Dynamic Macroeconomics - PhD Economics
Dynamic consumption theory (answers) - part 1

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PROBLEM 1. In order to study the relationship between *current* income and *permanent* income let's specify the behavior of y over time as AR(1) process:

$$y_{t+1} = \lambda y_t + (1-\lambda)\overline{y} + \varepsilon_{t+1} \tag{1}$$

where $E_t\varepsilon_{t+1} = 0$ and $0 \leq \lambda \leq 1$.

Given y_{t+1}, we compute y_{t+2} and then take expectations:

$$\begin{aligned} y_{t+2} &= \lambda y_{t+1} + (1-\lambda)\overline{y} + \varepsilon_{t+2} \\ &= \lambda^2 y_t + \lambda(1-\lambda)\overline{y} + \lambda\varepsilon_{t+1} + (1-\lambda)\overline{y} + \varepsilon_{t+2} \\ &= \lambda^2 y_t + (1+\lambda)(1-\lambda)\overline{y} + \lambda\varepsilon_{t+1} + \varepsilon_{t+2} \end{aligned}$$

Taking expectations with respect to time $t+1$ we get:

$$E_{t+1}y_{t+2} = \lambda^2 y_t + (1+\lambda)(1-\lambda)\overline{y} + \lambda\varepsilon_{t+1} \tag{2}$$

since $E_{t+1}\varepsilon_{t+2} = 0$ and taking expectations with respect to time t we obtain:

$$E_t y_{t+2} = \lambda^2 y_t + (1+\lambda)(1-\lambda)\overline{y}$$

since $E_t\varepsilon_{t+1} = E_t\varepsilon_{t+2} = 0$.

The difference between the expectation terms is then:

$$E_{t+1}y_{t+2} - E_t y_{t+2} = \lambda\varepsilon_{t+1}$$

which is the revision in the agent's expectation on future income at time $t+2$ from period t to $t+1$.

In general, we can show that*:*

$$E_{t+1}y_{t+1+i} - E_t y_{t+1+i} = \lambda^i\varepsilon_{t+1} \ , \ \ \forall i \geq 0 \tag{3}$$

Now recall the expression for the change in permanent income (see lecture notes)

$$y_{t+1}^P - E_t y_{t+1}^P = y_{t+1}^P - y_t^P$$

$$y_{t+1}^P = y_t^P + r \underbrace{\left[\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i (E_{t+1} - E_t) y_{t+1+i} \right]}_{\substack{\text{"surprise" in human wealth} \\ H_{t+1} - E_t H_{t+1}}}$$

The evolution over time of consumption follows that of permanent income, therefore

$$c_{t+1} = c_t + r \underbrace{\left[\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i (E_{t+1} - E_t) y_{t+1+i} \right]}_{\text{"surprise" in human wealth}}$$

Plugging expression (3) into the consumption dynamic equation we obtain (recall that $\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$, for $q < 1$)

$$\begin{aligned} c_{t+1} &= c_t + r \left[\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i \lambda^i \varepsilon_{t+1} \right] \\ &= c_t + r \left[\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{\lambda}{1+r} \right)^i \varepsilon_{t+1} \right] \\ &= c_t + \frac{r}{1+r-\lambda} \varepsilon_{t+1} \qquad (4) \end{aligned}$$

or

$$\begin{aligned} c_{t+1} &= c_t + \eta_{t+1}, \qquad (5) \\ \text{where} \quad \eta_{t+1} &\equiv \frac{r}{1+r-\lambda} \varepsilon_{t+1} \end{aligned}$$

So consumption follows a martingale process and, given the assumed stochastic process describing the evolution of income, we can give some economic interpretation to the error term η_{t+1}, which depends on the innovation in income at time $t+1$.

Consider now the two polar cases: $\lambda = 0$ and $\lambda = 1$.

a) $\lambda = 0$. In this case,

$$y_{t+1} = \overline{y} + \varepsilon_{t+1} \tag{6}$$

It can be observed that the increase in current income is only *transitory, i.e.* an income shock at time $t+1$ does not affect future income levels. For the sake of simplicity, let us assume that

$$y_{t-i} = \overline{y}, \quad \forall i \geq 0$$

i.e. in all periods before y_t we have income at its average level $\overline{y}$. At time t, the agent starts from a consumption level:

$$c_t = \overline{y} \tag{7}$$

Since $y_t = \overline{y}$, and $c_t = \overline{y}$, then also savings, $s_t = 0$ implying, $A_{t+1} = 0$. (Recall that y and c are realized at the end of period while A is set at the beginning of the period). Up to period t, the agent consumes all her income, and thus enters period $t+1$ with no financial wealth.

