Dynamic Macroeconomics
PhD Economics
Dynamic investment models (answers) - part 1

January 2024

PROBLEM 1.

a) The firm's cash flow in t can be written as:

$$F(t) = R(t, K(t), N(t)) - P_k(t) \cdot G(I(t), K(t)) - w(t)N_t$$

When capital is the only factor of production, investment costs depend only on the flow of investment $I(t)$ and $P_k = 1\ \forall t$, we have

$$F(t) = R(K(t)) - G(I(t))$$

The firm's optimization problem then becomes:

$$\begin{aligned}
Max\ V(0) &\equiv \int_0^\infty e^{-rt}F(t)dt = \int_0^\infty e^{-rt}[R(K(t)) - G(I(t))]dt \\
s.t.\ \dot{K}(t) &= I(t) - \delta K(t) \\
K(0) &= K_0,\ \text{given} \\
\lim_{t\to\infty} \lambda(t) \cdot K(t)e^{-rt} &= 0 \text{ (transversality condition for infinite horizon problems)}
\end{aligned}$$

The Hamiltonian function corresponding to this optimization problem is:

$$H(t) = \{[R(K(t)) - G(I(t))] + \lambda(t)[I(t) - \delta K(t)]\}\, e^{-rt}$$

where $\lambda(t)$ is the shadow price of capital at time t in current value terms, $I(t)$ is the *control* variable, and $K(t)$ is the *state* variable. The f.o.c. for this problem are:

1. control variable

$$\frac{\partial H_t}{\partial I_t} = 0 \Longrightarrow [-G'(I(t)) + \lambda(t)]e^{-rt} = 0 \Longrightarrow G'(I(t)) = \lambda(t) \qquad (1)$$

2. state variable

$$\begin{aligned}
\frac{\partial H_t}{\partial K_t} &= -\frac{\partial}{\partial t}[\lambda(t)e^{-rt}] \\
&\Longrightarrow [R'(K(t)) - \delta\lambda(t)]e^{-rt} = [-\dot{\lambda}(t) + r\lambda(t)]e^{-rt} \\
&\Longrightarrow r\lambda(t) = \dot{\lambda}(t) - \delta\lambda(t) + R'(K(t)) \qquad (2)
\end{aligned}$$

3. costate variable in current value terms ($\lambda(t)$)

$$\begin{aligned}
\frac{\partial H_t}{\partial \lambda_t} &= \dot{K}(t)e^{-rt} \\
&\Longrightarrow [I(t) - \delta K(t)]e^{-rt} = \dot{K}(t)e^{-rt} \\
&\Longrightarrow \dot{K}(t) = I(t) - \delta K(t) \qquad (3)
\end{aligned}$$

4. transversality condition

$$\lim_{t\to\infty} \lambda(t) \cdot K(t)e^{-rt} = 0, \quad K(0) = K_0 \text{ given} \qquad (4)$$

Concavity of the objective function subject to a linear constraint ensures that the problem has a unique internal solution identified by the first order conditions, with the second-order condition surely satisfied. Since the constraint is linear, we just need to make assumption relative to the concavity of the objective function

$$F(\cdot) = R(K) - G(I)$$

$F(\cdot)$ is concave in K if:

$$\frac{\partial^2 F}{\partial K^2} = \frac{\partial^2 R}{\partial K^2} \leq 0$$

$F(\cdot)$ is concave in I if:

$$\frac{\partial^2 F}{\partial I^2} = -\frac{\partial^2 G}{\partial I^2} \leq 0$$

Therefore, in our case the f.o.c. are also sufficient if:

$$\begin{aligned}
\frac{\partial^2 G}{\partial I^2} &\geqslant 0 \\
\frac{\partial^2 R}{\partial K^2} &\leq 0
\end{aligned}$$

