

# Dynamic Macroeconomics

## PhD Economics

### Growth theory (answers)

September 2024

**PROBLEM 1.**

$$Y_t = K(t)^\alpha L(t)^{1-\alpha} \quad 0 < \alpha < 1$$

$$L(t) = A(t)N(t)$$

$$\frac{\dot{A}_t}{A_t} = 0; \quad \frac{\dot{N}_t}{N_t} = g_N; \quad s, \delta \text{ given}$$

a)

$$\frac{Y(t)}{L(t)} = \frac{K(t)^\alpha L(t)^{1-\alpha}}{L(t)^\alpha L(t)^{1-\alpha}} = k(t)^\alpha = f(k(t)) = y(t)$$

where

$$k(t) = \frac{K(t)}{L(t)}$$

$$\dot{k}(t) = \frac{\dot{K}(t)L(t) - \dot{L}(t)K(t)}{L(t)^2} = \frac{\dot{K}(t)}{L(t)} - g_N k(t)$$

$$\implies \frac{\dot{K}(t)}{L(t)} = \dot{k}(t) + g_N k(t)$$

We know that:

$$I(t) = sY(t); \quad I(t) = \dot{K}(t) + \delta K(t)$$

$$C(t) = (1-s)Y(t)$$

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$$

$$\implies \frac{\dot{K}(t)}{L(t)} = f(k(t)) - c(t) - \delta k(t)$$

$$\begin{aligned}
\dot{k}(t) + g_N k(t) &= f(k(t)) - c(t) - \delta k(t) \\
\dot{k}(t) &= f(k(t)) - (1-s)f(k(t)) - \delta k(t) - g_N k(t) \\
&= sf(k(t)) - (\delta + g_N)k(t)
\end{aligned}$$

which is the fundamental equation of the Solow growth model.

In steady-state:  $\dot{k}(t) = 0$  :

$$sf(k(t)) = (\delta + g_N)k(t)$$

For the given production function:

$$\dot{k}(t) = sk(t)^\alpha - (\delta + g_N)k(t)$$

$$\begin{aligned}
k_t^{ss} &= \left( \frac{s}{\delta + g_N} \right)^{\frac{1}{1-\alpha}} \\
y_t^{ss} &= (k_t^{ss})^\alpha = \left( \frac{s}{\delta + g_N} \right)^{\frac{\alpha}{1-\alpha}} \\
c_t^{ss} &= (1-s)y_t^{ss} = (1-s) \left( \frac{s}{\delta + g_N} \right)^{\frac{\alpha}{1-\alpha}}
\end{aligned}$$

**b)** To find the saving rate which maximizes the steady-state consumption level, we set

$$\frac{d(c_t)^{ss}}{ds} = 0$$

$$\begin{aligned}
\Rightarrow (1-s) \frac{\alpha}{1-\alpha} \left( \frac{s}{\delta + g_N} \right)^{\frac{\alpha}{1-\alpha}-1} \left( \frac{1}{\delta + g_N} \right) - \left( \frac{s}{\delta + g_N} \right)^{\frac{\alpha}{1-\alpha}} &= 0 \\
\Rightarrow \left( \frac{s}{\delta + g_N} \right)^{\frac{\alpha}{1-\alpha}-1} \left( (1-s) \frac{\alpha}{1-\alpha} \frac{1}{\delta + g_N} - \frac{s}{\delta + g_N} \right) &= 0 \\
\Rightarrow (1-s) \frac{\alpha}{1-\alpha} \frac{1}{\delta + g_N} = \frac{s}{\delta + g_N} \\
\Rightarrow (1-s) \frac{\alpha}{1-\alpha} = s \\
\Rightarrow s = \alpha
\end{aligned}$$