At time $t+1$, her income becomes:

$$y_{t+1} = \overline{y} + \varepsilon_{t+1} \tag{8}$$

i.e. she receives an income shock ε_{t+1}.Under the optimal plan, consumption follows the dynamic

equation:

$$c_{t+1} = c_t + \frac{r}{1+r}\varepsilon_{t+1} = \overline{y} + \frac{r}{1+r}\varepsilon_{t+1} \geq c_t \tag{9}$$

from eq. [4], with $\lambda = 0$: the contemporaneous adjustment of consumption depends on the income shock.

We can compute savings at period $t+1$ as (recall $y_t^D \equiv \underbrace{y_t}_{\text{labor income}} + \underbrace{rA_t}_{\text{interest income}}$):

$$\begin{aligned}
s_{t+1} &= y_{t+1}^D - c_{t+1} \\
&= y_{t+1} - c_{t+1} \\
&= (\overline{y} + \varepsilon_{t+1}) - \left(c_t + \frac{r}{1+r}\varepsilon_{t+1}\right) \\
&= \frac{\varepsilon_{t+1}}{1+r} > 0
\end{aligned} \tag{10}$$

$$\begin{aligned}
A_{t+2} &= (1+r)A_{t+1} + y_{t+1} - c_{t+1} \\
&= \frac{\varepsilon_{t+1}}{1+r} > 0
\end{aligned}$$

In the next periods (and without any further innovation) current income returns to its average value ($y_{t+2} = y_{t+3} = ... = \overline{y}$), whereas consumption remains at its higher value (the value computed for period $t+1$):

$$c_{t+2} = c_{t+1} + \frac{r}{1+r}\varepsilon_{t+2} = c_{t+1} > c_t$$

Indeed, the return on financial wealth (accumulated in $t+1$) allows the agent to maintain the higher consumption level for all future periods.

Formally, we can compute y^D_{t+2} and s_{t+2} as follows:

$$\begin{aligned} y^D_{t+2} &= y_{t+2} + rA_{t+2} \\ &= \overline{y} + \frac{r}{1+r}\varepsilon_{t+1} \\ &= c_{t+1} = c_{t+2} \end{aligned}$$

The agent consumes all disposable income:

$$\begin{aligned} &\Rightarrow \qquad s_{t+2} = 0 \\ &\Rightarrow \qquad A_{t+3} = \frac{\varepsilon_{t+1}}{1+r} \end{aligned}$$

In general

$$s_{t+i} = 0 \quad \forall i \geq 2$$

The same computation applies to all periods *after* $t+2$: given the stochastic process for income in (6) with $\lambda = 0$ (*i.e. transitory* income shocks) there is *no saving* and the level of the agent's financial wealth stays constant at the level of A_{t+2}.

For given r, notice that

- $\dfrac{\partial s_{t+1}}{\partial \varepsilon_{t+1}} = \dfrac{1}{1+r} > 0$ a positive shock on y has a positive impact on savings
- $\dfrac{\partial s_{t+2}}{\partial \varepsilon_{t+1}} = 0$ in all subsequent periods savings are null
- $\dfrac{\partial y^D_{t+1}}{\partial \varepsilon_{t+1}} = 1; \dfrac{\partial y^D_{t+2}}{\partial \varepsilon_{t+1}} = \dfrac{r}{1+r} > 0$ disposable income increases due to previous period's savings accumulated into assets

b) $\lambda = 1$:

$$y_{t+1} = y_t + \varepsilon_{t+1} \tag{11}$$

In this case all the increase in income is *permanent* and, to be consistent with optimal behavior, should be entirely consumed. As in the previous case, the consumption level increases from t to $t+1$. However, unlike the previous case, there is no need to save in order to keep the higher consumption level in the future as the higher income level will be enjoyed in all subsequent periods

Similarly as before, assume no income shocks after period $t+1$ such that:

$$y_{t+2} = y_{t+1} = y_t + \varepsilon_{t+1} \tag{12}$$

Innovation at $t+1$ affects *all* subsequent labor incomes (the increase in income is *permanent*). Substituting $\lambda = 1$ in the equation for consumption (4) we obtain:

$$\begin{aligned} c_{t+1} &= c_t + \varepsilon_{t+1} > c_t \\ c_{t+2} &= c_{t+1} > c_t \end{aligned} \tag{13}$$

consumption increases from t to $t+1$, and remains higher than c_t in all subsequent periods.

Similarly, we can compute savings in period $t+1$:

$$\begin{aligned} s_{t+1} &= y_{t+1}^D - c_{t+1} \\ &= y_{t+1} - c_{t+1} \\ &= y_t + \varepsilon_{t+1} - c_t - \varepsilon_{t+1} \\ &= 0 \end{aligned} \tag{14}$$

$$A_{t+2} = 0$$

(recall that $c_t = y_t = \overline{y}$). Furthermore, for period $t+2$:

$$\begin{aligned} s_{t+2} &= y_{t+2}^D - c_{t+2} \\ &= y_t + \varepsilon_{t+1} - c_t - \varepsilon_{t+1} \\ &= 0 \end{aligned} \tag{15}$$

Given that the increase in income is permanent, consistently with optimal consumption behavior, the agent has no need to save in order to keep the increased consumption in the future.

