Dynamic Macroeconomics - PhD Economics
Dynamic consumption theory (answers) - part 2

September 2024

PROBLEM 3. Recall that
$$c_t = r(H_t + A_t) \equiv y_t^P \tag{1}$$
where y_t^P is permanent income at time t; *i.e.*, consumption at time t is the *annuity value* of total wealth.

At time $t+1$:
$$y_{t+1}^P = r(A_{t+1} + H_{t+1}) \tag{2}$$
Evaluate the value of y_{t+1}^P expected at time t:
$$E_t y_{t+1}^P = rE_t A_{t+1} + rE_t H_{t+1} \tag{3}$$
Subtracting [3] from [2], we obtain:
$$\begin{aligned} y_{t+1}^P - E_t y_{t+1}^P &= \underbrace{rA_{t+1} - rE_t A_{t+1}}_{\text{zero}} + r(H_{t+1} - E_t H_{t+1}) \\ &= r \quad (\underbrace{H_{t+1} - E_t H_{t+1}}_{\text{"surprise" in human wealth}}) \end{aligned} \tag{4}$$

Indeed, from the period budget constraint we know that:
$$A_{t+1} = (1+r)A_t + y_t - c_t \tag{5}$$
Take the expected value of both sides:
$$E_t A_{t+1} = E_t[(1+r)A_t + y_t - c_t] = (1+r)A_t + y_t - c_t = A_{t+1} \tag{6}$$

Given the identity in (1) and ($E_t c_{t+1} = c_t$) (recall the implicit assumptions behind the martingale property of consumption: 1) $r = \rho$, 2) quadratic utility), we deduce that permanent income follows:
$$E_t y_{t+1}^P = y_t^P$$

which implies that:

$$y_{t+1}^{P}=y_{t}^{P}+r(H_{t+1}-E_{t}H_{t+1}) \tag{7}$$

>From the definition of H_{t+1} :

$$H_{t+1}=\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}E_{t+1}y_{t+1+i}$$

Substitute this expression into (7) :

$$\begin{aligned}
y_{t+1}^{P} &= y_{t}^{P}+r\left\{\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}E_{t+1}y_{t+1+i}-E_{t}\left[\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}E_{t+1}y_{t+1+i}\right]\right\} \\
&= y_{t}^{P}+r\left\{\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}E_{t+1}y_{t+1+i}-\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}E_{t}y_{t+1+i}\right\} \\
&= y_{t}^{P}+\underbrace{r\left\{\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}(E_{t+1}-E_{t})y_{t+1+i}\right\}}_{\equiv\mu_{t+1}} \qquad (8)
\end{aligned}$$

The "surprise" in human wealth in $t+1$ is expressed as the revision in expectations on future income.

In sum:

$$\begin{aligned}
\Delta c_{t+1} &= \Delta y_{t+1}^{p} \qquad (9) \\
&= r(H_{t+1}-E_{t}H_{t+1}) \\
&= \frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}(E_{t+1}-E_{t})y_{t+1+i}
\end{aligned}$$

Notice that the *change in consumption* between t and $t+1$ cannot be foreseen as of time t; it depends on information available at time $t+1$

a) Assume that income follows:

$$y_{t+1}=\overline{y}+\varepsilon_{t+1}-\delta\varepsilon_{t} \tag{10}$$

where: $\overline{y}$ is the mean income, $E_{t}\varepsilon_{t+1}=0$, and $\delta>0$.

Using (9) compute the effects on consumption of a change in mean income ($\Delta\overline{y}$):

$$\begin{aligned}
\Delta c_{t+1} &= \Delta y_{t+1}^{p}=\frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}(E_{t+1}-E_{t})y_{t+1+i} \qquad (11) \\
&= \frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}\Delta\overline{y} \\
&= \Delta\overline{y} \qquad (12)
\end{aligned}$$

The increase in mean income represents a variation in permanent income for the consumer. Therefore, both her permanent income and consumption increase (the size of the increase is $\Delta\overline{y}$). Since the variation in income is entirely permanent, savings do not change.

b) Note that the exercise assumes $\varepsilon_{t-i} = 0\ \forall i \geq 0$, and $\varepsilon_{t+i} = 0\ \forall i > 1$ (only $\varepsilon_{t+1} > 0$).

