# PhD Economics
# **Dynamic Macroeconomics**

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## Notes on: Growth theory

**General references:**
Romer (2012) *Advanced Macroeconomics,* fourth edition, ch. 1, 2 (Part A), 3
Bagliano-Bertola (2007) *Models for dynamic macroeconomics*, chapter 4

**Specific references:**
Mankiw N.G., D. Romer, D. Weil (1992) "A contribution to the empirics of economic growth", *Quarterly Journal of Economics*, May, 407-437
Romer P. (1994) "The origins of endogenous growth", *Journal of Economic Perspectives*, 8, 1, 3-22
Barro-Sala-i-Martin (1995) *Economic Growth*, ch. 1, 2, 4
Aghion-Howitt (1998) *Endogenous Growth Theory,* ch. 1

**Topics:**

1. review of basic model of growth with exogenous saving rate, to explain main observed long-run features (Kaldor):

$$\left.\begin{array}{c} \frac{Y}{N} \text{ increases (at a non-decreasing rate)} \\ \frac{K}{Y} \text{ stable} \end{array}\right\} \Rightarrow \frac{K}{N} \text{ displays upward trend;}$$

2. application of dynamic optimization methods to basic model (to endogenize saving choice by representative agent);

3. decentralization of production and consumption decisions in a perfectly competitive economy;
4. introduction to endogenous growth mechanisms: "learning-by-doing" effects.

**1. Growth model with exogenous saving rate** (Solow-Swan)

Main assumptions:

- one-good, closed economy with no government consumption:

$$Y(t) = C(t) + I(t)$$

- (aggregate) production function (with *labour-augmenting* -or Harrod-neutral - technical progress):

$$Y(t) = F(K(t), L(t)) = F(K(t), A(t)\, N(t))$$

  with constant returns to scale in $K$ and $L \equiv AN$:

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

  In terms of units of labour (measured as $L$):

$$y(t) \equiv \frac{Y(t)}{L(t)} = \frac{F(K(t), L(t))}{L(t)} \underset{\substack{\uparrow \\ \text{by} \\ \text{C.R.S.}}}{=} F\left(\frac{K(t)}{L(t)}, 1\right) \equiv f(k(t))$$

  where $k \equiv K/L$.

- exogenous labour dynamics:

$$L(t) = L(0)\, \mathrm{e}^{g\,t} \;\Rightarrow\; \frac{\dot{L}(t)}{L(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{N}(t)}{N(t)} \equiv g_A + g_N = g$$

$\Rightarrow$ accumulation of capital in the aggregate $\dot{K}(t)$:

$$\dot{K}(t) = I(t) - \delta\, K(t)$$

and per unit of labour $\dot{k}(t)$:

$$\dot{k}(t) \equiv \frac{d}{dt}\left(\frac{K(t)}{L(t)}\right) = \underbrace{\frac{\dot{K}(t)}{L(t)}}_{\frac{I}{L}-\delta k} - \underbrace{\frac{\dot{L}(t)}{L(t)}}_{g}\underbrace{\frac{K(t)}{L(t)}}_{k}$$

$$\Rightarrow \quad \dot{k}(t) = \frac{I(t)}{L(t)} - (g+\delta)\, k(t)$$

- exogenous saving rate $s$:

$$C(t) = (1-s)\, Y(t) \Rightarrow I(t) = s\, Y(t) \Rightarrow \frac{I(t)}{L(t)} = s\, y(t) = s\, f(k(t))$$

$\Rightarrow$ stable (non-linear) dynamic equation for $k$:

$$\dot{k}(t) = \underset{\text{savings}}{s\, f(k(t))} - \underset{\text{“break-even” investment}}{(g+\delta)\, k(t)}$$

In steady state:

$$s\, f(k_{SS}) = (g+\delta)\, k_{SS} \;\Rightarrow\; k_{SS} = k(s, g+\delta)$$

$K$, $L$ and $Y$ grow at the same rate $g$ so that $k$ and $y$ are constant ($\rightarrow$ “balanced growth path”).