Now define:

$$q(t) = \frac{\lambda(t)}{P_k(t)}$$

Since $P_k = 1$, $q(t)$ is equal to $\lambda(t)$ and $\dot{q}(t) \equiv \dot{\lambda}(t)$. Call $i(\cdot)$ the inverse of:

$$\frac{\partial G(\cdot)}{\partial I} = \lambda = q \qquad \text{(from (1))} \qquad (5)$$

If G is not a function of K, $i(\cdot)$ will be a function of q (only):

$$i(\cdot) = i(q) = i(\lambda)$$

Example:

$$\begin{aligned}
G(I) &= I^2 \\
\frac{\partial G(\cdot)}{\partial I} &= 2I > 0 \Longrightarrow \frac{\partial G(\cdot)}{\partial I} = 2I = q \Longrightarrow I = \frac{q}{2} \\
\frac{\partial^2 G(\cdot)}{\partial I^2} &= 2 > 0
\end{aligned}$$

Plugging $I = i(q)$ into the accumulation constraint (3) yields:

$$\dot{K} = i(q) - \delta K$$

Since

$$\begin{aligned}
\lambda &= q \text{ and } \quad \dot{q} = \dot{\lambda}, \\
\dot{q} &= (r+\delta)q - R'(K) \quad \text{(condition (2))}
\end{aligned}$$

using the f.o.c. Then, we have obtained a system of two differential equations:

$$\begin{aligned}
\dot{K} &= i(q) - \delta K \\
\dot{q} &= (r+\delta)q - R'(K)
\end{aligned}$$

The dynamics of K and q can be studied using a phase diagram with q on the vertical axis and K on the horizontal axis. The locus where $\dot{q} = 0$ is:

$$\begin{aligned}
\dot{q} = 0 \ : \qquad & R'(K) = (r+\delta)q \\
\Longrightarrow \qquad & q = \frac{1}{r+\delta} \cdot R'(K)
\end{aligned}$$

This locus is negatively sloped if $R''(K) < 0$:

$$\left.\frac{\partial q}{\partial K}\right|_{\dot{q}=0} = \frac{1}{r+\delta} \cdot R''(K) < 0$$

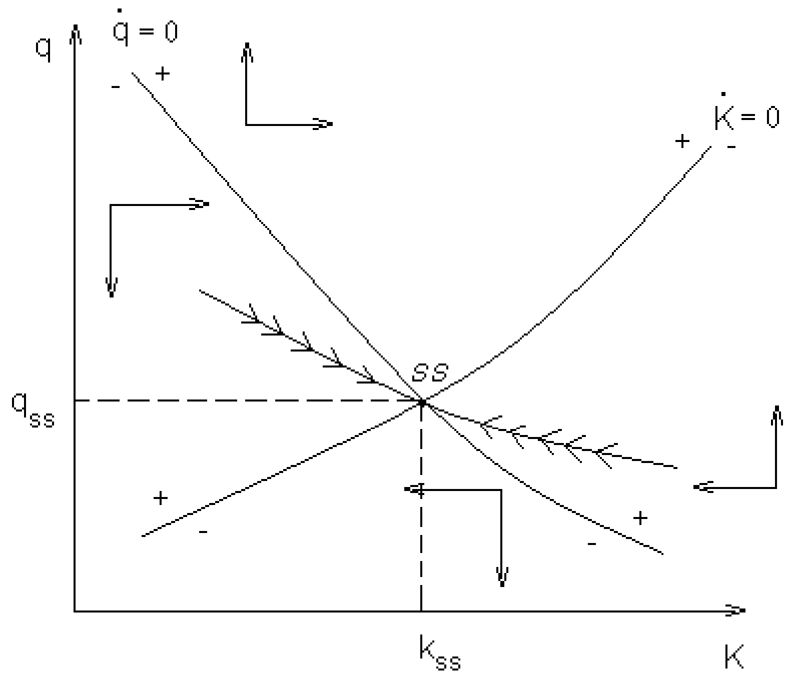
The stationary locus for K is derived as:

$$\dot{K} = 0 \qquad \qquad \iota(q) = \delta K$$

This locus is positively sloped if $\delta > 0$ (since $\iota' > 0$):*Example:*

$$\left.\frac{\partial q}{\partial K}\right|_{\dot{K}=0} = \frac{\delta}{\iota'} > 0$$

