## Dynamic Macroeconomics - PhD Economics Dynamic consumption theory (answers) - part 2

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PROBLEM 3. Recall that

$$
\begin{equation*}
c_{t}=r\left(H_{t}+A_{t}\right) \equiv y_{t}^{P} \tag{1}
\end{equation*}
$$

where $y_{t}^{P}$ is permanent income at time $t$; i.e., consumption at time $t$ is the annuity value of total wealth.

At time $t+1$ :

$$
\begin{equation*}
y_{t+1}^{P}=r\left(A_{t+1}+H_{t+1}\right) \tag{2}
\end{equation*}
$$

Evaluate the value of $y_{t+1}^{P}$ expected at time $t$ :

$$
\begin{equation*}
E_{t} y_{t+1}^{P}=r E_{t} A_{t+1}+r E_{t} H_{t+1} \tag{3}
\end{equation*}
$$

Subtracting [3] from [2], we obtain:

$$
\begin{align*}
y_{t+1}^{P}-E_{t} y_{t+1}^{P} & =\underbrace{r A_{t+1}-r E_{t} A_{t+1}}_{\text {zero }}+r\left(H_{t+1}-E_{t} H_{t+1}\right)  \tag{4}\\
& =r \underbrace{H_{t+1}-E_{t} H_{t+1}}_{\text {"surprise" in human wealth }})
\end{align*}
$$

Indeed, from the period budget constraint we know that:

$$
\begin{equation*}
A_{t+1}=(1+r) A_{t}+y_{t}-c_{t} \tag{5}
\end{equation*}
$$

Take the expected value of both sides:

$$
\begin{equation*}
E_{t} A_{t+1}=E_{t}\left[(1+r) A_{t}+y_{t}-c_{t}\right]=(1+r) A_{t}+y_{t}-c_{t}=A_{t+1} \tag{6}
\end{equation*}
$$

Given the identity in (1) and $\left(E_{t} c_{t+1}=c_{t}\right)$ (recall the implicit assumptions behind the martingale property of consumption: 1) $r=\rho, 2$ ) quadratic utility), we deduce that permanent income follows:

$$
E_{t} y_{t+1}^{P}=y_{t}^{P}
$$

which implies that:

$$
\begin{equation*}
y_{t+1}^{P}=y_{t}^{P}+r\left(H_{t+1}-E_{t} H_{t+1}\right) \tag{7}
\end{equation*}
$$

$>$ From the definition of $H_{t+1}$ :

$$
H_{t+1}=\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t+1} y_{t+1+i}
$$

Substitute this expression into (7) :

$$
\begin{align*}
y_{t+1}^{P} & =y_{t}^{P}+r\left\{\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t+1} y_{t+1+i}-E_{t}\left[\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t+1} y_{t+1+i}\right]\right\} \\
& =y_{t}^{P}+r\left\{\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t+1} y_{t+1+i}-\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t} y_{t+1+i}\right\} \\
& =y_{t}^{P}+r \underbrace{\left\{\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}\left(E_{t+1}-E_{t}\right) y_{t+1+i}\right\}}_{\equiv \mu_{t+1}} \tag{8}
\end{align*}
$$

The "surprise" in human wealth in $t+1$ is expressed as the revision in expectations on future income.

In sum:

$$
\begin{align*}
\Delta c_{t+1} & =\Delta y_{t+1}^{p}  \tag{9}\\
& =r\left(H_{t+1}-E_{t} H_{t+1}\right) \\
& =\frac{r}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}\left(E_{t+1}-E_{t}\right) y_{t+1+i}
\end{align*}
$$

Notice that the change in consumption between $t$ and $t+1$ cannot be foreseen as of time $t$; it depends on information available at time $t+1$
a) Assume that income follows:

$$
\begin{equation*}
y_{t+1}=\bar{y}+\varepsilon_{t+1}-\delta \varepsilon_{t} \tag{10}
\end{equation*}
$$

where: $\bar{y}$ is the mean income, $E_{t} \varepsilon_{t+1}=0$, and $\delta>0$.
Using (9) compute the effects on consumption of a change in mean income $(\Delta \bar{y})$ :

$$
\begin{align*}
\Delta c_{t+1} & =\Delta y_{t+1}^{p}=\frac{r}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}\left(E_{t+1}-E_{t}\right) y_{t+1+i}  \tag{11}\\
& =\frac{r}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} \Delta \bar{y} \\
& =\Delta \bar{y} \tag{12}
\end{align*}
$$