**Implication 1**: changes in $s$ do not affect the growth rate of $K$ (and $k$) in steady state but have an effect on the steady state level of $k$.

$$\frac{dk_{SS}}{ds} = -\frac{f(k_{SS})}{sf'(k_{SS}) - (g+\delta)} > 0$$

The effect on (per unit of labour) consumption $c_{SS}$ is ambiguous:

$$c_{SS} = (1-s)\, f(k_{SS}) = f(k_{SS}) - (g+\delta)\, k_{SS}$$

$$\Rightarrow \frac{dc_{SS}}{ds} = \underset{(\lesseqgtr 0)}{[f'(k_{SS}) - (g+\delta)]}\, \frac{dk_{SS}}{ds}$$

**Implication 2**: (conditional) *convergence* result: inverse relationship between growth rate and level of capital (per unit of labour):

$$\frac{d\left(\frac{\dot{k}}{k}\right)}{dk} < 0$$

This result is conditional on the determinants of the steady state ($s$ and $g + \delta$).

**Implication 3**: unlimited accumulation of capital (per unit of labour), i.e. $\dot{k} > 0$, is possible:

$$\begin{aligned} \dot{k} &= s\, f(k) - (g + \delta)\, k > 0 \text{ for } k \to \infty \Rightarrow s \lim_{k \to \infty} f'(k) \geq g + \delta \\ \Rightarrow &\quad \lim_{k \to \infty} f'(k) \geq \frac{g + \delta}{s} \end{aligned}$$

A positive *rate of growth* of $k$, i.e. $\dot{k}/k > 0$, is sustainable indefinitely if strict inequality holds $\Rightarrow$ *steady state in growth rates* (even without technological progress, i.e. $g_A = 0$).

Note: in the pure version of the Solow model the following (so-called "Inada" conditions) are assumed:

$$\lim_{k \to 0} f^{'}(k) = \infty \quad \text{and} \quad \lim_{k \to \infty} f'(k) = 0$$

**2. Optimal saving choice (Ramsey)**

**Problem.** Consider a representative consumer with infinite horizon maximizing the intertemporal utility function (with $c \equiv C/N$):

$$\max_{c(t)} \quad U = \int_0^\infty u(c(t))\, \mathrm{e}^{-\rho\, t}\, dt \qquad \rho > 0$$

subject to the accumulation constraint

$$\dot{k}(t) = f(k(t)) - c(t)$$

(we assume $g_A = g_N = \delta = 0$).

The Hamiltonian function of the problem is

$$H(t) = [u(c(t)) + \lambda(t)(f(k(t)) - c(t))]\ \mathrm{e}^{-\rho\, t}$$

where $\lambda(t)$ is the shadow value of capital at time $t$.

**Solution.** Standard f.o.c.:

$$\frac{\partial H}{\partial c} = 0 \ \Rightarrow\ u'(c(t)) = \lambda(t)$$

$$\begin{aligned} -\frac{\partial H}{\partial k} = \frac{d\,(\lambda(t)\, \mathrm{e}^{-\rho\, t})}{dt} \quad &\Rightarrow\ -\lambda(t)\, f'(k(t))\, \mathrm{e}^{-\rho\, t} = \dot{\lambda}(t)\, \mathrm{e}^{-\rho\, t} - \rho\, \lambda(t)\, \mathrm{e}^{-\rho\, t} \\ &\Rightarrow\ \rho\, \lambda(t) = f'(k(t))\, \lambda(t) + \dot{\lambda}(t) \\ &\Rightarrow\ \dot{\lambda}(t) = [\rho - f'(k(t))]\ \lambda(t) \end{aligned}$$

with the transversality condition $\lim_{t\to\infty} \lambda(t)\, \mathrm{e}^{-\rho\, t} k(t) = 0$ and the accumulation constraint.