The point where the two loci cross each other identifies the steady-state, and the system converges towards it along a negatively sloped saddle path.



b) Consider now the following functional forms for $R(K)$ and $G(I)$:

$$\begin{aligned} R(K) &= \alpha K \\ G(I) &= I + bI^2 \end{aligned}$$

that yield

$$\begin{aligned} F'(K) &= R'(K) = \alpha; \quad F''(K) = R''(K) = 0 \\ G'(I) &= 1 + 2bI; \; G''(I) = 2b \end{aligned}$$

Therefore the f.o.c. are also sufficient if $b > 0$. Substitute the f.o.c. for I (5):

$$\begin{aligned} G'(I) &= 1 + 2bI = q \\ I &= \frac{q-1}{2b} \equiv i(q) \end{aligned}$$

The dynamic equations of the system are therefore:

$$\begin{aligned} \dot{K} &= \frac{q-1}{2b} - \delta K \\ \dot{q} &= (r+\delta)q - \alpha \end{aligned}$$

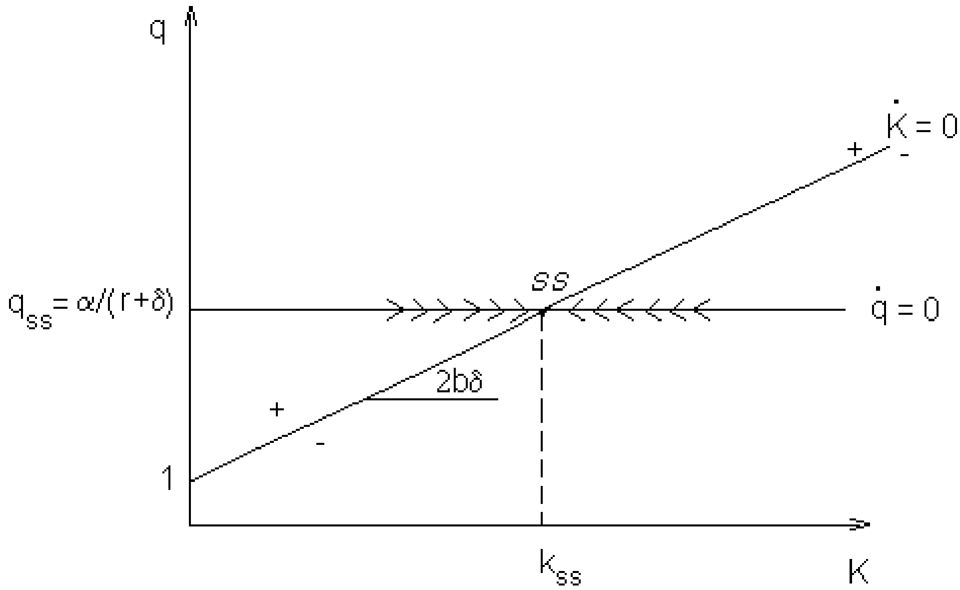
The $\dot{K} = 0$ locus is:

$$\dot{K} = 0 \ : \qquad\qquad q = 1 + 2b\delta K \ \ \text{(positively sloped if } \delta > 0\text{)}$$

and the $\dot{q} = 0$ locus*:*

$$\dot{q} = 0 \ : \qquad q = \frac{\alpha}{r + \delta}$$

does not depend on K (q is also independent of time).



The $\dot{q} = 0$ locus identifies a horizontal line. In the steady-state the shadow price of capital ($\lambda = q$) is constant and equal to the marginal present discounted (at rate $r + \delta$) contribution of capital to the firm's cash flow ($F'(K) = \alpha$). The saddlepath coincides here with the $\dot{q} = 0$ locus. The system must be on this path throughout its convergent trajectory.

