logic underlying the job ladder is that workers with low current wages are the most eager to get a job quickly, while the productive firms are the most eager to get workers quickly. If one side of the market matches quickly, it follows from the matching function that the other side matches slowly. Hence, efficiency requires that the most eager workers (the unemployed) search for the least eager firms (low-productivity firms), while the least eager workers (employed in the second most productive firm) search for the most eager firms (the most productive ones).

In the first part of the paper, we only allow for a finite number of firm types. In the resulting equilibrium, worker search is not completely ordered, in the sense that workers employed in the same firms may search for firms with different productivities. Furthermore, when the number of firm types increases, no clear convergence pattern emerges. We therefore extend the model and allow for a continuum of firm types. In the resulting equilibrium there is a one-to-one correspondence between the productivity of a worker's current employer and of the productivity of the firm she is searching for. Unemployed workers randomize over an interval of low productivity firms.

Our proof of existence of an equilibrium for continuous types relies on Schauders fixed point theorem. We do not employ the commonly used version found in Stokey and Lucas (1989), which requires equicontinuity. Instead we define equilibrium in terms of distribution functions, the distribution of workers over firms and the distribution of applications. Since Helly's selection theorem ensures that distribution functions form a compact and convex set (in the set of all functions on the unit interval), we can apply Schauder's fixed theorem in it's general topological formulation. This approach may be of broader interest, since it circumvents the technical complication of equicontinuity and makes Schauders fixed point theorem applicable to a larger class of equilibrium models.

The paper also delivers a method for mapping, continuously, the set of all distribution functions (which may be discontinuous) into a set of continuous distribution functions. As equilibrium in search models (and other models as well) often can be characterized by distribution functions, we believe that our methods may be useful in many applications.

In an extension section we explore in more details the role of convexity of the search costs for job-to-job transitions. We assume that search costs are linear, and introduce instead convex *hiring costs*. Hiring costs accrue conditional on firms' hiring, and are assumed to be convex in the intensity of the flow of new workers. We show that in the resulting equilibrium there is no wage dispersion, as the optimal wage is independent of firm productivity. Furthermore, there is no on-the-job search. We thus conclude that with directed search, convex vacancy costs are essential to obtain on-the-job search.

Our model is consistent with some stylized facts regarding job-to-job movements: 1) productivity differences across firms are large and persistent and different productivity level across firms coexist in the labor market, 2) on-the-job search is prevalent and worker flows between firms are large, and 3) more productive firms are larger and pay higher wages than less productive firms. In addition, our model gives rise to empirical predictions that differ substantially from those of existing models of on-the-job search. For instance, the Burdett-Mortensen (1998) model predicts a weak relationship between the wage before the job switch and the distribution of wages after the job switch. More specifically, the wage after successful on-the-job search is a draw from the wage offer distribution truncated at the wage in the previous job. According to our model, workers employed in firms offering relatively high wages (i.e., have high productivity) search for jobs that offer strictly higher wages than do workers employed in firms offering lower wages initially, and thus different workers face different distributions.

There is a substantial literature on job-to-job movements. First, Davis and Haltiwanger (1999) show that job-to-job flows are huge. Lentz and Mortensen (2005) find that reallocation of workers from low- to high productivity firms are important for economic growth. Postel-Vinay and Robin (2002) were among the first to structurally estimating a search model with on-the-job search. Other recent papers on on-the-job search include Lentz and Mortensen (2008, 2012), Bagger and Lentz (2014), Bagger et al (2013), Lise and Robin (2013), and Lamadon et al (2013). We deliver an alternative framework, based on directed search, suitable for empirical analysis.

There exist papers with directed on-the-job search. Moen and Rosen (2004) analyse human capital investments in the presence of on-the-job search. Shi (2009) and Menzio and Shi (2010, 2011) study directed on-the-job search. These model have similar structure to our theory in terms of employment contracts, but there are fundamental differences. Most importantly, in these papers firms are identical and differences in productivity are match specific. Furthermore, as firms are identical in the search phase, the equilibrium in Menzio and Shi is block recursive, essentially implying that a zero profit condition applies in all submarkets.

Our key motivating factor is that productivity difference across firms are large and persistent (Lentz and Mortensen, 2008). Hence, the model should capture this fact. In our model firms productivity are indeed permanently heterogeneous. After sinking a cost K, firms draw productivity y_i and will keep this productivity until they exit the market. Hence a zero profit condition only applies *ex ante*, not when firms are searching for workers. This dramatically changes the nature of equilibrium and breaks the block recursivity of Monzio and Shi (2011). Delacroix and Shi (2006) analyse an urn-ball model of the labour market with on-the-job search, and show that equilibrium is characterized by a job ladder. Again it is assumed that firms are identical. Furthermore, on-the-job search is inefficient, and solely caused by imperfect contracting between workers and firms. Our modeling of firms are similar to that of Kaas and Kircher (2013), but they do not allow for on-the-job search. Finally, our paper is related to Mortensen and Wright (2002), who analyze competitive search equilibrium when workers differ in income during unemployment.

The paper proceeds as flows. Section 2 presents the model with a discrete firm type space, while we allow for a continuous type space in section 3. In section 4 we introduce linear search cost and convex hiring costs. Section 5 discusses the empirical implications of our model, while the last section concludes.

2 Model with finite number of firm types

The labor market is populated by a measure 1 of identical, risk neutral and infinitely lived workers with discount rate r. Workers can search on and off the job at no cost.[1]

Ex ante identical and risk neutral firms enter at cost $K > 0$. Conditional upon entry, the firm learns its productivity $y \in \{y_1, y_2, ..., y_n\}$, with $y_1 < y_2 < ... < y_n$. The probability of obtaining productivity y_i is denoted by α_i with $\Sigma_i \alpha_i = 1$. The productivity of a firm is fixed throughout its life. Unemployed workers have access to an income flow $y_0 < y_1$. Firms discount the future at the rate r, and die at rate δ. In addition, workers separate from firms at an exogenous rate s.

Firms post vacancies and wages to maximize expected profits. Vacancy costs $c(v)$ are convex in the number of vacancies posted. Unless otherwise stated, we assume that $c'(0) = c(0) = 0$.[2]

As will be clear below, the search market may endogenously separate into submarkets, consisting of a set of workers and firms with vacancies searching for each other. Firms can not simultaneously search in more than one sub-market. In each submarket, the flow of matches is determined by a constant-returns-to scale matching function. If a measure u of workers search for a measure v of vacancies, the flow of matches is $x(u, v)$. Let $\theta = v/u$, and define $p(\theta) = x(1, \theta)$ and $q(\theta) = x(1/\theta, 1)$. Finally, let $\eta = |q'(\theta)\theta/q|$ denote the absolute value of the elasticity of q with respect to θ. In order to ensure that a firm's profit as a function of the advertised wage has a unique maximum (for each

[1] We relax this assumption in Section 3.

[2] Convex hiring costs are often assumed in search models, see Bertola and Cabalero (1994) and Bertola and Garibaldi (2001). Convex hiring costs may be rationalized by decreasing returns to scale in the firm's recruitment department. Convex hiring costs can be seen as a generalization of Burdet Mortensen (1998), where the number of vacancies is exogenously fixed. Analogously, the search costs of workers are usually assumed to be convex (Pissarides 2000). Finally, our assumption of convex hiring costs have empirical support, see Yashiv (2000a,b).

type), we assume that $\eta(\theta)$ is non-decreasing in θ. It can be shown that this is not necessary for the equilibrium to exist.

We assume that a worker's on-the-job search behaviour is chosen so as to maximize the joint income of the worker and his employee. Thus, the worker, when choosing between different firms to search for, internalizes the loss for the employer if he quits. This assumption is convenient, and first made in Moen and Rosen (2004) and later in Shi (2009) and in Menzio and Shi (2010,2011). There are various wage contracts that implement this behavior, see Moen and Rosen (2004). In particular, the worker may buy the job from the firm, in which case the worker is the residual claimant. Or the wage contract may include a quit fee equal to the capital loss for the firm associated with losing the worker. In principle the worker and the firm may also contract directly upon the wages the worker should target for when doing on the-job-search.

Efficient on-the-job search implies that the wages paid to the worker in the current job do not influence her on-the-job search behavior. It follows that a worker employed in a firm of type i will never search for a job in another firm of type $j \leq i$. Such jobs cannot profitably offer a wage that exceeds the productivity in the current firm.

Firms advertise and workers search for contracts. For any given contract σ, let $W(\sigma)$ denote the associated net present income of the worker that obtains the job. As will be clear below, $W(\sigma)$ is a rather complicated object, as it includes the expected income to the worker from on-the-job search, which again depends on wages advertised by more productive firms and the probability rates of getting these jobs.

Consider an economy where a countable set of NPV wages $W_1, ..., W_l, ...$ are advertised, each by a strictly positive measure of firms. Let $\theta_1, ..., \theta_l, ...$ denote the associated vector of labor market tightness. The set of pairs (θ_l, W_l) is denoted by Ω, and is endogenously determined in equilibrium.

Let M_i $i = 0, 1, ..., n$ denote the joint expected discounted income flow of a worker and a job in a firm of type i, where the gains from on-the-job search is included. Since on-the-job search is efficient, it follows that M_i is given by

$$rM_i = y_i + (s + \delta)(M_0 - M_i) + \max_{l \in I} p(\theta_l)[W_l - M_i] \tag{1}$$

where I is the set of submarkets. The first term is the flow production value created on the job. The second term captures the expected capital loss due to job separation, which happens at rate $s + \delta$, and reduces the joint income to M_0 (since the firm then earns zero on this match). The last term shows the expected joint gain from on-the-job search. Since the current wage is a pure transfer from the employer to the worker, it does not appear in the expression.

From (1) it follows that the optimal search behaviour of a worker depends on her current position, as this influences M_i. Hence our model is characterized with what we refer to as endogenous worker heterogeneity, captured by the workers' output y_i in their current job. We refer to a worker that currently works in a firm of type i as of type $i-$searching worker or just *type i worker* (note that all worker " types" are equally productive, the difference in output reflects differences in the productivity of the current employer).