When  $s = \alpha$ , the *Golden Rule level of capital* (i.e. the level corresponding to the *saving rate* that maximizes consumption) is:

$$k_{GR} = \left( \frac{\alpha}{\delta + g_N} \right)^{\frac{1}{1-\alpha}}$$

**PROBLEM 2.**

$$Y_t = bK_t + BK_t^\alpha L_t^{1-\alpha}$$

(otherwise, same assumptions as in problem 1)

a) Check that the production function shows constant returns to scale:

$$\lambda Y_t = \lambda[bK_t + BK_t^\alpha L_t^{1-\alpha}]$$

$$\begin{aligned} F(\lambda K_t, \lambda L_t) &= b\lambda K_t + B(\lambda K_t)^\alpha (\lambda L_t)^{1-\alpha} \\ &= b\lambda K_t + B\lambda^\alpha K_t^\alpha \lambda^{1-\alpha} L_t^{1-\alpha} \\ &= b\lambda K_t + \lambda BK_t^\alpha L_t^{1-\alpha} \\ &= \lambda[bK_t + BK_t^\alpha L_t^{1-\alpha}] \end{aligned}$$

Since  $\lambda F(K, L) = F(\lambda K_t, \lambda L_t)$ ,  $Y_t$  displays CRS

$$f(k) = \frac{Y}{L} = b\frac{K}{L} + B\frac{K_t^\alpha L_t^{1-\alpha}}{L_t^\alpha L_t^{1-\alpha}} = bk + Bk^\alpha$$

$$f'(k) = b + \alpha Bk^{\alpha-1}, \quad \lim_{k_t \rightarrow \infty} f'(k) = b$$

Finally, we know that without technological progress, the dynamic equation for  $k$  is:

$$\begin{aligned} \dot{k}_t &= sf(k_t) - (\delta + g_N)k_t \\ \Rightarrow \dot{k}_t &= s(bk + Bk^\alpha) - (g_N + \delta)k_t \end{aligned}$$

b) Divide both sides by  $k_t$  and obtain:

$$\begin{aligned} \gamma_k = \frac{\dot{k}_t}{k_t} &= s(b + Bk^{\alpha-1}) - (g_N + \delta) \\ \frac{\partial \gamma_k}{\partial k_t} &= (\alpha - 1)sBk^{\alpha-2} < 0 \end{aligned}$$

The convergence result holds. In detail:

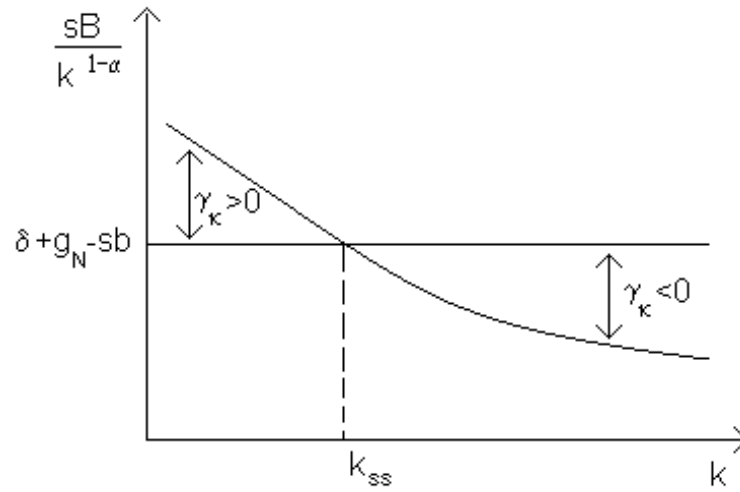
$$\gamma_k = \frac{\dot{k}_t}{k_t} = s\left(b + \frac{B}{k^{1-\alpha}}\right) - (g_N + \delta) = 0 \quad \text{in the steady-state equilibrium}$$

This implies:

$$s\left(b + \frac{B}{k^{1-\alpha}}\right) = g_N + \delta$$

or

$$s\frac{B}{k^{1-\alpha}} = g_N + \delta - sb$$



Assume that for each country the parameters  $\delta$ ,  $g_N$ ,  $s$ ,  $b$  and  $B$  are the same, meaning that all countries have the same aggregate technology, saving rate ( $s$ ), depreciation rate ( $\delta$ ), and the same growth rate of the number of workers ( $g_N$ ). In this case, the steady-state level of capital ( $k_{ss}$ ) is unique and equal across countries. The smaller is  $k$  the higher the equilibrium growth rate

c)

$$\lim_{k_t \rightarrow +\infty} \frac{\dot{k}_t}{k_t} = \lim_{k_t \rightarrow +\infty} s \left( b + \frac{B}{k^{1-\alpha}} \right) - (g_N + \delta) = sb - (g_N + \delta) > 0$$

if

$$sb > g_N + \delta \\ \Rightarrow s > \frac{\delta + g_N}{b} \quad \text{or} \quad b > \frac{\delta + g_N}{s}$$

Under this condition, unlimited accumulation of capital is possible under the assumed production function.

