

Dynamic Macroeconomics

PhD Economics

Dynamic investment models (answers) - part 2

January 2024

PROBLEM 3.

a) Let

$$\begin{aligned}R(K) &= K - \frac{1}{2}K^2 \\ \delta &= 0.25 \\ G(I) &= I + \frac{1}{2}I^2 \\ r &= 0.25\end{aligned}$$

and assume

$$P_k = 1 \quad \text{constant}$$

The firm solves the following general dynamic optimization problem:

$$\begin{aligned}\max_{\{I_t, K_t, N_t\}_{t=0}^{\infty}} V(0) &\equiv \int_0^{\infty} [R(K_t, N_t, t) - P_k(t) \cdot G(I_t, K_t) - w_t N_t] e^{-rt} \\ \text{s.t. } \dot{K}(t) &= I_t - \delta K \\ K(0) &= K_0, \text{ given} \\ \lim_{t \rightarrow \infty} \lambda(t) \cdot K(t) e^{-rt} &= 0\end{aligned}$$

The Hamiltonian function associated to this problem:

$$H_t = \{[R(K_t, N_t, t) - P_k(t) \cdot G(I_t, K_t) - w_t N_t] + \lambda_t [I_t - \delta K_t]\} e^{-rt}$$

where λ_t is in current value terms. The f.o.c. are:

$$\frac{\partial H}{\partial I} = 0 \implies [-P_k(t) \cdot \frac{\partial G(\cdot)}{\partial I} + \lambda(t)] e^{-rt} = 0 \implies P_k(t) \frac{\partial G(\cdot)}{\partial I} = \lambda(t) \quad (1)$$

(there is no labor in this problem, so we omit the condition $\frac{\partial H}{\partial N} = 0$)

$$\begin{aligned}\frac{\partial H}{\partial K} &= -\frac{\partial}{\partial t}[\lambda(t)e^{-rt}] \\ \implies r\lambda_t &= \dot{\lambda}_t + \frac{\partial R(\cdot)}{\partial K} - P_k(t)\frac{\partial G(\cdot)}{\partial K} - \delta\lambda(t)\end{aligned}\quad (2)$$

$$\begin{aligned}\frac{\partial H}{\partial \lambda} &= \dot{K}(t)e^{-rt} \\ \dot{K}(t) &= I(t) - \delta K(t)\end{aligned}\quad (3)$$

and

$$\lim_{t \rightarrow \infty} \lambda(t) \cdot K(t)e^{-rt} = 0$$

Using the functional forms proposed in the problem, we get:

$$\begin{aligned}\frac{\partial G(\cdot)}{\partial I} &= 1 + I \\ \frac{\partial G(\cdot)}{\partial K} &= 0 \\ \frac{\partial R(\cdot)}{\partial K} &= 1 - K\end{aligned}$$

and the f.o.c. become:

$$\begin{aligned}1 + I &= \lambda \quad \text{assuming } P_k = 1 \\ r\lambda &= \dot{\lambda} + (1 - K - \delta\lambda)\end{aligned}\quad (4)$$

Define

$$q(t) = \lambda(t) \implies \dot{q}(t) = \dot{\lambda}(t) \quad \text{given } P_k(t) = 1$$