The indifference curve of a worker of type i shows combinations of θ and W that gives a joint income equal to M_i. We can represent this as $\theta_i = f_i(W; M)$.[3] It follows that f_i is defined implicitly by the equation

$$rM_i = y_i + (s + \delta)(M_0 - M_i) + p(f_i(W, M))[W - M_i] \tag{2}$$

where M_i is the equilibrium joint income in firm i. It follows that for $M_i < W_i$

$$f_i(W; M) = p^{-1}\left(\frac{(r + s + \delta)M_i - y_i - (s + \delta)M_0}{W - M_i}\right) \tag{3}$$

The indifference curve is defined for all W, not only the values advertised in equilibrium. Define

$$f(W; M) = \min_{i \in \{0,1,\ldots,n\}} f_i(W; M) \tag{4}$$

The function $f(W; M)$ is thus the lower envelope of the set of functions $f_i(W; M)$. In equilibrium, $f(W; M)$ shows the relationship between the wage advertised and the labor market tightness in a submarket. Suppose that for a given W, the minimum in (4) is obtained for worker type i'. This worker type will then flow into the market up to the point where $\theta = f_{i'}(W; M)$. At this low labor market tightness, no other worker types want to enter this submarket. The labor market tightness is thus given by $f_{i'}(W; M)$, and only workers of type i' enter the market.

[3]Strictly speaking, f_i only depends on M_i and M_0, but we write it as a function of the vector M for convenience.

Then we turn to the firms. It follows that at any point in time, a firm of type j maximizes the value of search given by[4]

$$\pi_j = -c(v) + v_j q(\theta)[M_j - W_j]. \tag{5}$$

where W_j is the wages paid by the firm. The first part is the flow cost of posting vacancies, while the second part is the gain from search. The firm's maximization problem thus reads

$$\max_{v,W} \{-c(v) + vq(\theta)[M_j - W]\} \quad \text{s.t.} \quad \theta = f(W, M)$$

Denote the associated maximum profit flow by π_j^*. The expected profit of a firm entering the market as a type j firm is thus

$$\Pi_j = \frac{\pi_j^*}{r+\delta} \tag{6}$$

Denote the set of wages that solves j's maximization problem by $W_j(M)$. Below we show that W_j has a finite number of elements. Denote the optimal measure of vacancies by $v_j(M)$. Note that the net gain from search is the same for all $W \in W_j(M)$, the number of vacancies posted by firm j is the same for all advertised wages $W \in W_j(M)$.

Define $N = (N_0, N_1, ..., N_j, ...)$, where N_j denotes the measure of workers employed in type j firms. Let the vector $\widetilde{\tau}_j = (\tau_{j1}, \tau_{j2}, ...)$ denote the distribution of vacancies posted by firms of type j over the different submarkets. Similarly, let $\widetilde{\kappa}_j = (\kappa_{j1}, \kappa_{j2}, ...)$ denote the distribution of searching type j workers over the different submarkets. Finally, let k denote the total number of firms in the market. In steady state, inflow of workers into type j firms has to be equal to outflow, hence

$$k \sum_l \alpha_j v_j \widetilde{\tau}_{jl} q(\theta_l) = N_j[s + \delta + \sum_l p_{jl}(\theta_l)\widetilde{\kappa}_{jl}] \tag{7}$$

for all j. For unemployed workers, the corresponding inflow-outflow equation reads

$$(s+\delta)(1-N_0) = \sum_l p_{0l}(\theta_l)\widetilde{\kappa}_{0l} N_0 \tag{8}$$

The labor market tightness θ_l in market l is given by

$$\theta_k = k \frac{\sum_j \alpha_j \widetilde{\tau}_{jl} v_j}{\sum_j \widetilde{\kappa}_{jl} N_j} \tag{9}$$

We are now in a position to define the general equilibrium.

[4]At any point in time, the firm decides on the number of vacancies and the wages attached to them. This only influences profits through future hirings, and is independent of the stock of existing workers.

Definition 2.1 *General equilibrium is defined as a vector of asset values M^*,wages W^*, employment stocks N^*, labor market tightness θ^*, distributions of searching workers $\widetilde{\kappa}_j^*$, distributions of vacancies $\widetilde{\tau}_j^*$, and a number of firms k such that*

1. *Profit maximization: i) $W^* = \cup_{j=1}^{n} W_j(M^*)$ ii) $v_j = v_j(M^*)$ iii) If $\tau_{jl} > 0$, then $W_l^* \in W_j(M^*)$.*
2. *Optimal worker search: $rM_i^* \geq y_i + (s+\delta)(M_0^* - M_i^*) + p(\theta_l)[W_l^* - M_i^*]$, with equality if $\kappa_{il} > 0$.*
3. *Optimal entry: The expected profit of entering the market is equal to the entry cost K, i.e.,*

$$E\Pi_j = K$$

4. *Aggregate consistency: Equations (7), (8) and (9) are satisfied.*

In addition we make the following *equilibrium refinement*: if more than one allocation satisfies the equilibrium conditions, the market picks the equilibrium where aggregate output is highest.[5] This can be rationalized by assuming that a market maker sets up the markets (as in Moen 1997).

2.1 Characterizing equilibrium

Before we prove existence of equilibrium, we will derive some properties of the equilibrium (assuming that it exists). First we will derive properties for $f(W, M)$.

Consider an arbitrary set of submarkets Ω, and let M denote the corresponding vector of asset values defined by (1). Let W^s denote the highest wage in Ω. By construction, M exists and is unique. Furthermore, in the appendix we show that $y_0/r \leq M_0 < M_1 < ... < M_n$ and $(r+s+\delta)M_i - y_i$ is decreasing in i. If we restrict our attention to the case were $W^s < y_n$, it follows that $M_n = \frac{y_n + sM_0}{r+s}$.

If $M_j \geq W^s$, workers employed in firms of type j or higher do not search. In this case $(r+s+\delta)M_j = y_j + (s+\delta)M_0$, and it follows from (2) that$f_i(W, M) = 0$ for $W > M_j$. It follows that $f(W, M)$ is discontinuous at $W = M_j$. If $M_i < W^s$, type i-searching workers will search, hence $f_i(W, M) = f(W, M)$ for some W (at least the wages they actually search for). In this case $(r+s+\delta)M_i > y_i + (s+\delta)M_0$, and it follows from (3) that $f_i(W; M) > 0$ for all $W > M_i$.

[5] We cannot rule out that the first order conditions of the planner's maximization problem has more than one solution, in which case the equilibrium of the model may have more than one solution as well.

Lemma 2.1 *For any vector Ω and associated vector M defined by (1), the following holds*

a) Single crossing: For any i, j, $i < j$, $M_j < W^s$, the equation $f_i(W; M) = f_j(W; M)$ has exactly one solution, and at this point

$$|\frac{df_i(W;M)}{dW}| < |\frac{df_j(W;M)}{dW}|$$

b) Suppose $M_{n'-1} < W^s < M_{n'}$. For $i \in \{0, 1, 2..., n'-1\}$, define W^i as the solution to $f_i(W; M) = f_{i+1}(W; M)$. Then it follows that

i) If $W \in (W^i, W^{i+1})$, $i \leq n-1$, then $f(W; M) = f_i(W; M)$

ii) If $W > W^{n-1}$ then $f(W; M) = f_{n'+1}(W; M)$

ii) If $W > M_{n'}$ then $f(W; M) = f_{n'+1}(W; M) = 0$,

c) $f(\theta)$ is discontinuous at $M_{n'+1}$, as $\lim_{W \to M^-_{n'+1}} f(W; M) > 0$ while $\lim_{W \to M^+_{n'+1}} f(W, M) = 0$.

d) For all W in $(M_0, M_{n'})$, f is strictly decreasing in W, continuous, and continuously differentiable except at the points $(W^1, W^2, ..., W^{n'})$. At these points, $\lim_{W \to W^{i-}} |df(W; M)/dW| < \lim_{W \to W^{i+}} |df(W; M)/dW|$.

Property a) follows directly from the fact that workers employed in more productive firms are more willing to trade off a high job finding rate for a high wage than are workers employed in less productive firms. Property b) follows more or less directly from property a), and states that there are line segments $[W^i, W^{i+1}]$ such that firms advertising a wage $W \in [W^i, W^{i+1}]$ attracts workers hired in firms of type i only. In addition, if no wages above M^n are advertised in equilibrium, if a submarket with wages above M^n did open up it would obtain a labor market tightness of zero. Furthermore, at M_n is the function $f(W; M)$ discontinuous in W. Properties c) and d) state that except for this, $f(W; M)$ is continuous everywhere and differentiable everywhere except at the intersection points $(W^1, W^2, ..., W^{n'})$.

In order to characterize the equilibrium of the market, the following result is useful (recall that $\eta(\theta) = -q'(\theta)\theta/q(\theta)$):

Lemma 2.2 *a) In any non-empty submarket there is exactly one type of firms, say j, and one type of workers, say i (working in a type i firm). The equilibrium wage W^*_{ij} in this submarket is uniquely* *determined as*

$$\frac{\eta(\theta_l^*)}{1-\eta(\theta_l^*)} = \frac{W_{ij}^* - M_i^*}{M_j^* - W_{ij}^*} \tag{10}$$

where θ_l^* is the labor market tightness in that submarket.

The lemma simplifies characterization of equilibrium. Each worker-firm combination leads to at most one operating submarket, and each submarket can be attributed to exactly one worker-firm combination. Hence we can index by ij a submarket in which workers currently employed in firms of type i and firms of type j search for each-other. Furthermore, the vectors of distributions of searching workers on submarkets, $\widetilde{\kappa_j}$, can be described as an $n \times n$ matrix κ, where where κ_{ij} gives the fraction of workers employed in firms of type i that search in the ij-submarket. Note that $\kappa_{ij} = 0$ for all $j \leq i$. Similarly, the vector of distributions $\widetilde{\tau}_j$ of vacancies on submarkets as an $n \times n$ matrix τ, where τ_{ij} denote the fraction of firms of type j searching in the ij-submarket. Again $\tau_{ij} = 0$ if $i \geq j$. It follows trivially that the equilibrium satisfies the following conditions (with $M_{ij}^* = M_j^* - M_i^*$)

$$rM_i^* = y_i - (s+\delta)M_{0i}^* + \max_j p(\theta_{ij}^*)\eta M_{ij}^* \tag{11}$$

$$W_{ij}^* = M_i^* + \eta M_{ij}^* \qquad \text{for all } i,j|\kappa_{ij} > 0 \tag{12}$$

$$c'(v_j) = (1-\eta)M_{ij}^* q(\theta_{ij}^*) \qquad \text{for all } i,j|\kappa_{ij} > 0 \tag{13}$$

The first condition defines joint income and ensures efficient on-the-job search. The second equation defines the traditional efficient rent sharing in competitive search equilibrium, the Mortensen-Hosios condition. The third condition equates the marginal cost of vacancy posting to its expected benefit. Since the value of search is the same in all submarkets a firm operates, v_j is independent of i.