From the f.o.c. for $c$ and $k$:

$$\begin{aligned} \dot{\lambda}(t) &= \frac{d\,(u'(c(t)))}{dt} = u''(c(t))\,\dot{c}(t) \\ &\Rightarrow \;\; u''(c(t))\,\dot{c}(t) = [\rho - f'(k(t))]\; u'(c(t)) \\ &\Rightarrow \;\; \dot{c}(t) = \left(\frac{u'(c(t))}{-u''(c(t))}\right)\,[f'(k(t)) - \rho] \end{aligned}$$

Using the equation of motion for $k$, the optimal dynamics of $c$ and $k$ is described by the system:

$$\begin{aligned} \dot{c} &= \left(\frac{u'(c)}{-u''(c)}\right)\,[f'(k) - \rho] \\ \dot{k} &= f(k) - c \end{aligned}$$

Example: assume a CRRA utility function: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, with coefficient of relative risk aversion $\sigma > 0$:

$$\frac{u'(c)}{-u''(c)\,c} = \frac{1}{\sigma} \text{ measuring the “elasticity of intertemporal substitution”}$$

The dynamic equations become:

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\sigma}\,[f'(k) - \rho] \\ \dot{k} &= f(k) - c \end{aligned}$$

**Steady state and dynamics.**

- $\dot{c} = 0$ (stationary curve for $c$) $\Rightarrow f'(k) = \rho$
- $\dot{k} = 0$ (stationary curve for $k$) $\Rightarrow f(k) = c \Rightarrow \frac{dc}{dk}\Big|_{\dot{k}=0} = f'(k) > 0$

Assuming a CRRA utility function and linearizing the system around the steady state $c_{SS}$, $k_{SS}$ :

$$\begin{pmatrix} \dot{c} \\ \dot{k} \end{pmatrix} = \left( \begin{array}{cc} \overbrace{\tfrac{1}{\sigma}[f'(k_{SS}) - \rho]}^{(0)} & \overbrace{\frac{1}{\sigma} c_{SS} f''(k_{SS})}^{(-)} \\ -1 & \underbrace{f'(k_{SS})}_{(+)} \end{array} \right) \begin{pmatrix} c - c_{SS} \\ k - k_{SS} \end{pmatrix}$$

The determinant $\Delta = \frac{1}{\sigma} c_{SS} f''(k_{SS}) < 0$ ensures saddlepoint stability of the system.

**Optimal capital accumulation and savings.** Even in the absence of (exogenous) technical progress, it is possible for capital and consumption to grow forever at a non-decreasing rate.

Assume $f(k) = b\,k \Rightarrow f'(k) = b$ (with $b$ constant) and CRRA utility function. Then, from f.o.c.:

$$\frac{\dot{c}(t)}{c(t)} = \frac{b - \rho}{\sigma} \qquad \text{constant}$$

From the accumulation constraint:

$$\dot{k}(t) = b\,k(t) - c(t) \;\Rightarrow\; \frac{\dot{k}(t)}{k(t)} = b - \frac{c(t)}{k(t)}$$

$\Rightarrow$ $k$ grows at a constant rate if the $c/k$ ratio is constant (which necessarily holds along the optimal path leading to the steady state), implying

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{c}(t)}{c(t)} \quad \Rightarrow \quad \frac{b - \rho}{\sigma} = b - \frac{c(t)}{k(t)}$$

$$\Rightarrow \quad c(t) = \frac{b(\sigma - 1) + \rho}{\sigma} k(t)$$

Since $y(t) = b\,k(t)$, saving is a constant fraction $s$ of output:

$$1 - \frac{c}{y} \equiv s = 1 - \frac{b(\sigma - 1) + \rho}{\sigma} \frac{k(t)}{bk(t)} = 1 - \frac{b(\sigma - 1) + \rho}{b\sigma} = \frac{b - \rho}{b\sigma}$$

as in Solow's model (but here it depends on parameters of preferences and technology).