and call $\iota(\cdot)$ the inverse of

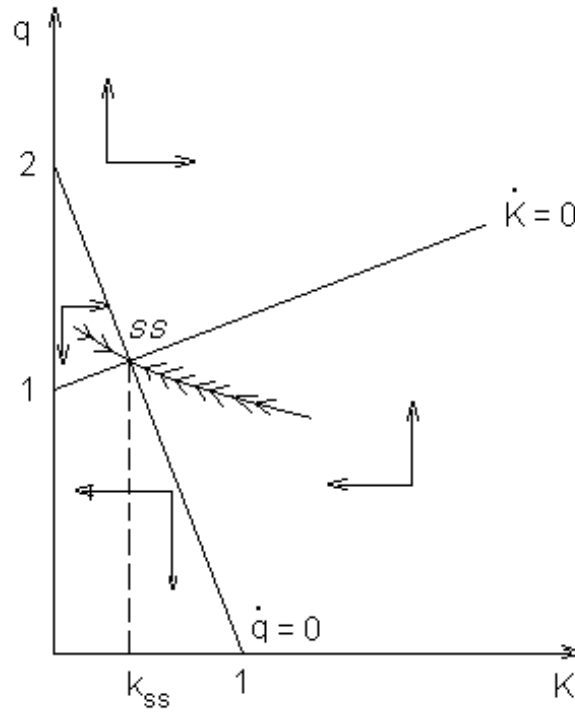
$$\begin{aligned}\frac{\partial G(\cdot)}{\partial I} &= 1 + I = q \\ \implies I &= q - 1 \equiv \iota(q)\end{aligned}$$

Plugging $\iota(\cdot)$ into the accumulation constraint:

$$\begin{aligned}\dot{K}(t) &= I(t) - \delta K(t) = q(t) - 1 - \delta K(t) \\ &= q(t) - 1 - 0.25K(t)\end{aligned}$$

From (4):

$$\begin{aligned}\dot{q}(t) &= (r + \delta)q(t) - (1 - K(t)) \\ &= 0.5q(t) - (1 - K(t))\end{aligned}$$



The $\dot{K} = 0$ locus is then:

$$\dot{K} = 0 : \quad q = 1 + 0.25K$$

and the $\dot{q} = 0$ locus:

$$\dot{q} = 0 : \quad q = 2(1 - K)$$

b) Now let

$$P_k = \frac{1}{2}$$

and see what happens to the $\dot{q} = 0$ and $\dot{K} = 0$ loci. The f.o.c. become:

$$\begin{aligned} \frac{1}{2}(1 + I) &= \lambda && \text{this condition changes (in terms of } \lambda) \\ r\lambda &= \dot{\lambda} + (1 - K - \delta\lambda) && \text{this condition does not change (in terms of } \lambda) \end{aligned} \quad (5)$$

Then, define:

$$\begin{aligned} q(t) &= \frac{\lambda(t)}{P_k(t)} = 2\lambda(t) \implies \dot{q}(t) \equiv 2\dot{\lambda}(t) \\ \frac{\partial G(\cdot)}{\partial I} &= \frac{\lambda(t)}{P_k(t)} \implies 1 + I = q(t) \implies I = q(t) - 1 \equiv i(q) \end{aligned}$$

$$\begin{aligned}\dot{K}(t) &= I(t) - \delta K(t) = q(t) - 1 - \delta K(t) \\ &= q(t) - 1 - 0.25K(t)\end{aligned}$$

- Notice that $\iota(\cdot)$ and $\dot{K}(t)$ do not change with respect to the previous situation.(in terms of $q(t)$). From (2):

$$\begin{aligned}\dot{q} &= (r + \delta)q - 2(1 - K) \\ &= 0.5q - 2(1 - K)\end{aligned}$$

with:

$$r + \delta = 0.5; \quad \lambda = \frac{q}{2}; \quad \dot{\lambda} = \frac{\dot{q}}{2}$$

- Notice also that \dot{q} has changed with respect to the previous situation

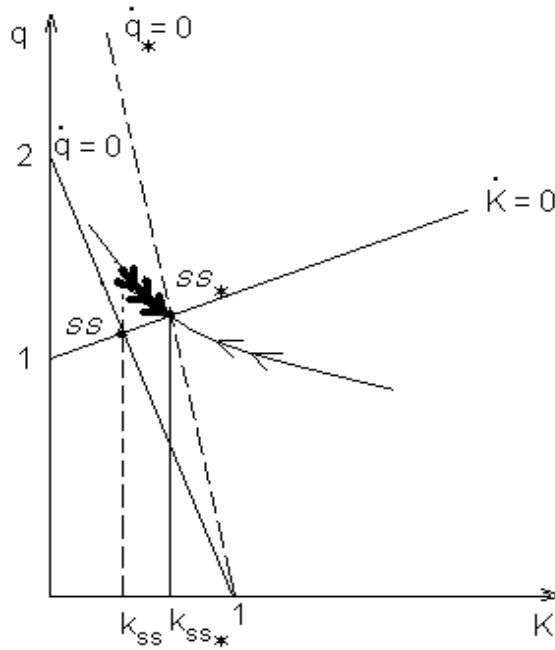
The $\dot{K} = 0$ locus does not change:

$$\dot{K} = 0 : \quad q = 1 + 0.25K$$

whereas the $\dot{q} = 0$ locus changes:

$$\dot{q} = 0 : \quad q = 4(1 - K)$$

in psrticular, the $\dot{q} = 0$ schedule rotates clockwise around its intersection with the horizontal axis and q jumps onto the new saddle path (if the reduction in P_k is permanent).



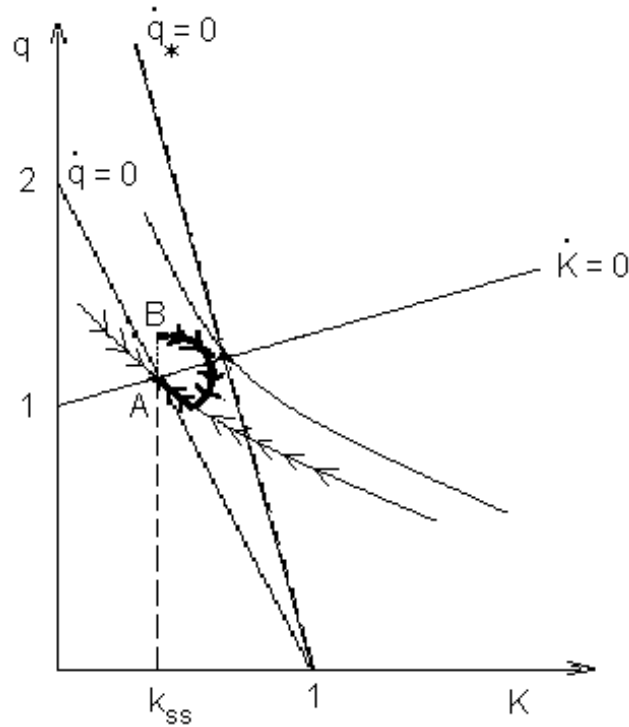
c) At time $t = 0$, when P_k is halved, it is also announced that at some future time $T > 0$ the interest rate will be tripled, so that $r(t) = 0.75$ for $t \geq T$.

Consider again the $\dot{q} = 0$ and $\dot{K} = 0$ loci of part (b) of the problem. The $\dot{K} = 0$ locus does not change:

$$\begin{aligned}\dot{K}(t) &= I(t) - \delta K(t) = q(t) - 1 - \delta K(t) \\ &= q(t) - 1 - 0.25K(t) \\ \dot{K} &= 0 : \quad q = 1 + 0.25K\end{aligned}$$

whereas the $\dot{q} = 0$ locus changes

$$\begin{aligned}\dot{q} &= (r + \delta)q - 2(1 - K) \\ \dot{q} &= 0 : \quad q = \frac{2(1 - K)}{r + \delta} = 2(1 - K), \text{ with } r + \delta = 1\end{aligned}$$



The only locus to change is the $\dot{q} = 0$ locus. From T onwards, the $\dot{q} = 0$ locus returns to its original position (the combination of the subsidy and higher interest rate exactly offset each other) When P_k is halved the system jumps from A to B. The magnitude of the jump depends on how far in the future is T (if $T \rightarrow 0$, there is no movement away from A) If at $t = 0$, when P_k is halved, it is also announced that there will be an increase in the interest rate (r), q jumps but does not reach

the saddle path (of the previous case). Its trajectory reaches and crosses the $\dot{K} = 0$ locus (and would diverge if parameters did not change again at T). At time T , the original saddle path is met, and the trajectory converges back to its starting point. (Intuitively, the firm finds it convenient to dilute over time the adjustment it foresees). The further in the future is T , the longer lasting is the investment increase. In the limit, as T goes to infinity, the trajectory tends to coincide with the saddle path of the previous case (case (b)), at least initially.