Remark 2.1 *Note that all firms with productivity strictly higher than y_0 are active in equilibrium. Since workers search equally well on and off jobs, the joint income of a worker and a firm of the lowest type, M_1, is then strictly greater than M_0. Since, by assumption $c'(0) = c(0) = 0$, firms of type 1 offers a wage strictly greater than M_0 and attracts workers.*

We are now ready to show that the equilibrium exists. In the standard competitive search equilibrium, a zero profit condition applies to all submarkets, and make the model block recursive (see

Menzio and Shi 2010). In this model, the zero profit condition only holds *ex ante*, not for each firm type separately.

Proposition 2.1 *The equilibrium exists.*

In Moen (1997) and Shimer (1996), it is shown that with no on-the-job search, wages are set so that search externalities are internalized. As a result, the equilibrium allocation is socially efficient.

We want to show that this result carries over to our model with on-the-job search. We say that the equilibrium allocation is efficient if it maximizes the net present income of the economy along the steady state path, where the net present income is defined as

$$W = \int_0^\infty [\sum_{j=0}^{n} N_j y_j - \sum_{j=1}^{n} \alpha_j k c(v_j) - aK] e^{-rt} dt$$

Proposition 2.2 *The equilibrium is efficient in the sense that it maximizes W given the law of motions of $N_0, N_1, ..., N_n$.*

Our next proposition characterizes wage distributions and search behavior of workers and firms

Proposition 2.3 *Maximum separation:*

a) Let $k < l$. Then workers in a firm of type l always search for jobs with strictly higher wages than workers employed in firms of type k. Firms of type l always offer a strictly higher wage than firms of type k.

b) Let I_k denote the set of worker types searching for firms of type k. Consider I_k and I_l, $k > l$. Then all elements in I_k are greater than or equal to all elements in I_l. Hence I_k and I_l have at most one common element.

From a) it follows that high-type firms grow quicker than low-type firms, even if they search for the same worker types. Thus, firms of different productivities may offer different wages and attract workers at different speeds, as an efficient response to search frictions. Furthermore, for all firm types, the hiring flow is constant, while the separation flow is proportional to size. Thus, even the most productive firms don't grow indefinitely, conditional upon survival the size converges to a steady state level.

From b) it follows that the market, to the largest extent possible, separates workers and firms so that the low-type workers search for the low-type firms. Note the similarity with the non-assortative matching results in the search literature (Shimer and Smith, 2001, Eeckout and Kirkcher, 2008). If the production technology is linear in the productivities of the worker and the firm, it is optimal that the high-type firms match with the low-type workers and vice versa. Similarly, workers in a firm with a high current productivity search for vacancies with high productivity, and vice versa.

From an efficiency point of view, the result can be understood by recalling that quick vacancy filling requires long worker queues, so that workers find jobs slowly. It is therefore optimal that the most "patient"workers, i.e., the workers employed in firms with high productivity, search for the most "impatient"firms, the firms with the highest productivity. Similarly, the most impatient workers (the unemployed workers) search for the most patient firms (with the lowest productivity).

3 Continuum of types

With a discrete distribution of firm types, our model does not give rise to a pure job ladder. Proposition 2.3 gives us some ordering of the search behaviour of workers and firms at different rings of the job ladder, but the ordering is not complete. Workers employed by firms with the same productivity initially, may end up in firms with different productivities after successful on-the-job search. Likewise, firms with the same productivity may attract and hire workers employed in firms with different productivities. Furthermore, as the number of firm types increases, no clear pattern of convergence of search strategies emerge. This calls for a model with a continuum of firm types.

In this subsection we therefore analyze the equilibrium of the model with a continuum of types. In this case there is a one-to-one mapping between the productivity of a worker's current employer and future employer. Our main objective is to show that such an equilibrium exists.

To this end, let $G(y)$ denote the cumulative distribution function of a continuous distribution on the set $Y \equiv [y^{\min}, y^{\max}]$, where $y^{\min} > y_0$, so that the lowest firm productivity is strictly larger than unemployed income. The associated density is denoted $g(y)$. We will define the equilibrium in terms of *cumulative distribution functions.*

To simplify the proofs we assume that each firm advertises one vacancy, hence the total measure of vacancies in the economy is equal to the measure of firms. While the number of firms is endogenous, the idea of a fixed measure of vacancy per firm is an extreme form of convexity, and it is thus coherent with the model of Section 2. Note that this does not imply that firms will have the same size; more productive firms will set a higher wage, attract workers more quickly, and hence grow more quickly than less productive firms.

In order to avoid technical issues at the top of the distribution, we assume that there is an (arbitrarily small) cost of worker search. We denote the cost by ε. For notational simplicity we assume that the firm incur the cost (for instance due to lower worker effort), and that y measures output net of search costs. A firm of type y where the worker does not search thus has output flow $y + \varepsilon$, while it has an output flow of y if the employees search.[6]

Suppose the market consists of a continuum of submarkets $(\theta(y), W(y))$, where as above i is an index. Since there is a cost associated with worker search, workers in firms with productivity above a certain threshold y^s will not search. We write the expected joint income of a worker and a job in a firm as

$$
\begin{aligned}
(r+s+\delta)M(y) &= y + \max_{y\in Y} p(\theta(y))\,[W(y) - M(y)] \quad \text{for } y < y^s && (14)\\
M(y) &= \frac{y+\varepsilon}{r+s+\delta} \qquad\qquad\qquad \text{for } y \geq y^s && (15)
\end{aligned}
$$

At the threshold y^s, the firm is indifferent between searching and not searching. Hence, the flow cost is equal to the flow gain from search. It follows that y^s is implicitly defined by the equation

$$
\varepsilon = \max_{y} p(\theta(y))(W(y) - M(y^s)) \tag{16}
$$

As above, define $\theta = f(W; M(y))$ as a family of indifference curves of searching workers of type y. Define $\mathbf{f}(W) \equiv \min_y f((W; M(y))$. As f and M are continuous functions, this minimum problem is well defined.

Firms set wages so as to maximize the profit flow. As in the finite case, the firms thus maximize

$$
\pi = q(\theta)[M(y) - W] \quad \text{S.T.} \quad \theta = \mathbf{f}(W)
$$

[6] Delacroix and Shi (2006) make a similar assumption.

For any y, let $y(W)$ denote the wage(s) that solves the maximization problem, and $z(y)$ the worker types that are attracted to the firm. Hence $z(y)$ solves

$$\mathbf{f}(W(y)) = f(W(y, M(z(y)))$$

It is straight-forward to show that $z(y)$ is single-valued for all y and continuous for all $y \in [y^{\min}, y^{\max}]$ except at one point y^u. The value y^u is defined as the highest productivity a firm will have and still attract unemployed workers. Thus, $z(y) = y_0$ for all $y \leq y^u$ while $z(y) > y^{\min}$ for all $y > y^u$ (with $\lim_{y \to y^{u}+} z(y) = y^{\min}$). As will be clear below, it follows by construction that $z(y)$ is differentiable on $(y^{\min}, y^{\max})$ except at the point y^u. Furthermore, $z(y)$ is strictly increasing in y and hence has an inverse on $(y^u, z(y^{\max}))$. Finally, $z(y^{\max}) = y^s$.

It follows easily that the optimal sharing rule still applies,

$$(r + s + \delta)M(y) = y + \eta p(\theta(z^{-1}(y))[M(z^{-1}(y)) - M(y)] \tag{17}$$

This optimal sharing rule implies that given the worker type y that firms attract, the wage is optimally set. Furthermore, from the envelope theorem it follows that

$$M'(y) = \frac{1}{r + s + \delta + p(\theta(z^{-1}(y))} \tag{18}$$

Note that (18) is a necessary condition for efficient on-the-job search from the worker side. From the sharing rule (14), W is strictly increasing in y, and hence that θ is strictly decreasing in y. The single-crossing property then ensures that this is also a sufficient condition for maximum. Hence it follows that $f(W(y), M(z^{-1}(y)) = \mathbf{f}(W(y))$.

Let $N(y)$ denote the cumulative distribution of workers on firms (including unemployment) in the economy. In other words, $N(y)$ is the fraction of workers either unemployed or employed in firms with productivity at most y. For notational convenience we denote the fraction of unemployed by u as before.