The increase in mean income represents a variation in permanent income for the consumer. Therefore, both her permanent income and consumption increase (the size of the increase is $\Delta \bar{y}$ ). Since the variation in income is entirely permanent, savings do not change.
b) Note that the exercise assumes $\varepsilon_{t-i}=0 \forall i \geq 0$, and $\varepsilon_{t+i}=0 \forall i>1$ (only $\left.\varepsilon_{t+1}>0\right)$.

Consequently, we know that:

$$
\begin{align*}
y_{t-i}= & \bar{y}, \text { for } i=0,1, \ldots \infty  \tag{13}\\
y_{t+1} & =\bar{y}+\varepsilon_{t+1}-\delta \varepsilon_{t} \\
& =\bar{y}+\varepsilon_{t+1} \\
y_{t+2} & =\bar{y}+\varepsilon_{t+2}-\delta \varepsilon_{t+1} \\
& =\bar{y}-\delta \varepsilon_{t+1}
\end{align*}
$$

and for all $i \geq 2$

$$
\begin{aligned}
y_{t+1+i} & =\bar{y}+\varepsilon_{t+i}-\delta \varepsilon_{t+i-1} \\
& =\bar{y}
\end{aligned}
$$

To calculate (11) in this case, again we need to compute $\left(E_{t+1}-E_{t}\right) y_{t+1+i}$ :
for $i=0$ :

$$
\begin{align*}
\left(E_{t+1}-E_{t}\right) y_{t+1} & =E_{t+1} y_{t+1}-E_{t} y_{t+1}  \tag{14}\\
& =\left(\bar{y}+\varepsilon_{t+1}\right)-\bar{y} \\
& =\varepsilon_{t+1}
\end{align*}
$$

for $i=1$ :

$$
\begin{align*}
\left(E_{t+1}-E_{t}\right) y_{t+2} & =\left(E_{t+1}-E_{t}\right)\left(\bar{y}-\delta \varepsilon_{t+1}\right)  \tag{15}\\
& =\left(\bar{y}-\delta \varepsilon_{t+1}\right)-(\bar{y}) \\
& =-\delta \varepsilon_{t+1}
\end{align*}
$$

for $i \geq 2$ :

$$
\begin{align*}
\left(E_{t+1}-E_{t}\right) y_{t+1+i} & =\left(E_{t+1}-E_{t}\right)\left(\bar{y}+\varepsilon_{t+1+i}-\delta \varepsilon_{t+i}\right)  \tag{16}\\
& =(\bar{y}-\bar{y}) \\
& =0
\end{align*}
$$

Now we can compute the change in consumption:

$$
\begin{align*}
\Delta c_{t+1} & =\frac{r}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i}\left(E_{t+1}-E_{t}\right) y_{t+1+i}  \tag{17}\\
& =\frac{r}{1+r}\left[\varepsilon_{t+1}-\left(\frac{1}{1+r}\right) \delta \varepsilon_{t+1}+\left(\frac{1}{1+r}\right)^{2} \cdot 0+\ldots\right] \\
& =\frac{r}{1+r}\left[\varepsilon_{t+1}-\left(\frac{\delta}{1+r}\right) \varepsilon_{t+1}\right] \\
& =\frac{r(1+r-\delta)}{(1+r)^{2}} \varepsilon_{t+1} \\
& \underbrace{\varepsilon_{t+1}}_{\text {(income innovation at time } t+1)} \tag{18}
\end{align*}
$$

- Consumption between $t$ and $t+1$ increases less than one-to-one with current income. This result is consistent with the temporary nature of the income innovation.
- The higher is $\delta$ the lower is the variation in consumption. A positive income innovation at $t+1\left(\varepsilon_{t+1}\right)$ is compensated by a negative income variation $\left(-\delta \varepsilon_{t+1}\right)$ in the next period!
c) The behavior of savings over time reflects the expectations about future variations in income:

For a given $\varepsilon_{t+1}$ and using the expression for the stochastic process for income (10):

$$
\begin{align*}
& \Delta y_{t+2}=y_{t+2}-y_{t+1}  \tag{19}\\
& = \\
& =\left(\bar{y}-\delta \varepsilon_{t+1}\right)-\left(\bar{y}+\varepsilon_{t+1}\right) \\
& =  \tag{20}\\
& \begin{aligned}
& \Delta y_{t+3}=y_{t+3}-y_{t+2} \\
&=\bar{y}-\left(\bar{y}-\delta \varepsilon_{t+1}\right) \\
&=\delta \varepsilon_{t+1} \\
& \begin{aligned}
\Delta y_{t+4} & =y_{t+4}-y_{t+3} \\
& =\bar{y}-\bar{y} \\
& =0
\end{aligned}
\end{aligned} . \begin{aligned}
& \\
&=0
\end{aligned} \\
&
\end{align*}
$$

In general,

$$
\begin{equation*}
\Delta y_{t+1+i}=0 \forall i \geq 3 \tag{22}
\end{equation*}
$$

After period $t+3$, income is not expected to vary any further.
Now let us have a look at savings

$$
\begin{aligned}
S_{t} \equiv & y_{t}^{D}-c_{t}=y_{t}^{D}-y_{t}^{P}=\left(y_{t}+r A_{t}\right)-r\left(A_{t}+H_{t}\right)=y_{t}-r H_{t}= \\
= & y_{t}-r\left[\frac{1}{1+r} \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t} y_{t+i}\right]=\frac{1}{1+r} y_{t}-\left(\frac{1}{1+r}-\left(\frac{1}{1+r}\right)^{2}\right) E_{t} y_{t+1} \\
& -\left(\left(\frac{1}{1+r}\right)^{2}-\left(\frac{1}{1+r}\right)^{3}\right) E_{t} y_{t+2}-\ldots=-\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t} \Delta y_{t+i}
\end{aligned}
$$

(Note that $\frac{r}{(1+r)^{i}}=\left(\frac{1}{1+r}\right)^{i-1}-\left(\frac{1}{1+r}\right)^{i}$ )
The behavior of savings in $t+1$ and $t+2$ is, therefore, as follows:

$$
\begin{align*}
S_{t+1} & =-\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t+1} \Delta y_{t+1+i}  \tag{23}\\
= & -\left\{\left(\frac{1}{1+r}\right) \cdot\left(-(1+\delta) \varepsilon_{t+1}\right)+\left(\frac{1}{1+r}\right)^{2} \delta \varepsilon_{t+1}+\left(\frac{1}{1+r}\right)^{3} \cdot 0+0+\ldots . .+0\right\} \\
= & -\left(\frac{1}{1+r}\right)\left\{-(1+\delta) \varepsilon_{t+1}+\left(\frac{1}{1+r}\right) \delta \varepsilon_{t+1}\right\} \\
= & {\left[\frac{1+r(1+\delta)}{(1+r)^{2}}\right] \varepsilon_{t+1}>0 } \\
S_{t+2} & =-\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^{i} E_{t+2} \Delta y_{t+2+i}  \tag{24}\\
& =-\left\{\left(\frac{1}{1+r}\right) \cdot \delta \varepsilon_{t+1}+\left(\frac{1}{1+r}\right)^{2} \cdot 0+\ldots . .+0\right\} \\
& =-\left(\frac{\delta}{1+r}\right) \varepsilon_{t+1}<0
\end{align*}
$$

- In $t+1$ part of the higher level of income is saved - since the consumer recognizes it's a temporary innovation. The agent forecasts further variations of income in the future $\left(y_{t+2}=\right.$ $\left.\bar{y}-\delta \varepsilon_{t+1}\right)$.
- In $t+2$, income is temporarily lower than its mean level $\left(y_{t+2}=\bar{y}-\delta \varepsilon_{t+1}\right)$ and the agent consumes the savings accumulated in the previous period: in $t+2$ savings are negative!