PROBLEM 4.

a) The cash flow of the firm IS:

$$\begin{aligned} F(t) &= R(t, K(t), N(t)) - P_k(t) \cdot G(I(t), K(t)) - w(t)N(t) \\ &= 2K^{\frac{1}{2}}N^{\frac{1}{2}} - I - \frac{1}{2}I^2 - wN \end{aligned}$$

The firm solves the problem:

$$\begin{aligned} \max_{\{I_t, K_t, N_t\}_{t=0}^{\infty}} V(0) &\equiv \int_0^{\infty} e^{-rt} F(t) dt = \int_0^{\infty} e^{-rt} [2K^{\frac{1}{2}}N^{\frac{1}{2}} - I - \frac{1}{2}I^2 - wN] dt \\ s.t. \quad \dot{K}(t) &= I_t - \delta K \\ \lim_{t \rightarrow \infty} \lambda(t) \cdot K(t) e^{-rt} &= 0 \quad \text{and} \quad K(0) = K_0, \text{ given} \end{aligned}$$

with the associated Hamiltonian function:

$$H_t = \left\{ [2K^{\frac{1}{2}}N^{\frac{1}{2}} - I - \frac{1}{2}I^2 - wN] + \lambda[I - \delta K] \right\} e^{-rt}$$

The f.o.c. are:

$$\begin{aligned} \frac{\partial H}{\partial N} &= \frac{\partial R(\cdot)}{\partial N} - w = 0 \implies K^{\frac{1}{2}}N^{-\frac{1}{2}} = w \\ \frac{\partial H_t}{\partial I_t} &= 0 \implies 1 + I = \lambda \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{\partial H}{\partial K} &= -\frac{\partial}{\partial t} [\lambda(t) e^{-rt}] \\ \implies [K^{-\frac{1}{2}}N^{\frac{1}{2}} - \delta\lambda] e^{-rt} &= [-\dot{\lambda} + r\lambda] e^{-rt} \\ \implies K^{-\frac{1}{2}}N^{\frac{1}{2}} - \delta\lambda &= -\dot{\lambda} + r\lambda \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{\partial H}{\partial \lambda} &= \dot{K} e^{-rt} \\ \implies [I - \delta K] e^{-rt} &= \dot{K}(t) e^{-rt} \\ \implies \dot{K} &= I - \delta K \end{aligned} \tag{8}$$

$$\lim_{t \rightarrow \infty} \lambda(t) \cdot K(t) e^{-rt} = 0, \quad K(0) = K_0 \text{ given}$$

b) From the f.o.c. (6) we get:

$$N = \frac{K}{w^2}$$

Hence:

$$\implies \frac{\partial F(t)}{\partial K(t)} = K^{-\frac{1}{2}} N^{\frac{1}{2}} = \frac{1}{w(t)}$$

Recall that:

$$\lambda(0) = \int_0^{\infty} e^{-(r+\delta)t} \frac{\partial R(\cdot)}{\partial K(t)} dt$$

This means that the shadow value of capital at time $t = 0$ ($\lambda(0)$) is equal to the discounted value of all the marginal contributions of the capital stock to the firm's cash flows from now ($t = 0$) to infinity. Discounting is done at the rate $(r + \delta)$, to take into account the depreciation (δ). In our case

$$\lambda(0) = \int_0^{\infty} e^{-(r+\delta)t} \frac{1}{w(t)} dt$$

c) If

$$w(t) = \bar{w} = \text{constant}$$

$$\lambda(0) = \frac{1}{(r + \delta)\bar{w}} = \lambda(t) = \lambda \quad \forall t$$