Let $X(y)$ denote the measure of workers searching for jobs with productivity at most y. Unemployed workers randomize over which firms to search for on the interval $[y^{\min}, y^u]$. For $y^u < y < y^{\max}$, it follows that $X(y)$ can be written as

$$X(y) = u + \int_{y^u}^{z(y)} n(y)dy$$

Denote the associated density by $x(y)$ Let $\theta(y)$ denote the labor market tightness in the market in which firms of type y recruits. Note that the workers in this submarket is of type $z(y)$. It follows that

$$\theta(y) = \frac{kg(y)}{x(y)} \tag{19}$$

Finally, firms enter up to the point where the net present value of expected profits equals the entry cost K,

$$\int_{y^{\min}}^{y^{\max}} \frac{\pi(y)}{r+s+\delta} dy = K \tag{20}$$

Definition 3.1 *The equilibrium of the model is two distribution functions $X(y)$ and $N(y)$ with densities $x(y)$ and $n(y)$, a wage distribution $W(y)$, a labor market tightness distribution $\theta(y)$, a distribution of joint incomes $M(y)$, a search function $z(y)$, and numbers y^u, y^s and k such that*

1. *$M(y)$ satisfies (14)*

2. *Optimal wages: $W(y)$ maximizes the profit flow $q(\theta)(M(y) - W(y))$ subject to $\theta = \mathbf{f}(W)$,*

3. *Optimal search: $\mathbf{f}(W(y)) = f(W(y), M(z(y)))$ for $y \leq y^s$.*

4. *For $y \geq y^s$, employees do not search. The threshold y^s is implicitly defined by (16).*

5. *Zero profit: The zero profit condition (20) is satisfied.*

6. *Labor market consistency: $\theta(y) = \frac{kg(y)}{x(y)}$.*

7. *Inflow equals outflow in all markets:*

$$\begin{aligned} kgq(\theta(y)) &= [s+\delta]n(y) \quad \textit{for } y > y^s \qquad (21) \\ kgq(\theta(y)) &= [s+\delta+p(\theta(z^{-1}(y))n]n(y) \quad \textit{for } y \in [y^{\min}, y^s] \qquad (22) \\ (1-u)(s+\delta) &= \int_{y_{\min}}^{y^u} x(y)p(\theta(y))dy \end{aligned}$$

8. Aggregate consistency:

$$\int_{y^{\min}}^{y^u} x(y)dy = u \tag{23}$$

$$\int_{y^u}^{z(y)} n(y)dy = u + X(y) \tag{24}$$

$$u + \int_{y^{\min}}^{y^{\max}} n(y)dy = 1 \tag{25}$$

Our goal is to define a mapping of which a fixed point is an equilibrium. Before entering that task, we point out two important properties of cumulative distribution functions which form the basis of our existence proof.

Property 1

Cumulative distributions are monotone functions, and thus by Lebesgue's Theorem (Royden Fitzpatrick p. 112) are differentiable almost everywhere. This implies that any cumulative distribution function gives rise to a probability density function (with potential mass points).

Property 2

The subset of cumulative distribution functions on a closed interval $[a, b]$*, is contained in the topological vector space of all functions (with the sup norm) on* $[a, b]$*. This subset is obviously convex. Furthermore, Helly's selection theorem (Helly 1912, Surhone et al., 2010) gives that this subset is compact.*

The first property ensures the existence of the distribution functions $x(y)$, $n(y)$ or $z'(y)$ almost everywhere (since $z(y)$ is a distribution).

The second property is important when applying Schauder's fixed point theorem. The theorem asserts that if K is a convex subset of a topological vector space V and Γ is a continuous mapping of K into itself so that $\Gamma(K)$ is contained in a compact subset of K, then T has a fixed point. See also Istratescu (2001), which provides an equivalent definition.

In many economic applications, the mapping is defined on the set of continuous functions, and this set is not compact. It is therefore usual to apply a variant of Schauder's fixed point theorem that does not require that $\Gamma(K)$ is contained in a compact subset of K, but instead requires that $\Gamma(K)$ is an equicontinuous family of functions , see Stokey and Lucas (1989), p 520. Showing equicontinuity of $\Gamma(K)$ is often cumbersome. In our case, the set of cumulative distributions do not form an equicontinuous family. However, the set of cumulative distributions have two redeeming properties that we recalled above. Therefore, the requirements of Schauder's fixed point theorem is trivially satisfied as long as Γ is continuous.

Below we will construct a mapping

$$\Gamma : (X(y), N(y), k, M_0, y^s) \to (\hat{X}(y), \hat{N}(y), \hat{k}, \hat{M}_0, \hat{y}^s),$$

where $X(y) \in CD[y^{\min}, y^{\max}]$, $N(y) \in CD[y_0, y^{\max}]$, $k \in [0, k^{\max}]$, $M_0 \in [0, M_{0,\max}]$ and $y^s \in [y^{\min}, y^{\max}]$.

The mapping will make sense for all pairs of distributions $X(y)$ and $N(y)$, but we will restrict the subset of pairs where $N(y) > X(y)$ (which also is a convex set), as the search technology is directed towards more productive firms.

Although Schauders's fixed point theorem does not require the elements in the domain to be continuous functions (although the mapping as such has to be continuous), it is convenient in the updating algorithm (which we will state below) to have continuity. We solve this technical challenge by mapping a given cumulative distribution to a continuous cumulative distribution satisfying a growth constraint. In technical terms the continuous cumulative distributions have a derivative bounded by a fixed positive number a. In other words, we construct a mapping $\Phi : CD[b, c]$ to $CCD[b, c]$, where $CCD[b, c]$ denotes the set of all continuous distributionson the interval $[b, c]$. As the mapping Φ will be part of the $\Gamma : (X(y), N(y), k, M_0, y^s) \to (\hat{X}(y), \hat{N}(y), \hat{k}, \hat{M}_0, \hat{y}^s)$, the map Φ itself needs to be continuous. This severely restricts the list of candidate maps. In particular, *any* mapping that leaves already continuous distributions unchanged, and only removes discontinuities for discontinuous distributions, is necessarily discontinuous. A proof of this result is given in the appendix.

Before we give the formal definition of Φ, it may be helpful to give the intuition behind the construction. Consider a cumulative distribution F defined on the interval $[b, c]$. Imagine that we start at b and construct a cumulative distribution that meets the growth constraint a, in the following way: whenever F grows faster than ax, we divide F into two parts, F_s^{aL}, a continuous function with derivative equal to a, and store the excess probability mass in a function T_F^a. That is, $F(y) =$

$F_s^{aL}(y) + T_F^a(y)$ for all y. As soon as the growth rate falls below a again, we start to redistribute the accumulated probability mass by letting F_s^{aL} have the maximal derivative a until the "tank" of excess probability mass is empty. This redistribution, which we may say is redistribution from the left, may give a mass point in c, but not elsewhere. Moreover, redistribution from the right by starting in c instead of b can be constructed in the same way. For expository purposes will we consider only redistribution from the left and introduce the notation F_s^{aL}. The corresponding results for F_s^{aH}, redistribution from the right, follows by symmetry. Note that in this case of redistribution from the right the continuous distribution may have a mass point at b.

To ease the notation let

$F(x^-) = lim_{t \to x^-} F(t)$ and $F_{\text{jump}}(x) = F(x) - lim_{t \to x^-} F(t)$.

Definition 3.2 *Let* $F \in CD[b,c]$ *and define* $T_F^{aL}(Y)$ *by*

i) $T_F^{aL}(b) = F(b)$

ii) if F *is differentiable at* x*, then*

$T_F^{aL\prime}(x) = F'(x) - a$ *if* $F'(x) - a > 0$ *or* $T_F^{aL}(x) > 0$*, and* $T_F^{aL\prime}(x) = 0$ *otherwise.*

iii) if F *not differentiable at* x*, then*

$T_F^{aL}(x) = T_F^{aL}(x^-) + F_{jump}(x)$

Note that at any point the difference $F(y) - T_F^{aL}(y)$ is by construction a continuous function obeying the growth constraint a.

Definition 3.3 *Let*

$$\Phi^{aL} : CD[b,c] \longrightarrow CCD[b,c],$$

where $CCD[b,c]$ *denotes the set of continuous distributions defined on the interval* $[b,c]$*, be given by* $\Phi^{aL}(F) = F_s^{aL} = F - T_F^{aL}$.

In order to use Schauder's fixed point theorem, we need continuity of Φ^{aL} (and Φ^{aH}) :

Proposition 3.1 *The mapping* $\Phi^{aL} : CD[b,c] \longrightarrow CCD[b,c]$ *is continuous.*

The proof of this proposition is given in the appendix. We will use the maps Φ^{aL} and Φ^{aH} to ensure that the distributions $X(y)$ and $N(y)$ are at desired form. In particular, that $X(y)$ be a distribution without mass points, and $N(y)$ has potentially a mass point in y_0, no probability mass between y_0 and $y^{\min}$ and is continuous elsewhere.

We are now ready to describe the mapping

$\Gamma : (X(y), N(y), k, M_0, y^s) \rightarrow (\hat{X}(y), \hat{N}(y), \hat{k}, \hat{M}_0, \hat{y}^s)$

1. We replace $X(y)$ by $X(y) = \Phi^{aL}(\Phi^{aH}(X(y)))$. Note that a redistribution from the left followed by a redistribution from the right gives a distribution without mass points (if $a \geq 1$).

   The replacement of $N(y)$ is done in two steps. First consider only the part of $N(y)$ defined on $[y^{\min}, y^{\max}]$, $N\text{emp}(y)$, and compute $\Phi^{aH}(N\text{emp}(y))$.

   Second we define $N(y) = \Phi^{aH}(N\text{emp}(y^{\min}))$ for all y in $[y^0, y^{\min}]$. With this replacement $N(y)$ has a potential mass point in y^0, no probability mass between y^0 and $y^{\min}$), is continuous for all y in $[y^0, y^{\max}]$ and $N(y) = \Phi^{aH}(N(y))$ for all y in $< y^{\min}, y^{\max}]$.

2. Note that $u = N(y_0)$. Let y^u be determined by $u = X(y^u)$. (The unemployed randomly search jobs in the interval $[y^{\min}, y^u]$.) Furthermore let $\theta(y) = \min(\theta_{\max}, kg(y)/x(y))$. Let $\theta(y) = \theta_{\max}$ when $x(y) = 0$. Note that θ is naturally bounded from above ($\theta_{\max}$) and below ($\theta_{\min}$). These bounds are explicitly given in the appendix.)

   Finally, determine $z(y)$ by the equality $X(y) = N(z(y))$ for $y > y^u$.