In steady-state, imposing $\dot{K} = 0$, we have:

$$1 + 2b\delta K = \frac{\alpha}{r + \delta} \Longrightarrow K_{ss} = \frac{\alpha - (r + \delta)}{(r + \delta)2b\delta} \tag{6}$$

So $Kss > 0$ iff $\alpha > (r + \delta)$. The firm's capital stock is an increasing function of the difference between α (the marginal revenue product of capital) and $r+\delta$ (the financial and depreciation cost of each installed unit of capital). If $\alpha > (r + \delta)$ the steady state capital stock is positive provided that $b\delta > 0$. If $\alpha < (r + \delta)$ revenues afforded by capital are smaller than its opportunity cost and it's never optimal to invest.

If $\delta \to 0$, the $\dot{K} = 0$ locus is horizontal (likewise the $\dot{q} = 0$ locus) and the steady-state is ill-defined. Equation (6) above implies that:

$$\begin{aligned} Kss &\to +\infty, \quad if \quad \alpha > r \\ Kss &\to -\infty, \quad if \quad \alpha < r \end{aligned}$$

($Kss \to 0$ imposing an obvious non-negativity constraint)

$$Kss \text{ is undetermined if } \alpha = r$$

PROBLEM 2. Let

$$\begin{aligned} Y(t) &= \alpha\sqrt{K(t)} + \beta\sqrt{L(t)} \\ G(I) &= I + \frac{\gamma}{2}I^2 \\ P_y &= 1, \ P_x = 1 \quad \text{(given)} \end{aligned}$$

The revenue that the firm gets from selling output is:

$$R(t, K(t), L(t)) = P_y(t) * Y(t) = \alpha\sqrt{K(t)} + \beta\sqrt{L(t)} \qquad \text{(given } P_y = 1)$$

The firm's cash flow is:

$$\begin{aligned} F(t) &= R(t, K(t), L(t)) - P_k(t) \cdot G(I(t), K(t)) - w(t)L(t) \\ &= \alpha\sqrt{K(t)} + \beta\sqrt{L(t)} - [I(t) + \frac{\gamma}{2}I^2(t)] - w(t) \cdot L(t) \end{aligned}$$

a) Set up the Hamiltonian function:

$$H(t) = \left\{\alpha\sqrt{K(t)} + \beta\sqrt{L(t)} - I(t) - \frac{\gamma}{2}I(t)^2 - w(t) \cdot L(t) + \lambda(t) \cdot [I(t) - \delta K(t)]\right\} e^{-rt}$$

with control variables: $L(t)$, $I(t)$, state variable: $K(t)$, co-state variable in current value terms $\lambda(t)$. Now write down the f.o.c. of the optimization problem:

1. control variables (I and L) :

$$\begin{aligned} \frac{\partial H_t}{\partial I_t} &= 0 \Longrightarrow (\underbrace{-1 - \gamma I(t)}_{-\frac{\partial G(\cdot)}{\partial I}} + \lambda)e^{-rt} = 0 \\ &\Longrightarrow 1 + \gamma I(t) = \lambda(t) \end{aligned}$$

The marginal investment cost $(1 + \gamma I(t))$ should be equal to the shadow price of capital.

$$\begin{aligned} \frac{\partial H_t}{\partial L_t} &= 0 \Longrightarrow (\underbrace{\frac{\beta}{2\sqrt{L}}}_{\frac{\partial Y}{\partial L}} - w)e^{-rt} = 0 \\ &\Longrightarrow \frac{\beta}{2\sqrt{L}} = w \end{aligned}$$

The marginal revenue product of labor should be equal to the wage rate.