**3. Growth in a decentralized economy.**

The optimal growth path derived above can be obtained in an economy where consumers and firms act in a decentralized and uncoordinated way, under the following set of assumptions:

- C.R.S. in production;
- all (product and factor) markets perfectly competitive;
- all consumers face the same optimization problem.

**Consumers.** Each *household*, indexed by $i$, has one (constant) unit of labour ($L$) and solves the following problem:

$$\max_{c_i(t)} \int_0^\infty u(c_i(t))\, \mathrm{e}^{-\rho\, t} dt$$

subject to the accumulation constraint for financial wealth $a_i(t)$:

$$\dot{a}_i(t) = r(t)\, a_i(t) + w(t) - c_i(t)$$

where $r$ and $w$ are determined on competitive markets and taken as given by households.

To solve the problem, from the Hamiltonian:

$$H(t) = [u(c_i(t)) + \lambda_i(t)\ \left(r(t)\, a_i(t) + w(t) - c_i(t)\right)]\ \mathrm{e}^{-\rho\, t}$$

and the associated f.o.c. for $c_i$ and $a_i$ we get:

$$\left.\begin{array}{r} u'(c_i(t)) = \lambda_i(t) \\ \\ \dot{\lambda}_i(t) = [\rho - r(t)]\ \lambda_i(t) \end{array}\right\} \ \Rightarrow\ \dot{c}_i(t) = \left(\frac{u'(c_i(t))}{-u''(c_i(t))}\right)(r(t) - \rho)$$

with the transversality condition $\lim_{t\to\infty} \lambda_i(t)\, \mathrm{e}^{-\rho\, t} a_i(t) = 0$.
For example, if agents have a common CRRA utility function:

$$\frac{\dot{c}_i(t)}{c_i(t)} = \frac{r(t) - \rho}{\sigma} \quad \text{for all households } i$$

**Firms.** Each firm, indexed by $j$, employs the same C.R.S. production function $F(K_j(t), L_j(t))$ and solves the profit maximization problem:

$$\max_{K_j(t),L_j(t)} \underbrace{F(K_j(t), L_j(t))}_{\substack{L_j(t)\, f\left(\frac{K_j(t)}{L_j(t)}\right) \\ \text{by C.R.S.}}} - r(t)\, K_j(t) - w(t)\, L_j(t)$$

The f.o.c.:

$$\begin{aligned} f'\left(\frac{K_j(t)}{L_j(t)}\right) &= r(t) \\ f\left(\frac{K_j(t)}{L_j(t)}\right) - \frac{K_j(t)}{L_j(t)} f'\left(\frac{K_j(t)}{L_j(t)}\right) &= w(t) \end{aligned}$$

are valid for all firms $j$ (being $r$ and $w$ common to all firms), which may differ only as to the scale of production

$$\Rightarrow \frac{K_j(t)}{L_j(t)} = k(t) \quad \forall j$$

In the aggregate:

$$Y = \sum_j F(K_j, L_j) = \sum_j L_j f\left(\frac{K_j}{L_j}\right) = \left(\sum_j L_j\right) f(k) = Lf(k) = F(K, L)$$

Aggregate financial wealth must be equal to the aggregate capital stock:

$$\sum_i a_i(t) = La(t) = K(t) \Rightarrow a(t) = k(t)$$

since households are identical (so $a_i = a \ \forall i$). The financial wealth accumulation constraint becomes (using the f.o.c. from the firm's problem):

$$\begin{aligned} \dot{a}(t) &= r(t)\, a(t) + w(t) - c(t) \\ \Rightarrow \dot{k}(t) &= r(t)k(t) + \left[ f\ (k(t)) - k(t) \underbrace{f'\ (k(t))}_{r(t)} \right] - c(t) \\ &\Rightarrow \ \dot{k}(t) = f\ (k(t)) - c(t) \end{aligned}$$

The optimal paths of consumption and capital accumulation derived from decentralized optimization coincides with those obtained from the "centralized" (or "social planner's") solution.