3. For $y > y^s$, $M(y) = (y + \varepsilon + (s+\delta)M_0)/(r+s+\delta)$. For $y \leq y^s$

$$M(y) = \frac{y^s + \varepsilon + (s+\delta)M_0}{r+s+\delta} - \int_y^{y^s} \frac{dy}{r+s+\delta+p(\theta(z^{-1}(y))}$$

4. Update M_0 as follows:

$$r\widehat{M_0} = y_0 + p(\theta(y^{\min})\eta(M(y^{\min}) - M_0).$$

5. Update of k. Calculate the profit flow $\pi(y) = q(\theta(y))\eta(M(y) - M(z(y))$ and the npv profit $\Pi(y) = \dfrac{\pi(y)}{r+\delta}$, and update k as follows:

$$\widehat{k} = \frac{E\Pi(y)}{K}k$$

6. Update of y^s. Calculate $\Delta = q(\theta(y^{\max}))\eta(M(y^{\max}) - M(y^s))$. Update y^s as follows

$$\widehat{y^s} = y^s + \frac{\Delta - \epsilon}{q(\theta(y^{\max}))\eta M(y^{\max})}(y^{\max} - y^s)$$

7. Update of $N(y)$. This goes in several steps. First define define $\widetilde{\theta}(y)$ on $y \in [y^{\min}, y^{\max}]$ as

$$(r + s + \delta)M(z(y)) = y + p((\widetilde{\theta}(y)))\eta[M(y) - M(z(y))] + (s + \delta)M(y_0)$$

Given $\widetilde{\theta}(y)$, calculate $\widetilde{n}(y)$ and $\widehat{u}$ from the formula

$$\begin{aligned} \widetilde{k}q(\widetilde{\theta}(y)) &= [s + \delta + p(\widetilde{\theta}(z^{-1}(y))]\widetilde{n}(y) \\ (1 - \widehat{u})s &= \int_{y^{\min}}^{\widetilde{y}^u} x(y)p(\widetilde{\theta}(y))dy \end{aligned} \tag{26}$$

and define

$$\hat{N}(y) = \frac{\int_{y^{\min}}^{y} \widetilde{n}(y)dy + \hat{u}}{\int_{y^{\min}}^{y\max} \widetilde{n}(y)dy + \hat{u}}$$

8. Update $X(u)$. First define

$$\widetilde{x}(y) = \widehat{k}\widetilde{\theta}(y)$$

Then update as follows:

$$\hat{X}(y) = \frac{\int_{y^{\min}}^{y} \widetilde{x}(y)dy}{\int_{y^{\min}}^{y\max} \widetilde{x}(y)dy}$$

Note that each step of updating algorithm is continuous in the arguments of Γ. Moreover, that Schauder's fixed point theorem ensures existence of a fixed point. Denote all the variables at the fixed-point by an asterix. First, at the fixed-point, $M^*(y)$ satisfies (18). Furthermore, θ^* satisfies (17). Hence the search behaviour of workers and firms is optimal. By construction, expected profit is zero, and the consistency requirements 6-8 are all satisfied.

Theorem 3.1 *The equilibrium with continuous types exists.*

4 Linear adjustment costs

In the previous section we construct a competitive flavoured model for which the optimal solution prescribes on-the-job search, and where high-productivity firms offer higher wages and grow faster than low-productivity firms. In this section we analyse whether efficient on-the-job search obtains with linear adjustment costs and possibly convex hiring costs. For simplicity, we work with a discrete number of states.

With linear vacancy costs, the highest type of firms will open vacancies up to the point where the marginal value of posting a vacancy is equal to the cost. With constant returns to scale in production, the marginal value of a worker is independent of firm size. It follows that when the marginal value of a vacancy is equal to the marginal cost for the most efficient firms, then no other firm types will find it profitable to open vacancies. Hence, in steady state, $N_1 = N_2 = ... = N_{n-1} = 0$. Thus there will be no on-the-job search, for the trivial reason that only one firm type will be active in equilibrium. Hence there will be no on-the-job search.

In order to have linear hiring costs and more than one firm type active in equilibrium, other parts of the model has to be "convexified". One way of doing this is to assume decreasing returns to scale in production at the firm level. New-born firms will then adjust immediately to to their desired employment level by posting infinitely many vacancies. In this section, we follow Lucas (1978), and Sargent (1987), who argued that sluggish employment adjustment may be due to convex hiring costs. Adjustment costs are organizational costs and training costs that firms incur when hiring new workers, and in contrast with search costs they are not influenced by external labor market conditions. We assume that the adjustment costs depend on gross hiring $h = qv$, and that the costs can be written as $\gamma(h)$, where $h = qv$. Furthermore, we assume that $\gamma(0) = \gamma'(0) = 0$, that $\gamma'()$ and $\gamma''()$ are strictly positive for $h > 0$, and that $\lim_{h\to\infty} \gamma'(h) = \infty$.

Consider a firm of type j that searches for workers and offers a wage W_j. The profit flow from hiring can then be written as

$$\begin{aligned}\pi_j &= -c_0 v + vq(\theta)(M_j - W_j) - \gamma(vq(\theta)) \\ &= hM_j - \gamma(h) - h[\frac{c_0}{q(\theta)} + W] \qquad (27)\end{aligned}$$

where $\theta = f(W)$. The firm maximizes profit with respect to W and H. The first order condition for W reads

$$-\frac{c}{q^2}q'(f(W))f_w(W) = 1$$

The condition is independent of the productivity of the firm, as well as of the hiring rate h. Denote the solution by W^* and define $\theta^* = f(W^*)$. The first order condition for h reads

$$M_j - [\frac{c_0}{q(\theta^*)} + W^*] = \gamma'(h) \tag{28}$$

The left-hand side is the value of filling a position less the associated search-and wage costs associated with filling it (which is unique even if W^* is not). The right-hand side shows the hiring costs. Given our assumptions on γ, the solution is unique. The rest of the model is unaltered. In particular, efficient on-the-job search implies that workers search so as to maximize joint income defined by (1). Hence the equilibrium of the model is as defined above, with the profit function (5) replaced by (27).

It follows that in equilibrium, if it exists, there will be no on-the-job search. Suppose on-the-job search does take place. Suppose workers in a firm of type i searches for a job in a firm of type j. Let W_{ij} denote the wage in the associated submarket. Efficient on-the-job search requires that $W_{ij} > M_i$. Since the type-i firm hired the worker in the first place, $M_i > \frac{c_0}{q(\theta^*)} + W^*$. Hence it would be more profitable for the j-firm to follow the same hiring strategy as the i-firm, a contradiction.

Given that no on-the-job search takes place, it is trivial to show that the equilibrium exists, and the proof is omitted. The next proposition follows

Proposition 4.1 *Suppose the vacancy costs are linear while hiring costs are convex in the hiring rate $h = vq$. Then the following is true*

a) The optimal wage is independent of the productivity of the firm.

b) The optimal hiring rate is increasing in firm productivity

c) If K is sufficiently high, all firm types open vacancies.

d)) There is no on-the-job search, all firms hire workers from the unemployment pool.

The get intuition for the absence of on-the-job search, note that although firms have different productivity, the value of a match net of the hiring costs are equalized across firms. Hence there is no welfare gain from reallocating workers from low-type to high-type firms.

5 Empirical implications

In this section we will briefly discuss testable differences in predictions between our model and some other important models of on-the-job search. To this end, let $D_w(w|w^o)$ denote the distribution of wages obtained after successful on-the-job search of a worker with a wage w^o prior to the job switch. Let $D_f(w|w^n)$ denote the distribution of wages *prior* to the job switch for a worker that obtains a wage w^n after successful on-the-job search. Finally, $D_p(y|y_0)$ denote the distribution of productivities in the new firms contingent on the productivity of the employer prior to the job change.

The Burdett-Mortensen (BM) model (Burdett-Mortensen, 1998). In the BM model, search is random. With identical firms, firms play with mixed strategy, and a distribution of wages arise endogenously. With heterogeneous firms, there is a one-to-one correspondence between wages and productivities, as high-productivity firms pay more.

Since workers and firms match randomly, the distribution of wages D_w after successful on-the-job search is equal to the wage offer distribution, truncated at previous wage w^o. If the wage distribution over vacancies is denoted by $F^w(w)$, it follows that

$$D_w(w|w^o) = \frac{F^w(w) - F^w(w^o)}{1 - F^w(w^o)}$$

The support of the distribution D is $[w^o, w^s]$, where w^s is the supremum of the support of advertised wages. Let $D^{w \geq w_j}(w|w_i^o)$ denote the distribution function of new wages w, contingent on $w \geq w_j$, as a function of the old wage w_i^o. Then for any $w_j \geq w_h^o$,

$$\begin{aligned} D_w^{w \geq w_j}(w|w_i^o) &= \frac{D_w(w|w_i^o) - D_w(w_j|w_i^o)}{1 - D_w^{w \geq w_j}(w_j|w_i^o)} \\ &= \frac{F^w(w) - F^w(w_j)}{1 - F^w(w_j)} \end{aligned}$$

independently of w_i^o. Hence $D_w^{w \geq w_j}(w|w_l^o) = D_w^{w \geq w_j}(w|w_h^o)$ for any $w_j > w_o^h$. In words, the distribution of new wages, contingent on being above w^j, is independent on the previous wage, as long as the previous wage is below w^j.

Similarly, the distribution of prior wages $D_f(w|w^n)$ is equal to the distribution of wages over employees (including unemployment benefit) truncated at $w \leq w^n$. Consider two wages w_l^n and w_h^n, $w_l^n \leq w_h^n$, and let $D_f^{w \leq w_j}(w|w^n)$ denote the distribution of the prior wage w prior to the job change. It follows that as long as $w_j \leq w_l^n$, $D_f^{w \leq w_j}(w|w_l^n) = D_f^{w \leq w_j}(w|w_h^n)$.

If firms are heterogeneous, there will be a one-to-one correspondence between a firm's wage and its wage offer. Hence, as with wages, the productivity distribution in new firms, $D_p^{y \geq y_j}(y|y^0)$ will be independent of y^0 as long as $y^0 < y^j$.

The Postel-Vinay and Robin (PR) wage setting procedure. Postel-Vinay and Robin (2002) assume that after successful on-the-job search, the incumbent firm and the new firm compete for the worker in a Bertrand fashion. Furthermore, firms compete in NPV wages, hence a worker takes into account that expected future wages (after encountering another job offer) will be higher the higher is the productivity of the employer. The latter is referred to as the option value of the job.

Since workers meet firms randomly, and change employer if and only if she matches with a firm that is more productive than the current employer, the results for the BM model on productivity distributions carry over to the PR model. In particular, the productivity distribution in new firms, $D_p^{y \geq y_j}(y|y^0)$ will be independent of y^0 as long as $y^0 < y^j$

The distribution of wages after a job shift is less clear. Betrand competition will tend to increase wages after a job swithc. On the other hand, the fact that the option value is increasing in productivity implies that, given the productivity of the current employer, there is a *negative* relationship between wages in the new job and the productivity of the new employer.