**PROBLEM 3.** Consider the following aggregate production function

$$Y_t = F(K, \bar{L}) = A_0 \bar{L} + 2B\sqrt{K\bar{L}}$$

This technology displays constant returns to scale in  $L$  and  $K$ .

a)

$$\begin{aligned} \frac{Y}{\bar{L}} &= \frac{A_0 \bar{L}}{\bar{L}} + \frac{2B\sqrt{K\bar{L}}}{\bar{L}} \\ &= A_0 + 2B\sqrt{k} \equiv f(k) \end{aligned}$$

The aggregate budget constraint is:

$$Y = C + \dot{K} + \delta K$$

which, in per-capita terms, becomes:

$$\frac{Y}{\bar{L}} = \frac{C}{\bar{L}} + \frac{\dot{K}}{\bar{L}} + \delta \frac{K}{\bar{L}}$$

$$y \equiv f(k) = c + \frac{\dot{K}}{L} + \delta k \quad c \equiv \frac{C}{L}$$

In order to determine  $\frac{\dot{K}}{L}$ , start with  $k$ :

$$k = \frac{K}{L} \implies \dot{k} = \frac{\dot{K}L - L\dot{K}}{L^2} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k = \frac{\dot{K}}{L} \text{ since } L \text{ is constant.}$$

Therefore:

$$\begin{aligned} f(k) &= c + \dot{k} + \delta k \\ \implies \dot{k} &= f(k) - c - \delta k \end{aligned}$$

$$\dot{k} = A_0 + 2B\sqrt{k} - c - \delta k$$

The Hamiltonian function is

$$H_t = \left\{ \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t [A_0 + 2B\sqrt{k_t} - c_t - \delta k_t] \right\} e^{-\rho t}$$

and the f.o.c. of the dynamic optimization problem are derived as

1.

$$\begin{aligned} \frac{\partial H_t}{\partial c_t} &= 0 \implies (c_t^{-\sigma} - \lambda_t) e^{-\rho t} = 0 \\ \implies c_t^{-\sigma} &= \lambda_t \end{aligned}$$

Taking logs:

$$-\sigma \log c_t = \log \lambda_t$$

and differentiating with respect to time, we get:

$$\begin{aligned} -\sigma \frac{\dot{c}_t}{c_t} &= \frac{\dot{\lambda}_t}{\lambda_t} \\ \implies \frac{\dot{c}_t}{c_t} &= \underbrace{\frac{1}{\sigma}}_{IES} \left[ -\frac{\dot{\lambda}_t}{\lambda_t} \right] \end{aligned}$$

$$IES = \frac{-u'(c)}{u''(c)c} = \frac{1}{\sigma}$$

2.

$$\begin{aligned} \frac{\partial H_t}{\partial K_t} &= -\frac{\partial}{\partial t} [\lambda_t \cdot e^{-\rho t}] \implies \lambda_t [Bk_t^{-\frac{1}{2}} - \delta] e^{-\rho t} = -[\dot{\lambda}_t - \rho \lambda_t] e^{-\rho t} \\ \implies -\frac{\dot{\lambda}_t}{\lambda_t} &= \underbrace{Bk_t^{-\frac{1}{2}} - \delta - \rho}_{\text{excess return on alternative investment}} \end{aligned}$$

3.

$$\frac{\partial H_t}{\partial \lambda_t} = \dot{k}_t e^{-\rho t} \implies \dot{k}_t = A_0 + 2B\sqrt{k_t} - c_t - \delta k_t$$

Law of motion of per-capita capital.

4.