**4. Endogenous growth mechanisms.**

Unbounded growth without "exogenous" technological progress is possible even in the simple (Solow) model with exogenous savings. To obtain long-run (per capita) growth *and* convergence of growth rates, assume the following production function:

$$Y(t) = \underbrace{bK(t)}_{\text{"proportional" part}} + \underbrace{BK(t)^{\alpha}L(t)^{1-\alpha}}_{\text{Cobb-Douglas part}} \qquad 0 < \alpha < 1$$

displaying C.R.S. and decreasing marginal productivities for $K$ and $L$.

$$\Rightarrow f(k) = bk + Bk^{\alpha} \ , \ f'(k) = b + \alpha Bk^{\alpha-1} \ , \ \lim_{k\to\infty} f'(k) = b$$

Capital accumulation process in level and growth rate:

$$\begin{aligned} \dot{k} &= s\ (bk + Bk^{\alpha}) - (g_N + \delta)\, k \\ \frac{\dot{k}}{k} &= s\ \left(b + Bk^{\alpha-1}\right) - (g_N + \delta) \\ &\Rightarrow \lim_{k\to\infty} \frac{\dot{k}}{k} = s\, b - (g_N + \delta) > 0 \end{aligned}$$

If $b > \frac{g_N+\delta}{s}$ endogenous, steady state growth occurs.
Note, however, that if production inputs are rewarded at their marginal productivity rate, labour income share tends to vanish:

$$\begin{aligned} \frac{\partial Y}{\partial L} &= (1-\alpha)BK^{\alpha}L^{-\alpha} = (1-\alpha)Bk^{\alpha} \\ \lim_{k\to\infty} \frac{\partial Y}{\partial L}\frac{L}{Y} &= \frac{\partial Y/\partial L}{f(k)} = \frac{(1-\alpha)Bk^{\alpha}}{bk + Bk^{\alpha}} = 0 \end{aligned}$$

**Model with "involuntary" technological progress** ("learning-by-doing" and "knowledge spillovers", Romer 1986**).** Simple economic mechanism linking efficiency growth directly to production activity: technological progress does not

require specific use of productive resources. The production input "technology" is non-rival and not rewarded (as labour $N$ and physical capital $K$).
Consider a decentralized, perfectly competitive economy with the following *aggregate* production function:

$$Y(t) = F(K(t), A(t)N(t)) = K(t)^{\alpha} \left[A(t)\, N(t)\right]^{1-\alpha} \qquad 0 < \alpha < 1$$

with

$$A(t) = A\left(\frac{K(t)}{N}\right) = a\,\frac{K(t)}{N} \quad \text{with } N \text{ constant and } a > 0$$

*Firms.* Each individual firm $j$ employs the C.R.S. production function:

$$Y_j(t) = F(K_j(t), A(t)\, N_j(t)) = K_j(t)^{\alpha} \left[A\left(\frac{K(t)}{N}\right) N_j(t)\right]^{1-\alpha}$$

where $A(t)$ depends only on *aggregate* quantities. In maximizing profits, the firm takes the aggregate ratio $K/N$ (and therefore $A$) as given, together with the interest rate $r$ and the wage rate $w$.
The f.o.c. for optimal choice of $K_j$ and $N_j$ are:

$$\begin{aligned}
\frac{\partial F}{\partial K_j} = r \quad &\Rightarrow \quad \alpha\, K_j^{\alpha-1}\left(a\,\frac{K}{N}\,N_j\right)^{1-\alpha} = r \\
&\Rightarrow \quad \alpha\, a^{1-\alpha}\left(\frac{K_j}{N_j}\right)^{\alpha-1}\left(\frac{K}{N}\right)^{1-\alpha} = r \\
\frac{\partial F}{\partial N_j} = w \quad &\Rightarrow \quad (1-\alpha)\, K_j^{\alpha}\left(a\,\frac{K}{N}\,N_j\right)^{-\alpha}\left(a\,\frac{K}{N}\right) = w \\
&\Rightarrow \quad (1-\alpha)\, a^{1-\alpha}\left(\frac{K}{N}\right)^{1-\alpha}\left(\frac{K_j}{N_j}\right)^{\alpha} = w
\end{aligned}$$