PROBLEM 2. In this solution we derive the intertemporal budget constraint faced by the consumer (IBC), using the period-by-period budget constraint (BC):

$$\begin{gathered} A_{t+i+1} = (1+r)A_{t+i} + y_{t+i} - c_{t+i} \qquad \forall i \geq 0 \\ A_t \quad \text{given} \end{gathered} \tag{16}$$

Starting from BC we compute (16) for $i = 0, 1, ...3$:

$$A_{t+1} = (1+r)A_t + y_t - c_t \tag{17}$$

$$\begin{aligned} A_{t+2} &= (1+r)A_{t+1} + y_{t+1} - c_{t+1} \\ &= (1+r)^2 A_t + (1+r)y_t - (1+r)c_t + y_{t+1} - c_{t+1} \end{aligned}$$

$$\begin{aligned} A_{t+3} &= (1+r)A_{t+2} + y_{t+2} - c_{t+2} \\ &= (1+r)^3 A_t + (1+r)^2 y_t - (1+r)^2 c_t + (1+r)y_{t+1} - (1+r)c_{t+1} + y_{t+2} - c_{t+2} \end{aligned}$$

$$\begin{aligned} A_{t+4} &= (1+r)A_{t+3} + y_{t+3} - c_{t+3} \\ &= (1+r)^4 A_t + (1+r)^3 y_t - (1+r)^3 c_t + (1+r)^2 y_{t+1} \\ &\quad -(1+r)^2 c_{t+1} + (1+r)y_{t+2} - (1+r)c_{t+2} + y_{t+3} - c_{t+3} \end{aligned}$$

Rearranging terms conveniently:

$$\begin{gathered} A_{t+4} + (1+r)^3 c_t + (1+r)^2 c_{t+1} + (1+r)c_{t+2} + c_{t+3} = \\ (1+r)^4 A_t + (1+r)^3 y_t + (1+r)^2 y_{t+1} + (1+r)y_{t+2} + y_{t+3} \end{gathered}$$

Divide both sides of this expression by $(1+r)^4$ to obtain:

$$\begin{aligned}
&\left(\frac{1}{1+r}\right)^4 A_{t+4} + \left(\frac{1}{1+r}\right) c_t + \left(\frac{1}{1+r}\right)^2 c_{t+1} + \left(\frac{1}{1+r}\right)^3 c_{t+2} + \left(\frac{1}{1+r}\right)^4 c_{t+3} \\
= \; & A_t + \left(\frac{1}{1+r}\right) y_t + \left(\frac{1}{1+r}\right)^2 y_{t+1} + \left(\frac{1}{1+r}\right)^3 y_{t+2} + \left(\frac{1}{1+r}\right)^4 y_{t+3}
\end{aligned}$$

In general, for $j \geq 1$:

$$\left(\frac{1}{1+r}\right)^j A_{t+j} + \frac{1}{1+r}\sum_{i=0}^{j-1}\left(\frac{1}{1+r}\right)^i c_{t+i} = A_t + \frac{1}{1+r}\sum_{i=0}^{j-1}\left(\frac{1}{1+r}\right)^i y_{t+i} \tag{18}$$

Therefore, as $j \to \infty$ (infinite horizon case):

$$\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i c_{t+i} = \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i y_{t+i} + A_t \qquad IBC \tag{19}$$

since

$$\lim_{j\to\infty}\left(\frac{1}{1+r}\right)^j A_{t+j} = 0 \tag{20}$$

by the *transversality condition.*

The IBC must hold also in expectations, such that:

$$\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t c_{t+i} = \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} + A_t \qquad IBC(E) \tag{21}$$

Under the assumption of $r = \rho$ and of a quadratic utility function: $c_{t+1} = c_t + \eta_{t+1}$ and therefore $E_t c_{t+1} = E_t c_{t+2} = ... = E_t c_{t+i} = c_t$, for $i \geq 1$ (*i.e.*, expected future consumption in each period is equal to current consumption). Thus, we can recast the *IBC(E)* as:

$$\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i c_t = \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} + A_t$$

$\Rightarrow$

$$\frac{1}{1+r} c_t \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i = \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} + A_t$$

$\Rightarrow$

$$\frac{1}{1+r} c_t \left(\frac{1}{1-\frac{1}{1+r}}\right) = \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} + A_t$$

$$\frac{1}{r} c_t = \underbrace{\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i}}_{\equiv H_t \text{ (human capital)}} + A_t$$

(where the third line is valid given that $\frac{1}{1+r} < 1$).