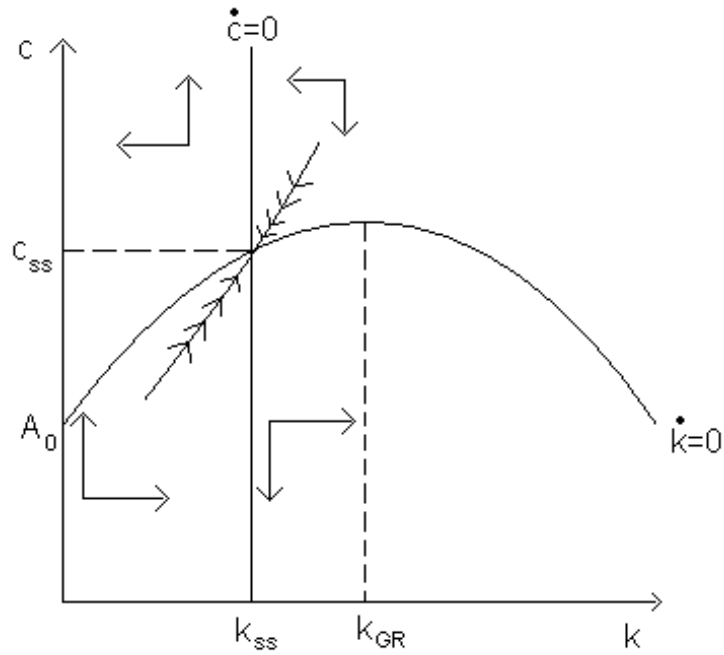
$$\lim_{t \rightarrow +\infty} \lambda_t \cdot K_t \cdot e^{-\rho t} = 0, \quad k(0) = k_0 \text{ given}$$

transversality condition.

Combining the f.o.c.

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \left[ \underbrace{Bk_t^{-\frac{1}{2}}}_{\equiv f'(k)} - \delta - \rho \right]$$

b) Setting  $\dot{c} = 0$  and  $\dot{k} = 0$  into the dynamic equations for consumption and the capital stock respectively, we obtain the stationary loci for  $c$  and  $k$ . The dynamics of  $c$  and  $k$  outside the stationary loci is plotted in the phase diagram below, together with the convergent saddlepath:



c) At time  $t_0$  there is an unexpected permanent increase in  $A$  from  $A_0$  to  $A_1 > A_0$ . The locus  $\dot{c}_t = 0$  is not affected. Indeed:

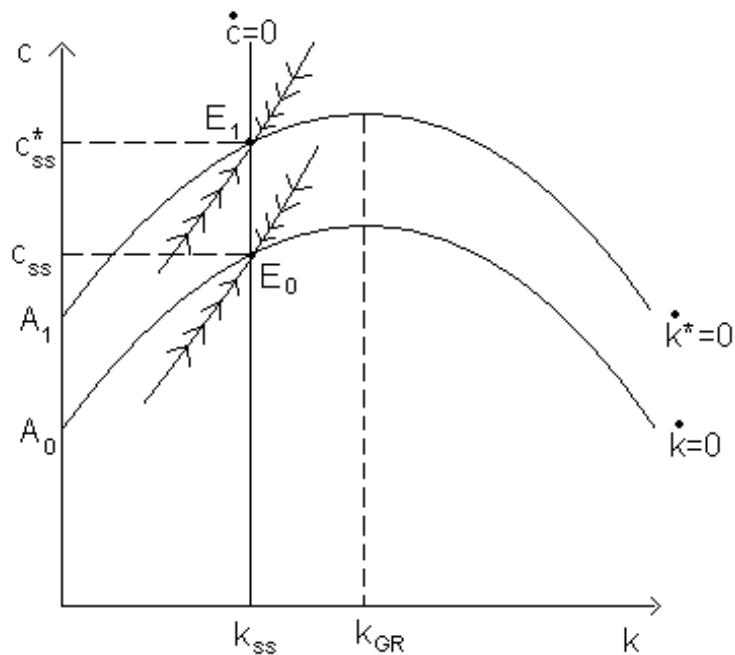
$$\dot{c}_t = 0 \implies k_t^{ss} = \left[ \frac{B}{\delta + \rho} \right]^2$$

independent of  $A$ . The locus  $\dot{k}_t = 0$  does change, since

$$\dot{k}_t = 0 \implies c_t = A_1 + 2B\sqrt{k_t} - \delta k_t$$

Note that the Golden Rule level of the capital stock  $k_{GR}$  (obtained by maximizing consumption along the stationary locus  $\dot{k} = 0$ ) does not change

$$\left. \frac{dc_t}{dk_t} \right|_{\dot{k}=0} = 0 \implies k_{GR} = \left( \frac{B}{\delta} \right)^2 > k_{ss}$$



In the transition from  $t_0$  to  $t_1$ , consumption jumps immediately from  $E_0$  to  $E_1$ . An exogenous positive productivity shock makes labour more productive and people "richer". This induces people to consume more instantaneously. The optimal level of  $k$  depends uniquely on the marginal productivity of capital and on parameters  $\delta$  and  $\rho$

d) Assume that all markets (final output and factor inputs markets) are competitive. Since the aggregate production function is *linearly homogenous*, the Euler Theorem applies and each input is compensated according to its own marginal productivity.