In other words λ is constant (being independent of time) and does not depend on K . As usual, let us define

$$q \equiv \frac{\lambda}{P_k} \implies q = \lambda$$

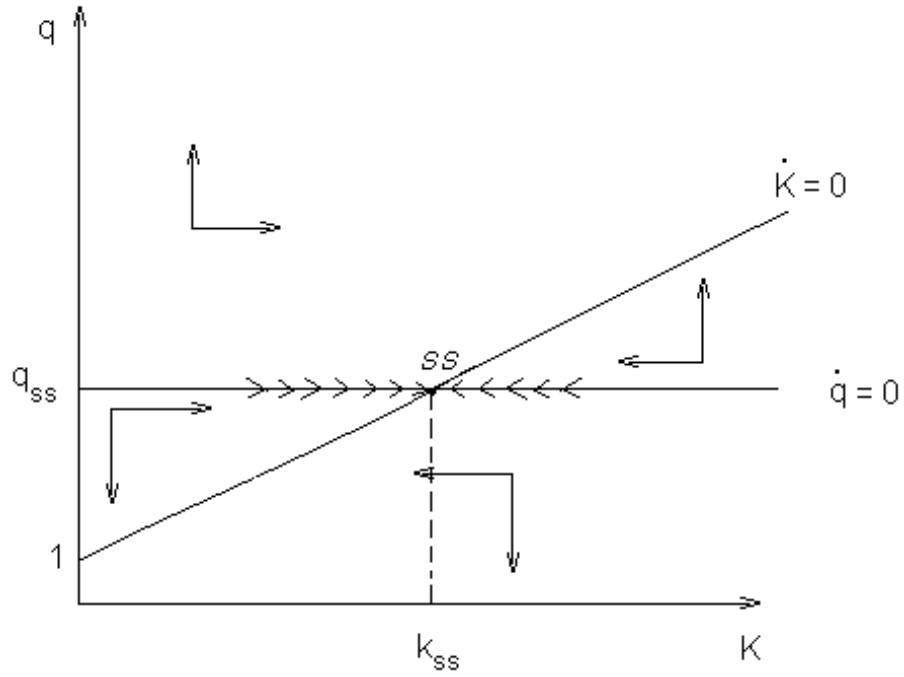
$$\frac{\partial G(\cdot)}{\partial I} = 1 + I = q \implies I = q - 1 \equiv \iota(q)$$

where $\iota(q)$ is the inverse of $\frac{\partial G(\cdot)}{\partial I} = q$. Plug $\iota(\cdot)$ into the accumulation constraint to get:

$$\dot{K}(t) = q - 1 - \delta K(t)$$

The dynamic equation for q then becomes

$$\dot{q} = (r + \delta)q - K^{-\frac{1}{2}} N^{\frac{1}{2}} = (r + \delta)q - \frac{1}{\bar{w}}$$



The two stationary loci are: \Rightarrow

$$\begin{aligned}\dot{K} = 0 & : & q &= 1 + \delta K \\ \dot{q} = 0 & : & q &= \frac{1}{r + \delta} \frac{1}{\bar{w}}\end{aligned}$$

Note: In this case, the saddle path coincides with

$$q = \lambda = \frac{1}{(r + \delta)\bar{w}}$$

and

$$\begin{aligned}1 + \delta K_{ss} &= \frac{1}{(r + \delta)\bar{w}} \\ \Rightarrow K_{ss} &= \left[\frac{1}{(r + \delta)\bar{w}} - 1 \right] \frac{1}{\delta}\end{aligned}$$

d) For investment to be a function of the average value of capital one should be sure that $R(\cdot)$ and $G(\cdot)$ are linearly homogeneous in N, K and I . Let us check:

$$R(K, N) = 2K^{1/2}N^{1/2}$$

is linearly homogeneous since

$$R(\lambda K, \lambda N) = \lambda R(K, N)$$

The function $G(I)$ is not homogeneous of degree one in I . If the $G(\cdot)$ function were, instead, modified in the following way

$$G(K, I) = I + \frac{I^2}{2K}$$

then it would be linearly homogeneous in (I, K) , in fact

$$G(\lambda K, \lambda I) = \lambda G(K, I)$$

In this case, investment would be a function of the average value of capital (Tobin's *average* q)