Competitive on-the-job search (CS). Our model is not a model of wages, but rather of NPV wages. However, we can assume that the wage that the worker obtains in a firm is constant, and that the workers' search behaviour is contracted upon directly. With a continuum of types, the wage before he job switch is an increasing function of the wage before the job switch. It follows that the distribution $D_w(w|w^o)$ then has a spike, at a discrete distance above w^o, and the wage at the spike will be increasing in w^o. The same is true for the productivity distribution $D_y(y|y^o)$. Another prediction from competitive on-the-job search (with discrete types) is that more productive firms pay higher wages than less productive firms, even if they attract workers from firms with the same productivity.

Thus, the BM model and the competitive search model have different predictions regarding the relationship between wages before and after a job change. The same is true for the productivity of the previous employer relative to the new one. In the PR model, the productivity distributions before and after the wage change is as in the BM model, and thus very different from the predictions of the CS model. The relationship between wages before and after wage changes is more involved in the PR model. Still there is one clear difference. The competitive search model predicts that if two firms with different productivities attract workers with equally productive employers, the high-productivity firm pays the higher wage. The PR model predicts the opposite.

6 Conclusion

We have developed a competitively flavored matching model where on-the-job search is an optimal response to productivity differences between firms in the presence of search frictions. The equilibrium features a job ladder, where workers gradually moves to jobs with higher wages. With a continuum of firm types, the job ladder is strict. Unlike existing models of labor turnover, the model predicts a strong relationship between the productivity of the present and future employer.

The papers also contributes methodologically. When proving existence of equilibrium, we do not follow the standard route, which is to apply a version of Schauder's fixed point theorem presented in Stokey and Lucas (1989). Instead we utilize that the equilibrium is formulated in terms of distribution functions, which allows for a different approach. We believe that our methodology may be a usefull tool for showing existence of equilibrium in search models more generally.

Finally, we show that if the costs of maintaining vacancies are linear, there will be no on-the-job search, even if we include convex hiring costs. Hence convex hiring costs seem to be necessary in order to explain the stylized facts of on-the-job search.

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Appendix 1, finite number of firm types

Proof of claim prior to lemma 2.1

We want to show that $y_0/r \leq M_0 < M_1 < ... < M_n$ and $(r + s + \delta)M_i - y_i$ is decreasing in i. An unemployed worker obtains y_0/r, hence $y_0/r \leq M_0$. A type n worker cannot gain from on-the-job search since $W \leq y_n$, and hence $M_n = y_n/(r+s+\delta)$. A worker and a firm of type j (hereafter referred to as a worker of type j) can always obtain a strictly higher joint income than a worker of type $i < j$ by following exactly the same search strategy as the type i worker, hence $M_j > M_i$. Analogously, a worker of type i can always mimic the search strategy of a worker of type $j > i$. Let the associated joint income be denoted M_i'. From (1),

$$(r + s + \delta)(M_j - M_i') = y_j - y_i$$

or

$$(r + s + \delta)M_j - y_j = (r + s + \delta)M_i' - y_i$$

Since $M_i' \leq M_i$ it follows that $(r + s + \delta)M_j - y_j \leq (r + s + \delta)M_i - y_i$.

Proof of lemma 2.1

a) We want to show that the indifference curve has the following single crossing property: Suppose $i < j \leq n'$. Then there exists a wage W' such that $f_i(W', M) = f_j(W', M)$, $f_i(W, M) < f_j(W, M)$ for all $W < W'$, and $f_i(W, M) > f_j(W, M)$ for all $W > W'$.

Note that $\lim_{W \to M_j^+} f_j = \infty$ while $f_i(M_j, M) < \infty$. Thus, for W close to but above M_j we have that $f_i(W, M) < f_j(W, M)$. The ratio of $p(f_i)$ to $p(f_j)$ reads

$$\begin{aligned} \frac{p(f_i)}{p(f_j)} &= \frac{(r+s+\delta)M_i - y_i - (s+\delta)M_0}{(r+s+\delta)M_j - y_j - (s+\delta)M_0} \frac{W - M_j}{W - M_i} \\ \lim_{W \to \infty} \frac{p(f_i)}{p(f_j)} &= \frac{(r+s+\delta)M_i - y_i - (s+\delta)M_0}{(r+s+\delta)M_j - y_j - (s+\delta)M_0} < 1 \end{aligned}$$

(from a in this lemma). Thus, for sufficiently large values of W, $p(f_i(W; M)) < p(f_j(W, M))$, and hence $f_i(W; M) < f_j(W, M)$. Since f_i and f_j are continuous it follows that there exists a value W' such that $f_i(W'; M) = f_j(W', M)$.

b) Suppose now that $i < j < k$. Let W^{ik} be the unique solution to $f_i(W; M) = f_k(W; M)$. Suppose $W^{ik} < W^{ij}$. From the single-crossing property just arrived it follows that $f_i < f_j$ for all $W < W^{ij}$ and that $f_k < f_j$ for all $W > W^{ik}$. Since, by assumption,$W^{ik} < W^{ij}$, it follows that hence $f_j(W; M) > f(W, M)$. Since all workers types at or below except the highest search this is a contradiction.

For wages below M_1, only type zero workers will apply, hence $f(W, M) = f_0(W; M)$. At $W^1 = W^{01}$, $f_0(W^1; M) = f_1(W; M)$. Let $W^2 = W^{12}$. Then $f(W) = f_2(W; M)$ for $W \in [W^1, W^2]$ (which may have zero measure) and so forth. Furthermore, by definition type n' gains from search, and hence searches for a job with a wage $W \leq W^s$. Hence there must be an interval $[W, W^s]$ at which $f_{n'} = f_n$

c) For all $i \leq n'$, each of the functions $f_i(W; M)$ is continuously differentiable for $W > M_i$. It follows that $f(W; M) = \min_i f_i(W; M)$ is continuous and piecewise differentiable for all $W < M_{n'+1}$. However, as the nominator in (3) is zero for $i = n'+1$, it follows that $f_{n'+1}$ is zero for any $W > M_{n'+1}$. Hence f is discontinuous at $M_{n'+1}$,as it jumps down to zero at this point.

Proof of lemma 2.2

Suppose a submarket attracts $i-$workers and j-workers, $i < j$. Denote the npv wage in the submarket by W'. Then $f_i(W', M) = f_j(W', M) = f(W', M)$. From lemma (2.1) this can only be the case if $j = i + 1$, that is, if $W' = W^i$.

From lemma 2.1 it follows that

$$\begin{aligned}
\lim_{W \to W^{i-}} \frac{\partial q(\theta(W^i))}{\partial W} &= \lim_{W \to W^{i-}} q'(\theta(W_i)) \frac{\partial \theta(W^i)}{\partial W} \\
&< \lim_{W \to W^{i-}} q'(\theta(W_i)) \frac{\partial \theta(W^i)}{\partial W} \\
&= \lim_{W \to W^{i-}} \frac{\partial q(\theta(W^i))}{\partial W}
\end{aligned}$$

It follows that $W = W^i$ cannot be a solution to any firm's maximization problem and hence cannot be an equilibrium wage. Hence a submarket cannot contain two different worker types.

Suppose then that two firm types i and j, $i < j$ offer the wage W'. The optimal wage for firm j solves

$$\max_{v,W} -c(v) + vq(f(W))[M_j - W]$$

with first order condition for W given by

$$\frac{q'(f(W))f'(W)}{q} = \frac{1}{M_j - W} \tag{29}$$

The left-hand side is independent of j, while the right hand side is increasing in j. It follows that the first order conditions cannot be satisfied for two different firm types simultaneously.

In order to derive (10), first note that

$$\frac{dp^{-1}(\theta)}{d\theta} = \frac{1}{p'(\theta)} = \frac{1}{q + \theta q'(\theta)}$$

(since $p(\theta) = \theta q(\theta)$). From (3) it thus follows that

$$f'(W) = -\frac{1}{q + \theta q'(\theta)} \frac{\theta q(\theta)}{W - M_i} \tag{30}$$

which inserted into (29) gives

$$-\frac{q'(\theta)}{q + \theta q'(\theta)} \frac{\theta q(\theta)}{W - M_i} = \frac{q}{M_j - W}$$

Inserting $\eta = -q'(\theta)\theta/q(\theta)$ and reorganizing slightly gives

$$\frac{\eta}{1 - \eta} = \frac{W - M_i}{M_j - W}$$

By assumption, the left-hand side is decreasing and the right-hand side is increasing in W. Thus, for given M_i and M_j the equation uniquely defines W.

Proof of existence of equilibrium

The strategy for the proof is to construct a mapping for which the equilibrium of the model is a fixed point, and then apply Brouwer's fixed point theorem to show existence.

The domain of the mapping is κ, a matrix describing submarket choices of workers κ_{ij}, $\Sigma_{i=0}^{n}\kappa_{ij} = 1$, a matrix θ consisting of labor market tightnesses θ_{ij}, and a real number k denoting the measure of firms in the economy. We impose that $0 \leq \theta_{ij} \leq \theta^{\max}$ for all $i < j$, and that $k \leq k^{\max}$, where $\theta^{\max}$ and $k^{\max}$ will be defined below. It follows that the domain $D^n \in R^{2n^2+1}$ of $(\kappa, \overline{\theta}, k)$ is closed and convex.

We construct the mapping $\Gamma : D^n \to D^n$ as follows: Let $\overline{p}$ denote the matrix of transition probabilities $p_{ij} = p(\theta_{ij})$. Define

$$(r+s+\delta)M_i = y_i + (s+\delta)M_0 + \max_j p_{ij}\eta(\theta_{ij})(M_j - M_i) \tag{31}$$

$$(r+s+\delta)M_{ij} = y_i + (s+\delta)M_0 + p_{ij}\eta(\theta_{ij})(M_l - M_j) \text{ for any } j > i \tag{32}$$

Let $\overline{M}$ denote the matrix of values M_{ij}. Given the matrix $p(\overline{\theta})$, the matrix $\overline{M} = M(\overline{\theta})$ is uniquely defined as a continuous function of $\overline{\theta}$. To see this, first note that M_n is independent of $\overline{\theta}$. Suppose M_i is uniquely defined and continuous functions of $\overline{\theta}$ for all $i > i'$. But then it follows from (31) that $M_{i'}$ is uniquely defined and continuous in $\overline{\theta}$ as well. The claim thus follows.