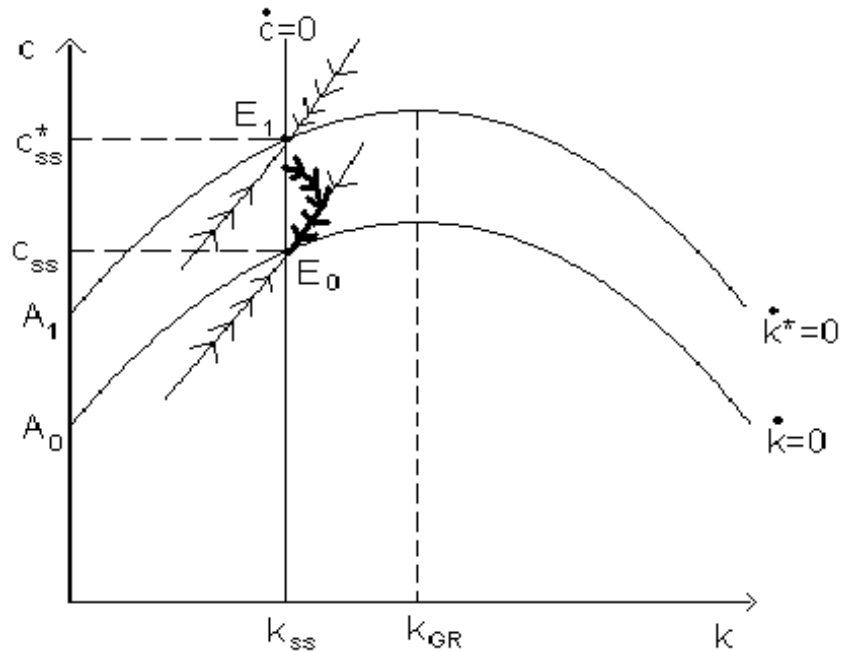
$$\frac{\partial Y(\cdot)}{\partial K_t} = \frac{BL}{\sqrt{KL}} = r + \delta \quad \text{in steady-state}$$

$$\frac{\partial Y(\cdot)}{\partial L} = A_0 + \frac{BK}{\sqrt{KL}} = w \quad \text{in steady-state}$$

An increase in  $A$  from  $A_0$  to  $A_1$  increases the wage rate ( $w$ ) and keeps unchanged the interest rate ( $r$ ) in the new steady-state.

e) The increase in  $A$  is only temporary (from  $t_0$  to  $t_1$ ):





From equilibrium  $E_0$ , consumption jumps immediately (at  $t_0$ ), but does not reach  $E_1$ . The original saddle-path trajectory is reached at  $t_1$  (with consumption decreasing and capital increasing). After  $t_1$  both capital and consumption decrease and reach the old equilibrium  $E_0$  asymptotically.

**PROBLEM 4.** Consider the following aggregate production function

$$Y_t = F(K, \bar{L}) = a\bar{L} + bK(t)^\alpha \bar{L}^{1-\alpha}$$

a) Start from the aggregate budget constraint:

$$Y = C + \dot{K} + \delta_0 K$$

which, in per-capita terms, becomes:

$$\frac{Y}{L} = \frac{C}{L} + \frac{\dot{K}}{L} + \delta_0 \frac{K}{L}$$

$$y \equiv f(k) = c + \frac{\dot{K}}{L} + \delta_0 k \quad k \equiv \frac{K}{L}$$

In order to determine  $\frac{\dot{K}}{L}$ , start with  $k$ :

$$k = \frac{K}{L} \implies \dot{k} = \frac{\dot{K}L - L\dot{K}}{L^2} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L}k = \frac{\dot{K}}{L} \quad \text{since } L \text{ is constant.}$$