The consumption function then follows as:

$$c_t = r(H_t + A_t) \equiv y_t^P \tag{22}$$

Now, we know that income is generated by the following $AR(1)$ process:

$$\begin{aligned} y_{t+1} &= \lambda y_t + (1-\lambda)\overline{y} + \varepsilon_{t+1} && (23) \\ E_t\varepsilon_{t+1} &= 0;\ E_t\varepsilon_{t+2} = 0; ... && (24) \end{aligned}$$

from which:

$$E_t y_{t+1} = \lambda y_t + (1-\lambda)\overline{y}$$

$$\begin{aligned} y_{t+2} &= \lambda y_{t+1} + (1-\lambda)\overline{y} + \varepsilon_{t+2} \\ &= \lambda^2 y_t + \lambda(1-\lambda)\overline{y} + \lambda\varepsilon_{t+1} + (1-\lambda)\overline{y} + \varepsilon_{t+2} \\ &= \lambda^2 y_t + (1+\lambda)(1-\lambda)\overline{y} + \lambda\varepsilon_{t+1} + \varepsilon_{t+2} \end{aligned}$$

$$E_t y_{t+2} = \lambda^2 y_t + (1+\lambda)(1-\lambda)\overline{y}$$

and in general:

$$\begin{aligned} E_t y_{t+i} &= \lambda^i y_t + (1+\lambda+\lambda^2+...+\lambda^{i-1})(1-\lambda)\overline{y} \\ &= \lambda^i y_t + \frac{(1-\lambda^i)}{(1-\lambda)}(1-\lambda)\overline{y} && (25) \\ &= \lambda^i y_t + (1-\lambda^i)\overline{y} && (26) \end{aligned}$$

Plugging the value of $E_t y_{t+i}$ into the definition of human capital (H_t) yields:

$$\begin{aligned} H_t &= \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} \\ &= \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i (\lambda^i y_t + (1-\lambda^i)\overline{y}) \\ &= \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i (\lambda^i y_t + \overline{y} - \lambda^i\overline{y}) \\ &= \frac{1}{1+r}\left[\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i \lambda^i y_t + \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i \overline{y} - \sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i \lambda^i\overline{y}\right] \\ &= \frac{1}{1+r}\left[\frac{1+r}{1+r-\lambda}y_t + \frac{1+r}{r}\overline{y} - \frac{1+r}{1+r-\lambda}\overline{y}\right] \\ &= \frac{1}{1+r}\left[\frac{1+r}{1+r-\lambda}y_t + \frac{(1+r)(1+r-\lambda)-r(1+r)}{r(1+r-\lambda)}\overline{y}\right] \\ &= \frac{1}{1+r}\left[\frac{1+r}{1+r-\lambda}y_t + \frac{(1+r)(1-\lambda)}{r(1+r-\lambda)}\overline{y}\right] \\ &= \frac{1}{1+r-\lambda}y_t + \frac{(1-\lambda)}{r(1+r-\lambda)}\overline{y} && (27) \end{aligned}$$

Thus:

$$c_t = r(A_t + H_t) = rA_t + \frac{r}{1+r-\lambda} y_t + \frac{(1-\lambda)}{(1+r-\lambda)} \overline{y} \tag{28}$$

Given the stochastic process followed by income (23), the consumer's optimal behavior is to follow this rule to decide on his consumption level c_t as a function of A_t and y_t.

Now consider the polar cases: $\lambda = 0$ and $\lambda = 1$.

a) $\lambda = 1$

$$\begin{aligned} y_{t+1} &= y_t + \varepsilon_{t+1} \\ y_{t+2} &= y_t + \varepsilon_{t+1} + \varepsilon_{t+2} \end{aligned} \tag{29}$$

Recall from the previous problem that, in this case, innovations to income are *permanent.* Furthermore:

$$E_t y_{t+i} = y_t \quad \forall i \geq 1 \tag{30}$$

i.e., the agent's best forecast of future incomes is equal to *current* income y_t. Thus,

$$c_t = rA_t + y_t \tag{31}$$

consumption at time t is equal to *total income* (interest income + labor income) in the same period.

b) $\lambda = 0$,

$$y_{t+1} = \overline{y} + \varepsilon_{t+1} \tag{32}$$

$$y_{t+2} = \overline{y} + \varepsilon_{t+2}$$

Similarly, recall that in this case, innovations in current income are only *temporary*, which also implies:

$$E_t y_{t+i} = \overline{y} \quad \forall i \geq 1 \tag{33}$$

i.e., the agent's best forecast of future income is equal to *average* income $\overline{y}$.

In this case, substituting $\lambda = 0$ in (28) and rearranging

$$c_t = rA_t + \overline{y} + \left(\frac{r}{1+r}\right)(y_t - \overline{y}) \tag{34}$$

where the last term corresponds to the annuity value, computed at the beginning of period t, of the income innovation at time t.