The gross income flow of a firm of type j of posting a vacancy in submarket i is given by

$$\rho_{ij} = q(\theta_{ij})(1-\eta(\theta_{ij}))(M_j - M_{ij}) \tag{33}$$

$$\rho_j = \max_i \rho_{ij}. \tag{34}$$

Now define θ^a_{ij} implicitly by the function

$$q(\theta^a_{ij})(1-\eta(\theta^a_{ij}))(M_j - M_{ij}) = \rho_j \tag{35}$$

Given our assumption that $\eta(\theta)$ is nondecreasing in θ we know that θ^a_{ij} is well defined. The equation thus shows values θ^a_{ij} such that the firm of type j is indifferent between searching in submarket i and in the income maximizing submarket given $\overline{\theta}$. Let v_j^a be defined by the equation

$$c'(v_j^a) = \rho_j$$

i.e., v_j^a is the optimal number of vacancies given ρ_j. It follows that both $\overline{\theta}^a$ and v_j^a are continuous functions of $\overline{\theta}$.

The expected profit of a firm of type j entering the market reads (from (34))

$$\Pi_j = \frac{1}{r+\delta}\left\{v_j^a \rho_j - c(v_j^a)\right\}$$

Define

$$E\Pi = \sum_j \alpha_j \Pi_j$$

The updating rule for k reads

$$\widehat{k} = k\frac{E\Pi}{K} \tag{36}$$

unless the upper bound $k^{\max}$ binds, in which case $\widehat{k} = k^{\max}$.

Given the initial vector $\overline{\theta}$ and κ, we can calculate the distribution $N_0, N_1, ..., N_n$. The outflow of workers of type i to firms of type $j > i$ is $N_i p(\theta_{ij})\kappa_{ij}$. Hence, the inflow equal to outflow requirement can be written as

$$N_i(s + \delta + \Sigma_{j=i+1}^n p(\theta_{ij})\kappa_{ij}) = \Sigma_{j=0}^{i-1} p(\theta_{ij})\kappa_{ij} N_j \text{ for } i = 1, ..., n-1$$

which uniquely defines $N_1, ..., N_n$ as continuous functions of $\overline{\theta}$. The measure of unemployed workers can be defined residually, $N_0 = \Sigma_{j=1}^n N_j$.

Generically, consistency requires that for each firm type,

$$\sum_i N_i \kappa_{ij}\theta_{ij} \equiv \sum_i v_{ij} = k\alpha_j v_j$$

(where v_{ij} is the total measure of vacancies in the i, j market). Define the constant $\zeta_j(\overline{\theta})$ by the expression

$$\sum_i N_i \kappa_{ij}\theta_{ij}^a = \zeta_j \widehat{k}\alpha_i v_j^a \tag{37}$$

Define

$$\widehat{\theta}_{ij} = \zeta_j(\overline{\theta})\theta^a_{ij}(\overline{\theta})$$

This is our updating rule for rule for θ unless the upper bound $\theta^{\max}$ binds, in which case $\widehat{\theta}_{ij} = \theta^{\max}$.

Consider the searching workers. Let $\digamma_i$ denote the set of firm types j such that $M_{ij} = M_i$. For all i, j, $j \notin \digamma_i$, define

$$\kappa^a_{ij} = \frac{M_{ij}}{M_j}\kappa_{ij}$$

Note that κ^a_{ij} is continuous in $\overline{\theta}$ and κ_{ij}. Define the constant σ_i by the expression

$$\sigma_i \sum_{j \in \digamma_i} \kappa_{ij} + \sum_{j \notin \digamma_i} \kappa^a_{ij} = 1$$

For all i, j such that $j \in J_i$, the updating rule reads

$$\widehat{\kappa}_{ij} = \eta_i \kappa_{ij}$$

(It thus follows that $\sum_j \widehat{\kappa}_{ij} = 1$ for all i.)

We have thus constructed a mapping $\Gamma : D^n \to D^n$, which by construction is continuous. It follows from Brouwers fixed point theorem that the mapping has a fixed point.

Our next step is to show that a fixed point of Γ is an equilibrium of our model (given that the bounds are sufficiently large, see below). First, the asset values M^*_i and M^*_{ij} are determined using the optimal sharing rule (equation 31 and (32). Second, $\kappa^*_{ij} > 0$ only if $M^*_{ij} = M^*_i$, which ensures efficient on-the-job search. It follows that the other markets are empty. Firms are indifferent between which of the non-empty submarkets to enter, hence their search decisions are optimal.[7] Finally, (36) implies that $E\Pi^* = K$ at the fixed point, hence the entry condition is satisfied.

At the fixed point, $\zeta_j = 1$ for all j. Hence (37) is satisfied. However, this means that the weights τ_{ij} give us enough degrees of freedom to satisfy the consistency requirement $N^*_i \theta^*_{ij} \kappa^*_{ij} = v^*_j \tau^*_{ij} k^*$ in each submarket.

[7] The labor market tightness θ^*_{ij} in empty submarkets are pinned down by iso-profit curve of firms, see equation (35). Workers flow to the submarkets that maximize their income given the iso-profit curve of the workers. In the model, it is the indifference curve of the workers that defines θ_{ij} in empty submarkets. Since maximizing worker income given the iso-profit of firms and maximizing profits given the indifference curves of workers are dual problems, they give the same solution.

Finally, we characterize the upper bounds. First consider k. An upper bound on the expected profit of entering the market is given by

$$\widetilde{\Pi} = \frac{y_n}{\alpha_n}\frac{1}{k}$$

which converges to zero as $k \to \infty$. Hence it is trivial to pick an upper bound that is not binding at the fixed point. For instance, we may let $\widehat{k}$ be defined by $\frac{y_n}{\alpha_n}\frac{1}{\widehat{k}} = K/2$.

Then consider θ. The proof is by contradiction. Suppose θ_{ij} is infinite for some i, j, $i > j$. Then no firm of type $l < j$ will attract any workers. Without loss of generality we therefore assume that $i = 0$. Unless $\theta_{ik} = \infty$ for some $k > j$, $\kappa_{ij} = 1$, it is always be better to search further with higher present income. Suppose first that θ_{ik} is finite for $k > j$. Since outflow from unemployment has to be equal to inflow, we have that

$$x(u, v_{0j}) = x(0, v_{0j}) = 0 = s + \delta$$

a contradiction. Suppose then that $\theta_{ik} = \infty$ for some $k > j$. Then the argument can be repeated for the highest type for which θ is infinite. Hence θ_{ij} is finite for all ij, and hence has an upper bound $\theta^{\max}$. This completes the proof.

Proof of efficiency

The welfare function is defined as

$$W = \int_0^\infty [\sum_{j=0}^n N_j y_j - \sum_{j=1}^n \alpha_j k c(v_j) - aK] e^{-rt} dt$$

Where v_j is the number of vacancies of a firm of type j. The law of motions are

$$\begin{aligned} \dot{N}_j &= \sum_{i=0}^{j-1} x(\kappa_{ij} N_i, \alpha_j k \tau_{ij} v_j) - \sum_{k=j+1}^{n} x(\kappa_{jk} N_j, \alpha_i k \tau_{kj} v_j) - (s+\delta) N_j \\ \dot{k} &= a - \delta k \end{aligned}$$

The initial conditions take care of the requirement that $\sum_i N_i = 1$. The controls are a, κ_{ij}, τ_{ij} and v_j. All κ_{ij}, τ_{ij} have to be between zero and 1, this will be discussed later. The current-value Hamiltonian reads

$$\begin{aligned} H &= \sum_{j=0}^{n} N_j y_j - \sum_{j=1}^{n} \alpha_j k c(v_j) - aK \\ &+ \sum_{j=0}^{n} \lambda_j [\sum_{i=0}^{j-1} x(\kappa_{ij} N_i, \alpha_j k \tau_{ij} v_j) - \sum_{k=j+1}^{n} x(\kappa_{jk} N_j, \alpha_k k \tau_{jk} v_j) - (s+\delta) N_j] \\ &+ A[a - \delta f] \end{aligned}$$

The controls are chosen so as to maximize H. Note that $x_v = (1-\eta)q(\theta)$, where $\eta = -q'(\theta)\theta/q$.[8]The first order conditions for the other controls read

$$A = K \tag{38}$$

$$p_{ij}(\lambda_j - \lambda_i) = \max_k p_{ik}(\lambda_i - \lambda_k) \text{ if } \kappa_{ij} > 0 \tag{39}$$

$$q_{ij}(\lambda_j - \lambda_i) = \max_k q_{kj}(\lambda_j - \lambda_k) \text{ if } \tau_{ij} > 0. \tag{40}$$

We thus get the following first order conditions for vacancy creation:

$$c'(v_j) = (1-\beta)q(\theta_{ij})[\lambda_j - \lambda_i] \tag{41}$$

for all ij for which $\kappa_{ij>0}$ (note that the right-hand side is the same for all active submarkets). Finally, the value functions for the adjoint variables are given by (in steady state)

$$\begin{aligned} (r+s+\delta)\lambda_j &= y_j + \beta \max_{k>j} p_{jk}(\lambda_k - \lambda_j) + (s+\delta)\lambda_u \qquad (42) \\ (r+\delta)A &= \sum_{j=1}^{n} \alpha_j [(1-\beta)v_j \max_k q_{jk}(\lambda_k - \lambda_j) - c(v_j)] \end{aligned}$$

It follows that the first order conditions of the planner is exactly equal to the market solution. More than that, the maximization problem for the controls is exactly equal to the maximization problem of the firm. Thus, the planner's solution and the decentralized solution is the same.

Proof of proposition 2.3

a) Let $h > l$, and suppose a firm of type h advertise a wage W^h with job finding rate q^h, while the firm of type l advertise a wage W^l with job finding rate q^l. From worker indifference it follows that

[8]To see this, note that $q(\theta) = x(\frac{1}{\theta}, 1)$. From the Euler equation it follows that

$$x_u(\frac{1}{\theta}, 1)\frac{1}{\theta} + x_v = x(\frac{1}{\theta}, 1) = q$$

which gives the expression in the text.