Therefore, we have:

$$\begin{aligned} f(k) &= c + \dot{k} + \delta_0 k \\ \implies \dot{k} &= f(k) - c - \delta_0 k \end{aligned}$$

In this case

$$f(k) = \frac{aL}{L} + \frac{bK^\alpha L^{1-\alpha}}{L^\alpha L^{1-\alpha}} = a + bk^\alpha$$

from which we get

$$\dot{k} = a + bk^\alpha - c - \delta_0 k$$

Compute the returns to scale of the aggregate production function. Since  $\lambda F(K, L) = F(\lambda K, \lambda L)$ , the aggregate technology exhibits constant returns to scale and each productive unit may be rewarded according to its own marginal productivity (provided that the factor inputs markets are competitive). In an economy where the aggregate production function displays CRS and each productive input is rewarded according to its own marginal productivity, the "Decentralized Competitive Solution" coincides with the "Centralized Social Planner Solution".

The dynamic problem faced by a representative agent can be stated as:

$$\begin{aligned} \max_{\{c_t\}_{t=0}^{\infty}} U_0 &\equiv \int_0^{\infty} \left(1 - \frac{1}{c_t}\right) e^{-\rho t} dt \\ \text{s.t. } \dot{k}_t &= a + bk_t^\alpha - c_t - \delta_0 k_t \end{aligned}$$

The corresponding Hamiltonian function is:

$$H_t = \left\{ \left(1 - \frac{1}{c_t}\right) + \lambda_t [a + bk_t^\alpha - c_t - \delta_0 k_t] \right\} e^{-\rho t}$$

and the f.o.c. are derived as:

1.

$$\begin{aligned} \frac{\partial H_t}{\partial c_t} &= 0 \implies \left(\frac{1}{c_t^2} - \lambda_t\right) e^{-\rho t} = 0 \\ \implies \frac{1}{c_t^2} &= \lambda_t \end{aligned}$$

Taking logs:

$$-2 \log c_t = \log \lambda_t$$

and differentiating with respect to time:

$$\begin{aligned} -2 \frac{\dot{c}_t}{c_t} &= \frac{\dot{\lambda}_t}{\lambda_t} \\ \implies \frac{\dot{c}_t}{c_t} &= \underbrace{-\frac{1}{2}}_{IES} \left[ \frac{\dot{\lambda}_t}{\lambda_t} \right] \end{aligned}$$

$$IES = \frac{-u'(c)}{u''(c)c} = \frac{1}{2}$$

2.

$$\begin{aligned} \frac{\partial H_t}{\partial K_t} &= -\frac{\partial}{\partial t}[\lambda_t e^{-\rho t}] \implies \lambda_t [b\alpha k_t^{\alpha-1} - \delta_0] e^{-\rho t} = -[\dot{\lambda}_t - \rho \lambda_t] e^{-\rho t} \\ \implies -\frac{\dot{\lambda}_t}{\lambda_t} &= b\alpha k_t^{\alpha-1} - \delta_0 - \rho \end{aligned}$$

3.

$$\begin{aligned} \frac{\partial H_t}{\partial \lambda_t} &= \dot{k}_t \cdot e^{-\rho t} \implies (a + bk_t^\alpha - c_t - \delta_0 k_t) e^{-\rho t} = \dot{k}_t \cdot e^{-\rho t} \\ \implies \dot{k}_t &= a + bk_t^\alpha - c_t - \delta_0 k_t \end{aligned}$$

4.

$$\lim_{t \rightarrow +\infty} \lambda_t \cdot k_t \cdot e^{-\rho t} = 0$$

Combining the f.o.c. we get

$$\frac{\dot{c}_t}{c_t} = \frac{1}{2} \left[ \underbrace{b\alpha k_t^{\alpha-1}}_{\equiv f'(k)} - \delta_0 - \rho \right]$$

as the optimal path for consumption.