2. state variable

$$\begin{aligned}
\frac{\partial H_t}{\partial K_t} &= -\frac{\partial}{\partial t}[\lambda(t)e^{-rt}] = 0 \\
&\Longrightarrow \left[\frac{\alpha}{2\sqrt{K}} - \lambda\delta\right] e^{-rt} = [-\dot{\lambda} + r\lambda]e^{-rt} \\
&\Longrightarrow \dot{\lambda} - r\lambda = \lambda\delta - \frac{\alpha}{2\sqrt{K}} \qquad (7)
\end{aligned}$$

In other terms: marginal revenue product of capital$\left(\frac{\alpha}{2\sqrt{K}}\right)$ − depreciation costs $(\lambda\delta)$+ capital gains $(\dot{\lambda})$ = opportunity cost of funds $(r\lambda)$.

3. costate variable in current value terms $(\lambda(t))$

$$\begin{aligned}
\frac{\partial H_t}{\partial \lambda_t} &= \dot{K}e^{-rt} = 0 \\
&\Longrightarrow \dot{K} = I - \delta K
\end{aligned}$$

represents the law of motion of capital.

4. transversality condition

$$\lim_{t\to\infty} e^{-rt}\lambda(t)\cdot K(t) = 0 \qquad \text{Transversality condition}$$

The above f.o.c. are necessary and sufficient for the global maximisation of the objective function. Indeed, the constraint is linear and it is easy to check that:

$$\frac{\partial^2 F(\cdot)}{\partial L^2} = -\frac{1}{4}\frac{\beta}{L\sqrt{L}} < 0$$

$$\frac{\partial^2 F(\cdot)}{\partial I^2} = -\gamma < 0, \quad \text{for } \gamma > 0$$

$$\frac{\partial^2 F(\cdot)}{\partial K^2} = -\frac{1}{4}\frac{\alpha}{K\sqrt{K}} < 0$$

so the $F(\cdot)$ is concave in L, K and I.
Now define:

$$q(t) \equiv \lambda(t) \Longrightarrow \dot{q}(t) \equiv \dot{\lambda}(t) \quad \text{as } P_k = 1 \text{ by hypothesis}$$

and call $\iota(\cdot)$ the inverse of

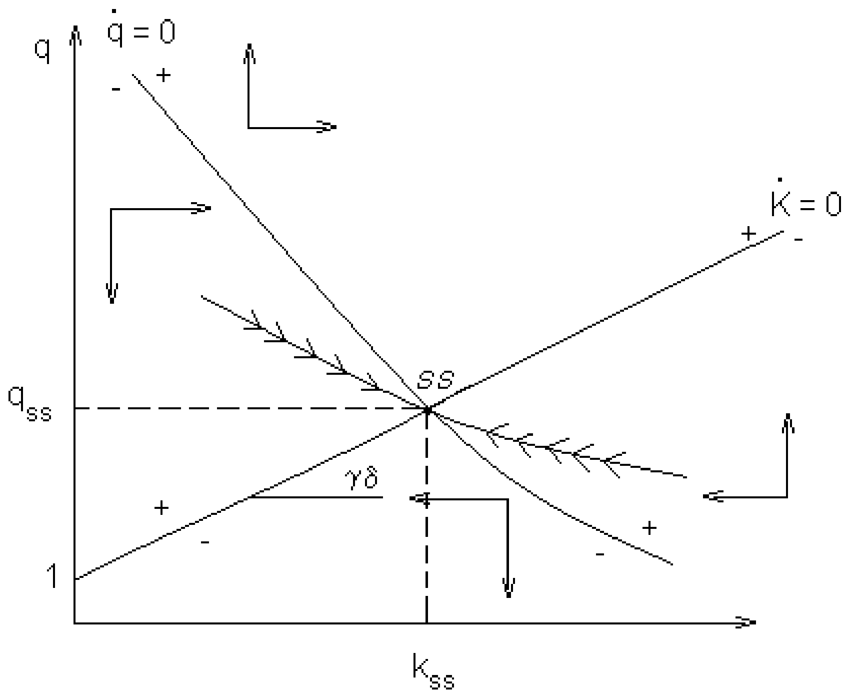
$$\begin{aligned}
\frac{\partial G(\cdot)}{\partial I} &= 1 + \gamma I = q \\
&\Longrightarrow I = \frac{q-1}{\gamma} \equiv \iota(q) = \frac{\lambda - 1}{\gamma}
\end{aligned}$$