Consequently, we know that:

$$y_{t-i} = \overline{y},\ \ for\ i = 0, 1, ...\infty \tag{13}$$

$$\begin{aligned} y_{t+1} &= \overline{y} + \varepsilon_{t+1} - \delta\varepsilon_t \\ &= \overline{y} + \varepsilon_{t+1} \end{aligned}$$

$$\begin{aligned} y_{t+2} &= \overline{y} + \varepsilon_{t+2} - \delta\varepsilon_{t+1} \\ &= \overline{y} - \delta\varepsilon_{t+1} \end{aligned}$$

and for all $i \geq 2$

$$\begin{aligned} y_{t+1+i} &= \overline{y} + \varepsilon_{t+i} - \delta\varepsilon_{t+i-1} \\ &= \overline{y} \end{aligned}$$

To calculate (11) in this case, again we need to compute $(E_{t+1} - E_t)y_{t+1+i}$:

for $i = 0$:

$$\begin{aligned} (E_{t+1} - E_t)y_{t+1} &= E_{t+1}y_{t+1} - E_t y_{t+1} \\ &= (\overline{y} + \varepsilon_{t+1}) - \overline{y} \\ &= \varepsilon_{t+1} \end{aligned} \tag{14}$$

for $i = 1$:

$$\begin{aligned} (E_{t+1} - E_t)y_{t+2} &= (E_{t+1} - E_t)(\overline{y} - \delta\varepsilon_{t+1}) \\ &= (\overline{y} - \delta\varepsilon_{t+1}) - (\overline{y}) \\ &= -\delta\varepsilon_{t+1} \end{aligned} \tag{15}$$

for $i \geq 2$:

$$\begin{aligned} (E_{t+1} - E_t)y_{t+1+i} &= (E_{t+1} - E_t)(\overline{y} + \varepsilon_{t+1+i} - \delta\varepsilon_{t+i}) \\ &= (\overline{y} - \overline{y}) \\ &= 0 \end{aligned} \tag{16}$$

Now we can compute the change in consumption:

$$\begin{aligned}
\Delta c_{t+1} &= \frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}(E_{t+1}-E_t)y_{t+1+i} && (17)\\
&= \frac{r}{1+r}\left[\varepsilon_{t+1}-\left(\frac{1}{1+r}\right)\delta\varepsilon_{t+1}+\left(\frac{1}{1+r}\right)^{2}\cdot 0+...\right]\\
&= \frac{r}{1+r}\left[\varepsilon_{t+1}-\left(\frac{\delta}{1+r}\right)\varepsilon_{t+1}\right]\\
&= \frac{r(1+r-\delta)}{(1+r)^2}\varepsilon_{t+1}\\
&< \underbrace{\varepsilon_{t+1}}_{\text{(income innovation at time } t+1)} && (18)
\end{aligned}$$

- Consumption between t and $t+1$ increases *less than one-to-one* with current income. This result is consistent with the temporary nature of the income innovation.
- The *higher* is δ the *lower* is the variation in consumption. A *positive* income innovation at $t+1$ (ε_{t+1}) is compensated by a *negative* income variation ($-\delta\varepsilon_{t+1}$) in the next period!

c) The behavior of savings over time reflects the expectations about future variations in income:

For a given ε_{t+1} and using the expression for the stochastic process for income (10):

$$\begin{aligned}
\Delta y_{t+2} &= y_{t+2}-y_{t+1} && (19)\\
&= (\overline{y}-\delta\varepsilon_{t+1})-(\overline{y}+\varepsilon_{t+1})\\
&= -(1+\delta)\varepsilon_{t+1}
\end{aligned}$$

$$\begin{aligned}
\Delta y_{t+3} &= y_{t+3}-y_{t+2} && (20)\\
&= \overline{y}-(\overline{y}-\delta\varepsilon_{t+1})\\
&= \delta\varepsilon_{t+1}
\end{aligned}$$

$$\begin{aligned}
\Delta y_{t+4} &= y_{t+4}-y_{t+3} && (21)\\
&= \overline{y}-\overline{y}\\
&= 0
\end{aligned}$$

In general,

$$\Delta y_{t+1+i} = 0 \ \forall i \geq 3 \qquad (22)$$

After period $t+3$, income is not expected to vary any further.
Now let us have a look at savings

$$
\begin{aligned}
S_t &\equiv y_t^D - c_t = y_t^D - y_t^P = (y_t + rA_t) - r(A_t + H_t) = y_t - rH_t = \\
&= y_t - r\left[\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i}\right] = \frac{1}{1+r}y_t - \left(\frac{1}{1+r} - \left(\frac{1}{1+r}\right)^2\right)E_t y_{t+1} \\
&\quad - \left(\left(\frac{1}{1+r}\right)^2 - \left(\frac{1}{1+r}\right)^3\right)E_t y_{t+2} - ... = -\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i E_t \Delta y_{t+i}
\end{aligned}
$$

(Note that $\dfrac{r}{(1+r)^i} = \left(\dfrac{1}{1+r}\right)^{i-1} - \left(\dfrac{1}{1+r}\right)^i$)

The behavior of savings in $t+1$ and $t+2$ is, therefore, as follows:

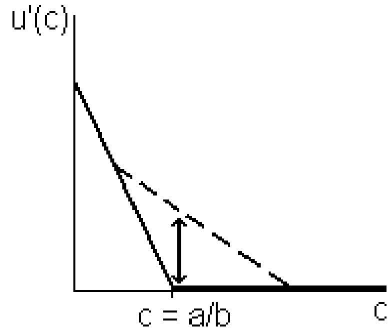
$$
\begin{aligned}
S_{t+1} &= -\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i E_{t+1}\Delta y_{t+1+i} \qquad (23) \\
&= -\left\{\left(\frac{1}{1+r}\right)\cdot(-(1+\delta)\varepsilon_{t+1}) + \left(\frac{1}{1+r}\right)^2 \delta\varepsilon_{t+1} + \left(\frac{1}{1+r}\right)^3 \cdot 0 + 0 + ... + 0\right\} \\
&= -\left(\frac{1}{1+r}\right)\left\{-(1+\delta)\varepsilon_{t+1} + \left(\frac{1}{1+r}\right)\delta\varepsilon_{t+1}\right\} \\
&= \left[\frac{1+r(1+\delta)}{(1+r)^2}\right]\varepsilon_{t+1} > 0
\end{aligned}
$$

$$
\begin{aligned}
S_{t+2} &= -\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i E_{t+2}\Delta y_{t+2+i} \qquad (24) \\
&= -\left\{\left(\frac{1}{1+r}\right)\cdot\delta\varepsilon_{t+1} + \left(\frac{1}{1+r}\right)^2 \cdot 0 + ... + 0\right\} \\
&= -\left(\frac{\delta}{1+r}\right)\varepsilon_{t+1} < 0
\end{aligned}
$$

- In $t+1$ part of the higher level of income is saved - since the consumer recognizes it's a temporary innovation. The agent forecasts further variations of income in the future ($y_{t+2} = \overline{y} - \delta\varepsilon_{t+1}$).
- In $t+2$, income is temporarily lower than its mean level ($y_{t+2} = \overline{y} - \delta\varepsilon_{t+1}$) and the agent consumes the savings accumulated in the previous period: in $t+2$ savings are *negative*!

EX 4. a) The marginal utility of consumption is given by:

$$
u'(c) = \left\{ \begin{array}{lll} a - bc & if & c < \frac{a}{b} \\ 0 & if & c \geq \frac{a}{b} \end{array} \right\}
$$



Graph the marginal utility function:

Note the *convexity* of the marginal utility function in the neighborhood of $c = \frac{a}{b}$, where it takes a zero value.

>From theory, recall that a convex marginal utility presumes a *precautionary motive for saving*; *i.e.,* the agent saves more in reaction to an increase in uncertainty (this reaction contrasts with the *certainty equivalence* property characterizing optimality in the case of a quadratic utility function).

b) Set up the expected-utility maximization problem faced by the agent:

$$\begin{aligned} \underset{\{c_{t+i},\ i=0,1...\}}{Max} \quad U_t &= E_t\left[\sum_{i=1}^{\infty}\left(\frac{1}{1+\rho}\right)^i u(c_{t+i})\right] \\ st \qquad A_{t+i+1} &= (1+r)A_{t+i} + y_{t+i} - c_{t+i} \qquad i = 0, 1, 2... \\ & \quad A_t \ given \end{aligned} \tag{25}$$

Solve the period budget constraint for c_{t+i} and plug the result into the objective function:

$$\underset{\{A_{t+i+1},\ i=0,1...\}}{Max} \quad U_t = E_t\left[\sum_{i=0}^{\infty}\left(\frac{1}{1+\rho}\right)^i u((1+r)A_{t+i} + y_{t+i} - A_{t+i+1})\right] \tag{26}$$

Since there are only two periods ($i = 0, 1$), rewrite (26) as:

$$U_t = E_t\left[u\left(\underbrace{(1+r)A_t + y_t - A_{t+1}}_{c_t}\right) + \left(\frac{1}{1+\rho}\right)u\left(\underbrace{(1+r)A_{t+1} + y_{t+1} - A_{t+2}}_{c_{t+1}}\right)\right] \tag{27}$$

Derive U_t with respect to A_{t+1} and set $\frac{\partial Ut}{\partial A_{t+1}} = 0$:

foc:

$$E_t\left[-u'(c_t)+\left(\frac{1}{1+\rho}\right)\cdot u'(c_{t+1})(1+r)\right]=0 \tag{28}$$

Rearranging this equation we obtain:

$$E_t\left[\left(\frac{1+r}{1+\rho}\right)\cdot u'(c_{t+1})\right]=u'(c_t) \tag{29}$$

or

$$\left(\frac{1+r}{1+\rho}\right)E_t u'(c_{t+1})=u'(c_t) \quad \textit{EULER EQUATION}$$

Assume $r=\rho=0$ and denote $t=1$ and $t+1=2$:

$$u'(c_1)=E_1\left[u'(c_2)\right] \tag{30}$$

(This is the required first order optimality condition.)