b) The two equations composing the dynamic system are:

$$\dot{c}_t = \frac{1}{2} [\alpha b k_t^{\alpha-1} - \delta_0 - \rho] c_t$$

$$\dot{k}_t = a + bk_t^\alpha - c_t - \delta_0 k_t$$

From which the stationary loci are derived:

$$\dot{c}_t = 0 \implies \underbrace{\alpha b k_t^{\alpha-1}}_{f'(k)} - \delta_0 - \rho = 0$$

$$\implies k_t^{ss} = \left[ \frac{\alpha b}{\delta_0 + \rho} \right]^{\frac{1}{1-\alpha}}$$

$$\dot{k}_t = 0 \implies c_t = a + bk_t^\alpha - \delta_0 k_t$$

$$\begin{aligned} \frac{dc_t}{dk_t} \Big|_{\dot{k}=0} &= 0 \Rightarrow \underbrace{\alpha b k_t^{\alpha-1}}_{f'(k)} - \delta_0 = 0 \\ \Rightarrow k_{GR} &= \left[ \frac{\alpha b}{\delta_0} \right]^{\frac{1}{1-\alpha}} > k_{ss} \end{aligned}$$

$k_{GR}$  is the consumption-maximising level of  $k$  in equilibrium (when  $\dot{k}_t = 0$ ). The stationary loci and the saddlepath are shown in the figure below.

c) An increase (unexpected and permanent) of the depreciation rate  $\delta$  from  $\delta_0$  to  $\delta_1 > \delta_0$  makes the loci  $\dot{k}_t = 0$  and  $\dot{c}_t = 0$  shift downwards and to the left respectively. In fact:

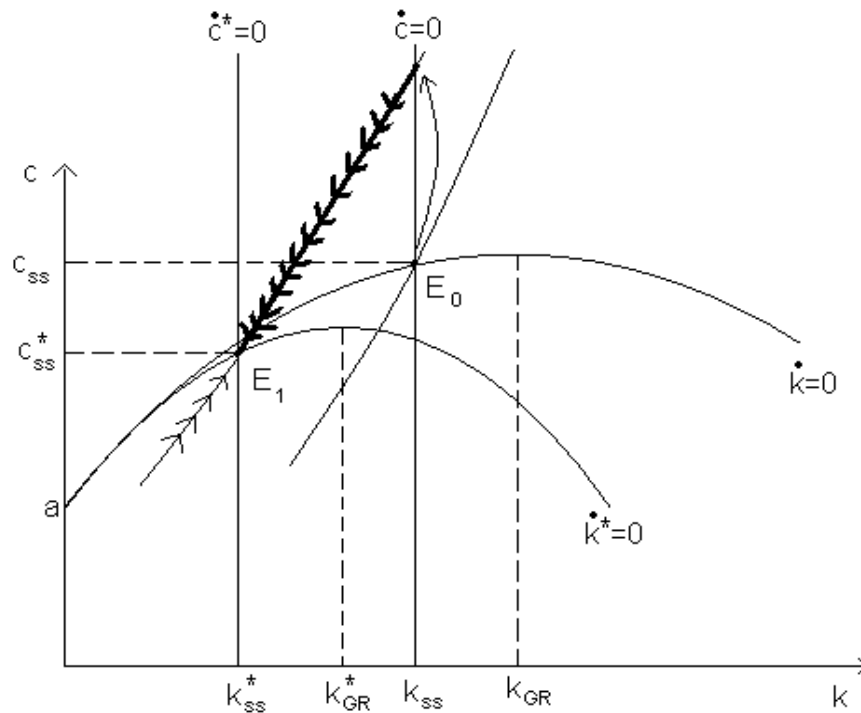
$$\dot{c}_t = 0 \Rightarrow k_t^{ss} = \left[ \frac{\alpha b}{\delta_1 + \rho} \right]^{\frac{1}{1-\alpha}} \downarrow$$

and

$$\dot{k}_t = 0 \Rightarrow c_t = a + b k_t^\alpha - \delta_1 k_t \downarrow$$

(with the same vertical intercept). Moreover

$$k_{GR} = \left[ \frac{\alpha b}{\delta_1} \right]^{\frac{1}{1-\alpha}} \downarrow$$



When there is the change in the depreciation rate output remains stable since  $k$  is unchanged. However since capital must be decumulated we start immediately dissaving by increasing consumption. When disinvestment is sufficiently high then consumption falls below the starting value  $c_{SS}$  until its new equilibrium value  $c_{SS}^*$  is attained. In the new steady state both consumption and capital will be at a lower level.