Insert $I(\cdot)$ in the accumulation constraint to get:

$$\dot{K}(t) = \frac{q(t)-1}{\gamma} - \delta K(t)$$

From the third f.o.c. (eq. (7)) we get:

$$\dot{\lambda}(t) = \dot{q}(t) = (r+\delta)q(t) - \frac{\alpha}{2\sqrt{K(t)}}$$



The $\dot{K} = 0$ locus is obtained as

$$\dot{K} = 0 \ : \qquad \lambda = q = \delta\gamma K + 1$$

and the $\dot{q} = 0$ locus as:

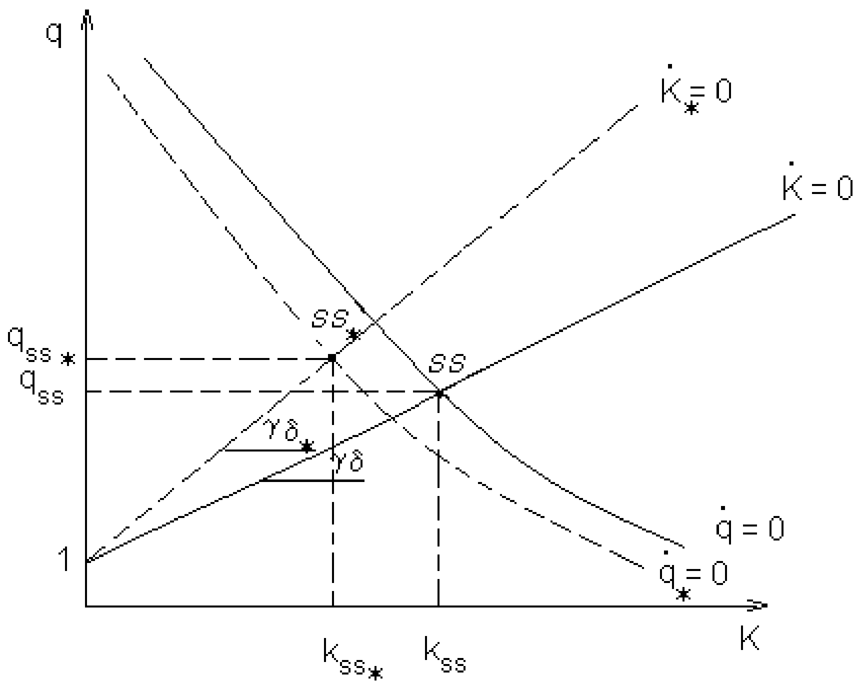
$$\dot{q} = 0 \ : \qquad q(t) = \frac{\alpha}{2(r+\delta)\sqrt{K(t)}}$$

The steady state level of the capital stock K_{ss} is such that:

$$\delta\gamma K_{ss} + 1 = \frac{\alpha}{2(r+\delta)\sqrt{K_{ss}}}$$

b) Effects of an increase in δ :

- the $\dot{q} = 0$ locus shifts downwards;
- the $\dot{K} = 0$ locus rotates upwards maintaing the same vertical intercept.



In the new steady-state the capital stock is unambiguously smaller ($K_{ss*} < K_{ss}$). Intuitively, a higher marginal revenue product is needed to offset the large cost of a higher replacement investment flow. The effect on capital's shadow price ($\lambda = q$) is ambiguous. It depends on the slope of the two curves in the relevant region. Recall that λ is the present discounted value of the capital's contribution to the firm's revenues ($F'(K) = \frac{\alpha}{2\sqrt{K}}$ and the discount factor is $\frac{1}{r+\delta}$) : in the new steady-state $F'(K_{ss*})$ is larger but it is more heavily discounted (at the rate $r + \delta$).