As stated in the exercise $y_1=\frac{a}{b}$ is known with certainty.

1. If $\sigma=0$,

$$y_1=y_2=\frac{a}{b}\equiv y \tag{31}$$

>From the marginal utility function, when
$c_1\geq y_1\equiv y=\frac{a}{b}\Longrightarrow u'(c_1)=0$
$c_2\geq y_2\equiv y=\frac{a}{b}\Longrightarrow E_1u'(c_2)=0$
Then

$$u'(c_1)=E_1\left[u'(c_2)\right]=0 \tag{32}$$

when $c_1=c_2=y=\frac{a}{b}$ is an equilibrium
Thus, the first order optimality condition is satisfied when consumption is equal to current income in each period and savings are equal to zero:

$$c_1=c_2=y=\frac{a}{b} \textit{ is an equilibrium} \tag{33}$$

2) Now suppose that $\sigma>0$ and $c_1=\frac{a}{b}$.

Income in period two is:

$$y_2=\left\{\begin{array}{l}\frac{a}{b}+\sigma=c_1+\sigma>c_1 \textit{ with probability } 1/2\\ \frac{a}{b}-\sigma=c_1-\sigma<c_1 \textit{ with probability } 1/2\end{array}\right\}$$

Continue to assume that total income is consumed in both periods (as in part 1):

$$y_1=c_1=\frac{a}{b}$$

$$y_2 = c_2 = \left\{ \begin{array}{l} \frac{a}{b} + \sigma \textit{ with probability } 1/2 \text{ (}\textit{zero}\text{ marginal utility)} \\ \frac{a}{b} - \sigma \textit{ with probability } 1/2 \text{ (}\textit{positive}\text{ marginal utility)} \end{array} \right\}$$

In period 2, the agent would consume $(\frac{a}{b} + \sigma)$, with zero marginal utility, in half of the cases and $(\frac{a}{b} - \sigma)$, with positive marginal utility, in the remaining half of the cases. Therefore,

$$\begin{aligned} E_1[u'(c_2)] &= \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot K > 0 \\ K &\equiv \text{arbitrary positive number} \end{aligned} \tag{34}$$

However, notice that if $E_1[u'(c_2)] > 0$ and $u'(c_1) = 0$, the first order optimality condition [30] is violated! This means that $c_1 = y_1$ and $c_2 = y_2$ is *not* an equilibrium with $\sigma > 0$. The presence of uncertainty ($\sigma > 0$) about future labor income induces the agent to consume *less* than $\frac{a}{b}$ in the first period (precautionary saving). But how much is c_1?

1. Notice the following:

$$c_2 = y_2 + \underbrace{(y_1 - c_1)}_{\text{savings at time 1}} = \left\{ \begin{array}{l} \left(\frac{a}{b} + \sigma\right) + y_1 - c_1 = \left(\frac{a}{b} + \sigma\right) + \frac{a}{b} - c_1 \equiv \overbrace{c_2^H(c_1) \textit{ with prob } 1/2}^{\text{zero marginal utility}} \\ \left(\frac{a}{b} - \sigma\right) + y_1 - c_1 = \left(\frac{a}{b} - \sigma\right) + \frac{a}{b} - c_1 \equiv \underbrace{c_2^L(c_1) \textit{ with prob} 1/2}_{\text{positive marginal utility}} \end{array} \right\}$$

2) Impose the first order optimality condition:

$$u'(c_1) = E_1[u'(c_2)]$$

$\Longrightarrow$

$$a - bc_1 = \frac{1}{2}[a - bc_2^L(c_1)] + \frac{1}{2} \cdot 0$$

$\Longrightarrow$

$$a - bc_1 = \frac{1}{2}[a - b\underbrace{(2\frac{a}{b} - \sigma - c_1)}_{c_2^L}]$$

$\Longrightarrow$

$$c_1 = \frac{a}{b} - \frac{\sigma}{3} = y_1 - \frac{\sigma}{3} \tag{35}$$

Consumption in period 1 is decreasing in σ : uncertainty about future labor income induces the agent to save for a "precautionary" motive.

In other terms:

$$\underbrace{\uparrow \sigma}_{\text{higher uncertainty}} \Rightarrow \downarrow c_1 : \ \uparrow \text{"}\textit{precautionary}\text{" } \textit{saving in period } 1 \tag{36}$$