EX 4. a) The marginal utility of consumption is given by:

$$
u^{\prime}(c)=\left\{\begin{array}{lcc}
a-b c & \text { if } & c<\frac{a}{b} \\
0 & \text { if } & c \geq \frac{a}{b}
\end{array}\right\}
$$



Graph the marginal utility function:

Note the convexity of the marginal utility function in the neighborhood of $c=\frac{a}{b}$, where it takes a zero value.
$>$ From theory, recall that a convex marginal utility presumes a precautionary motive for saving; i.e., the agent saves more in reaction to an increase in uncertainty (this reaction contrasts with the certainty equivalence property characterizing optimality in the case of a quadratic utility function).
b) Set up the expected-utility maximization problem faced by the agent:

$$
\begin{align*}
\underset{\left\{c_{t+i}, i=0,1 \ldots\right\}}{\operatorname{Max}} U_{t}= & E_{t}\left[\sum_{i=1}^{\infty}\left(\frac{1}{1+\rho}\right)^{i} u\left(c_{t+i}\right)\right]  \tag{25}\\
\text { st } \quad A_{t+i+1}= & (1+r) A_{t+i}+y_{t+i}-c_{t+i} \quad i=0,1,2 \ldots \\
& A_{t} \text { given }
\end{align*}
$$

Solve the period budget constraint for $c_{t+i}$ and plug the result into the objective function:

$$
\begin{equation*}
\underset{\left\{A_{t+i+1}, i=0,1 \ldots\right\}}{\operatorname{Max}} U_{t}=E_{t}\left[\sum_{i=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{i} u\left((1+r) A_{t+i}+y_{t+i}-A_{t+i+1}\right)\right] \tag{26}
\end{equation*}
$$

Since there are only two periods $(i=0,1)$, rewrite $(26)$ as:

$$
\begin{equation*}
U_{t}=E_{t}[u(\underbrace{(1+r) A_{t}+y_{t}-A_{t+1}}_{c_{t}})+\left(\frac{1}{1+\rho}\right) u(\underbrace{(1+r) A_{t+1}+y_{t+1}-A_{t+2}}_{c_{t+1}})] \tag{27}
\end{equation*}
$$

Derive $U_{t}$ with respect to $A_{t+1}$ and set $\frac{\partial U t}{\partial A_{t+1}}=0$ :
foc:

$$
\begin{equation*}
E_{t}\left[-u^{\prime}\left(c_{t}\right)+\left(\frac{1}{1+\rho}\right) \cdot u^{\prime}\left(c_{t+1}\right)(1+r)\right]=0 \tag{28}
\end{equation*}
$$

Rearranging this equation we obtain:

$$
\begin{equation*}
E_{t}\left[\left(\frac{1+r}{1+\rho}\right) \cdot u^{\prime}\left(c_{t+1}\right)\right]=u^{\prime}\left(c_{t}\right) \tag{29}
\end{equation*}
$$

or

$$
\left(\frac{1+r}{1+\rho}\right) E_{t} u^{\prime}\left(c_{t+1}\right)=u^{\prime}\left(c_{t}\right) \quad \text { EULER EQUATION }
$$

Assume $r=\rho=0$ and denote $t=1$ and $t+1=2$ :

$$
\begin{equation*}
u^{\prime}\left(c_{1}\right)=E_{1}\left[u^{\prime}\left(c_{2}\right)\right] \tag{30}
\end{equation*}
$$

(This is the required first order optimality condition.)
As stated in the exercise $y_{1}=\frac{a}{b}$ is known with certainty.

1. If $\sigma=0$,

$$
\begin{equation*}
y_{1}=y_{2}=\frac{a}{b} \equiv y \tag{31}
\end{equation*}
$$

$>$ From the marginal utility function, when
$c_{1} \geq y_{1} \equiv y=\frac{a}{b} \Longrightarrow u^{\prime}\left(c_{1}\right)=0$
$c_{2} \geq y_{2} \equiv y=\frac{a}{b} \Longrightarrow E_{1} u^{\prime}\left(c_{2}\right)=0$
Then

$$
\begin{equation*}
u^{\prime}\left(c_{1}\right)=E_{1}\left[u^{\prime}\left(c_{2}\right)\right]=0 \tag{32}
\end{equation*}
$$

when $c_{1}=c_{2}=y=\frac{a}{b}$ is an equilibrium
Thus, the first order optimality condition is satisfied when consumption is equal to current income in each period and savings are equal to zero:

$$
\begin{equation*}
c_{1}=c_{2}=y=\frac{a}{b} \text { is an equilibrium } \tag{33}
\end{equation*}
$$

2) Now suppose that $\sigma>0$ and $c_{1}=\frac{a}{b}$.