$W^h > W^l$ if and only if $q^h > q^l$, otherwise all workers would prefer to search for firm h rather than firn l. Profit maximization then implies that

$$\begin{aligned} q^h(M_h - W^h) &\geq q^l(M_h - W^l) \\ q^l(M_l - W^l) &\geq q^h(M_l - W^h) \end{aligned}$$

Combining the two gives

$$q^h(M_h - W^h) - q^h(M_l - W^h) \geq q^l(M_l - W^l) - q^l(M_h - W^l)$$

or

$$(q^h - q^l)(M_h - M_l) \geq 0 \tag{43}$$

The proof is by contradiction. Suppose $W^l > W^h$. Then $q^l > q^h$. Since $M_l < M_h$, this contradicts (43).

From lemma 2.1 we know that $f_i(W) = f(W)$ at the interval $[W^{i-1}, W^i]$. Furthermore, from the proof of lemma 2.2 we know that W^i cannot be an equilibrium point. It follows that a worker of type j always searches for higher wages than a worker of type $i < j$.

b) Suppose on the contrary that I_l has an element, say i, that is strictly greater than one element in I_k, say j. From a) it follows that worker i searches for strictly higher wages than worker j. Hence firm l advertises a wage that is strictly higher than a wage advertising by firm k, $l < k$. We know from a) that this is a contradiction.

Appendix 2, continuum of firm types

Proof of Proposition 3.1

The key ingredient is the following lemma.

Lemma 6.1 *Let $F, G \in CD[b, c]$, and let G be in a δ-neighborhood of F with the sup-norm, then $T_F^{aL}(y) - T_G^{aL}(y) < 2\delta$.*

Let $t(y)$ be defined by $F(y) + \delta = G(y) + t(y)$. In other words $t(y)$ is the distance to the "roof" of the δ-neighborhood for every y. Note that $\Delta F - \Delta G = \Delta t$ for all intervals in $[b, c]$. This implies that $F'(y) - G'(y) = t'(y)$, for all y where both F and G are differentiable. Moreover, this relation also applies to jump points, as by definition : $F_{\text{jump}}(y) = G_{\text{jump}}(y) + t_{\text{jump}}(y)$.

We want to show that $T_F^{aL\prime}(y) - T_G^{aL\prime}(y) \leq t'(y)$. Note that $T_F^{aL\prime}(y) - T_G^{aL\prime}(y) = F'(y) - a - (G'(y) - a) = F'(y) - G'(y) = t'(y)$ if $F'(x) - a > 0$ and $G'(x) - a > 0$. Similarly if only F surpass the growth constraint: $T_F^{aL\prime}(y) - T_G^{aL\prime}(y) = F'(y) - a \leq F'(y) - G'(y) = t'(y)$ since $G'(y) \leq a$. The inequality is trivially satisfied when only G surpass the growth constraint. The subcases regarding jump points follows by the same kind of computation. Intuitively, the difference in incremental growth at any point y, is precisely the change in the distance to the "roof" of the δ-neighborhood. Since this difference in incremental growth necessarily is the maximum difference of accumulated mass that may arise in any point, the inequality follows

Since this applies for any point y, it follows that

$$T_F^{aL}(y) - T_G^{aL}(y) = \int_b^y (T_F^{aL\prime}(x) - T_G^{aL\prime}(x))dx \leq \int_b^y t'(x)dx = t(y) - t(b) < 2\delta$$

The result now follows readily from the triangle inequality:

Proposition 6.1 *$\Phi^{aL} : CD[b, c] \longrightarrow CCD[b, c]$ is continuous.*

Let $F \in CD[b, c]$, and let G be a CD in a δ-neighborhood of F (with the sup-norm). Since $F(y) = F_s(y) + T_F^{aL}(y)$ and $G(y) = G_s(y) + T_G^{aL}(y)$, we get by the triangle inequality $|F_s(y) - G_s(y)| \leq |F - G| + |T_F^{aL}(y) - T_G^{aL}(y)| < \delta + 2\delta = 3\delta$. In other words, for any given ϵ, we will ensure $|F_s - G_s| < \epsilon$ provided we choose $|F - G| < \frac{\epsilon}{3}$.

Proposition 6.2 *Let $\Phi : CD[b, c] \longrightarrow CCD[b, c]$. IF $\Phi(G) = G$ whenever G is continuous, then Φ is discontinuous.*

Let Φ denote a mapping that leaves continuous distributions unchanged. Let G denote a distribution that is continuous everywhere except at at y_0, where it has a jump point. Construct a sequence

G_n of continuous functions by letting $G_n(y) = G(y)$ for all y except the interval $< y_0 - \frac{1}{n}, y_0 >$. In this interval the graph of the function G_n is the line between the points $(y_0 - \frac{1}{n}, G(y_0 - \frac{1}{n})$ and $(y_0, G(y_0))$. By construction G_n is continuous and converges to G as n approaches infinity. The contradiction now follows from invoking continuity of Φ, as continuity implies $\Phi(\lim_{n\to\infty} G_n) = \lim_{n\to\infty} \Phi(G_n)$. Hence, we get: $\Phi(G) = \Phi(\lim_{n\to\infty} G_n) = \lim_{n\to\infty} \Phi(G_n) = \lim_{n\to\infty} G_n = G$. But $\Phi(G) \neq G$, since $\Phi(G)$ is continuous and G is not.

Upper and lower bounds for key parameters in the updating algorithm of Γ

Definition of $\theta^{\min}$: A natural lower bound arise from solving $p(\theta)\frac{y^{\max}-y^0}{r+s+\delta} = \epsilon$. In other words, $\theta^{\min}$ is given by:

$$\theta^{\min} = p^{-1}(\frac{\epsilon(r+s+\delta)}{y^{\max} - y^0})$$

Definition of $k^{\max}$: The upper bound for the aggregate output in the economy is achieved if all workers work in the most productive firms. This gives rise to the following steady state condition:

$$y^{\max}/k = (r+\delta)K$$

Hence $k^{\max}$ is given by:

$$k^{\max} = \frac{y^{\max}}{(r+\delta)K}$$

Definition of $\theta^{\max}$: The highest value of $\theta^{\max}$ is realized in the submarket with the firms with the lowest productivities, since they offer the lowest wages. We will show that since $y^{min} > y_0$, the labour market tightness in this submarket has an upper bound.

First we derive a lower bound for the unemployment rate. For a given number of firms, the number of matches is maximized by if all submarkets have the same θ. Moreover, the unemployment rate is the lowest if all firms hire unemployed workers only. This give rise to the following steady state condition:

$$u^{\min}(s+\delta+p(\frac{k^{\max}}{u^{\min}}))=s+\delta$$

Let $\theta^h=\frac{K^{\max}}{u^{\min}}$, and p^h be the corresponding value of p. It follows that p^h is an upper bound for the average value of the hazard rate out of unemployment. Since $p(\theta(y^u))<p(\theta(y))$ for all $y<y^u$, it follows that $p(\theta(y^u))<p^h$. Suppose that the workers employed in firms with $y^{\min}$ join the same submarket as workers searching for the y^u-jobs. This is feasible, but not optimal. Let $\tilde{M}(y^{\min})$ denote the corresponding expected income. (Note that $\tilde{M}(y^{\min})<M(y^{\min})$) It follows that

$$(r+s+\delta)\tilde{M}(y^{\min})=y^{\min}+p(\theta(y^u))(W(y)-\tilde{M}(y^{\min})$$

Or

$$(r+s+\delta)(M(y^{\min})-M(y_0))>(r+s+\delta)(\tilde{M}(y^{\min})-M(y_0))>\frac{y^{\min}-y_0}{r+s+\delta+p^h}$$

Let

$$\Delta\equiv\frac{y^{\min}-y_0}{r+s+\delta+p^h}$$

An upper bound for $M(y_0)$, $M_0^{\max}$, is

$$M_0^{\max}=\frac{y^{\max}}{r+s}$$

Finally, define $\theta^{\max}$ by

$$(r+s+\delta)M_0^{\max}=y_0+\bar{p}(\theta^{\max}\eta(\theta^{\max})\Delta$$

Since $\eta(\theta)$ is nondecreasing in θ, this equation defines θ^* uniquely.

Endogenous number of vacancies per firm

Assume now that the number of vacancies is a (contiuous) choice variable of the firm, and that the maintainance cost is $c(v)$. We make the same assumptions as in the discrete case. In addition we assume that there exists an upper bound v^{max} on v, and that $\lim_{v \to v^{max-}} = \infty$.

Let $v(y)$ denote the number of vacancies posted by a firm of type y. The first order condition for $v(y)$ writes (analogue with (13) in the discrete case)

$$c'(v(y)) = (1-\eta)q(\theta)[M(y) - M(z(y))] \quad (44)$$

Define $\bar{v}$ as the aggregate number of vacancies, $\bar{v} = \int_{y^{min}}^{y^{max}} v(y)g(y)dy$. Define $\tilde{g}(y) \equiv g(y)v(y)/\bar{v}$. Define $\tilde{k} = k\bar{v}$. Finally, write the profits as $\pi(y) = vq(\theta)[M(y) - W] - c(v)$.

An equilibrium can now be defined as above, with $v(y)$ as an additional equilibrium object, with firms maximizing profit with respect to w and v (not only w), and with g and k replaced by $\tilde{g}$ and $\tilde{k}$ in the aggregate consistency equations. In all other respects the equilibrum conditions are unchanged.

The next step is to modify the mapping. Define $V(y) = \int_{y^{min}}^{y} v(y')dy'$. Clearly $V(y)$ is monotone and increasing, and bounded above by $\bar{V} = (y^{max} - y^{min})v^{max}$. Let $V(y)$ be an element of the mapping, and in the first step of the mapping, write $V(y) = \Phi^{aL}(\Phi^{aH}(V(y)))$ After step 5 in the updating procedure, update $v(y)$ by (44). It follows that the mapping is continuous, and that the conditions for using Schauder's fix-point theorem still applies. Hence the equilibrium exists.