Income in period two is:

$$
y_{2}=\left\{\begin{array}{l}
\frac{a}{b}+\sigma=c_{1}+\sigma>c_{1} \text { with probability } 1 / 2 \\
\frac{a}{b}-\sigma=c_{1}-\sigma<c_{1} \text { with probability } 1 / 2
\end{array}\right\}
$$

Continue to assume that total income is consumed in both periods (as in part 1):

$$
y_{1}=c_{1}=\frac{a}{b}
$$

$$
y_{2}=c_{2}=\left\{\begin{array}{l}
\frac{a}{b}+\sigma \text { with probability } 1 / 2(\text { zero marginal utility }) \\
\frac{a}{b}-\sigma \text { with probability } 1 / 2(\text { positive marginal utility })
\end{array}\right\}
$$

In period 2, the agent would consume $\left(\frac{a}{b}+\sigma\right)$, with zero marginal utility, in half of the cases and $\left(\frac{a}{b}-\sigma\right)$, with positive marginal utility, in the remaining half of the cases. Therefore,

$$
\begin{align*}
E_{1}\left[u^{\prime}\left(c_{2}\right)\right] & =\frac{1}{2} \cdot 0+\frac{1}{2} \cdot K>0  \tag{34}\\
K & \equiv \text { arbitrary positive number }
\end{align*}
$$

However, notice that if $E_{1}\left[u^{\prime}\left(c_{2}\right)\right]>0$ and $u^{\prime}\left(c_{1}\right)=0$, the first order optimality condition [30] is violated! This means that $c_{1}=y_{1}$ and $c_{2}=y_{2}$ is not an equilibrium with $\sigma>0$. The presence of uncertainty $(\sigma>0)$ about future labor income induces the agent to consume less than $\frac{a}{b}$ in the first period (precautionary saving). But how much is $c_{1}$ ?

1. Notice the following:
$c_{2}=y_{2}+\underbrace{\left(y_{1}-c_{1}\right)}_{\text {savings at time } 1}=\left\{\begin{array}{c}\left(\frac{a}{b}+\sigma\right)+y_{1}-c_{1}=\left(\frac{a}{b}+\sigma\right)+\frac{a}{b}-c_{1} \equiv c_{2}^{H}\left(c_{1}\right) \text { with marginal utility } \\ \left(\frac{a}{b}-\sigma\right)+y_{1}-c_{1}=\left(\frac{a}{b}-\sigma\right)+\frac{a}{b}-c_{1} \equiv c_{2}^{L}\left(c_{1}\right) \text { with prob } 1 / 2 \\ \text { positive marginal utility }\end{array}\right\}$
2) Impose the first order optimality condition:

$$
\left.\begin{array}{cc}
u^{\prime}\left(c_{1}\right)=E_{1}\left[u^{\prime}\left(c_{2}\right)\right] \\
\Longrightarrow & a-b c_{1}=\frac{1}{2}\left[a-b c_{2}^{L}\left(c_{1}\right)\right]+\frac{1}{2} \cdot 0 \\
\Longrightarrow & a-b c_{1} \\
\Longrightarrow & \frac{1}{2}[a-b \underbrace{\left(2 \frac{a}{b}-\sigma-c_{1}\right)}_{c_{2}^{L}}] \\
& c_{1}
\end{array}\right] \frac{a}{b}-\frac{\sigma}{3}=y_{1}-\frac{\sigma}{3}
$$

Consumption in period 1 is decreasing in $\sigma$ : uncertainty about future labor income induces the agent to save for a "precautionary" motive.

In other terms:

$$
\begin{equation*}
\underbrace{\uparrow \sigma}_{\text {higher uncertainty }} \Rightarrow \downarrow c_{1}: \uparrow " \text { precautionary" saving in period } 1 \tag{36}
\end{equation*}
$$