Since $\frac{K_j}{N_j} = \frac{K}{N}\ \forall j$:

$$\Rightarrow \begin{cases} r(t) = \alpha\, a^{1-\alpha} \text{ constant} \\ \\ w(t) = (1-\alpha)\, a^{1-\alpha}\,\frac{K(t)}{N} \end{cases}$$

*Households.* With CRRA utility function (risk aversion parameter $\sigma > 1$), the optimal rate of growth of aggregate consumption is:

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\sigma} = \frac{\alpha\, a^{1-\alpha} - \rho}{\sigma} \qquad \text{constant}$$

*Balanced growth path.* From the aggregate production function:

$$\begin{aligned} Y &= K^{\alpha}\left(a\frac{K}{N}N\right)^{1-\alpha} \\ &= a^{1-\alpha}K \end{aligned}$$

$\Rightarrow Y/K$ ratio constant $\Rightarrow \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K}$.

On a balanced growth path $\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\alpha\, a^{1-\alpha}-\rho}{\sigma} \Rightarrow \frac{C}{K}$ constant

$$\Rightarrow \frac{C}{K} = \underbrace{a^{1-\alpha}}_{\frac{Y}{K}} - \underbrace{\frac{\alpha\, a^{1-\alpha}-\rho}{\sigma}}_{\frac{\dot{K}}{K}} = \frac{(\sigma-\alpha)\, a^{1-\alpha}+\rho}{\sigma}$$

Income share of labour:

$$\frac{w(t)N}{Y(t)} = \frac{(1-\alpha)\, a^{1-\alpha}K}{a^{1-\alpha}K} = 1-\alpha \qquad \text{constant}$$

consistent with unbounded capital accumulation.

The basic endogenous growth mechanism is based here on an "externality" effect: decentralized, optimizing agents do not consider the effect of their choices on the dynamics of aggregate productivity. In fact, the "social" marginal productivity of capital is:

$$\begin{aligned} \frac{d}{dK}\left(F\left(K, A\left(\frac{K}{N}\right)N\right)\right) &= \frac{\partial F}{\partial K} + \frac{\partial F}{\partial L}A' \\ &= \alpha a^{1-\alpha} + (1-\alpha)a^{1-\alpha} = a^{1-\alpha} > \underbrace{\frac{\partial F}{\partial K_j} = \alpha a^{1-\alpha}}_{\text{"private" marg. prod. of } K} \end{aligned}$$

$\Rightarrow$ (balanced) growth rate lower than socially optimal rate.

# Problems

1. Consider the following Cobb-Douglas aggregate production function ($0 < \alpha < 1$):
$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$$
There is no technological progress, the number of workers $N$ grows at a rate $g_N$, the rate of capital depreciation is $\delta$ and the savings rate $s$ is exogenously given.

   a) Set up the dynamic equation for $k$ and find the steady-state levels of capital, output and consumption (all expressed per unit of labour);

   b) what saving rate $s$ is needed to maximize *consumption*? (yielding the *golden rule* level of capital)

2. Now consider an aggregate production function of the form:
$$Y(t) = bK(t) + BK(t)^{\alpha} L(t)^{1-\alpha} \qquad 0 < \alpha < 1$$
As in problem 1, there is no technological progress, the number of workers $N$ grows at a rate $g_N$, the rate of capital depreciation is $\delta$ and the savings rate $s$ is exogenously given.

   a) Check that the above production function has constant returns to scale and write $f(k)$ and the dynamic equation for $k$;

   b) show whether the *convergence* result (inverse relationship between the growth rate and the level of $k$) holds;

   c) is it possible for this economy to sustain indefinitely a positive rate of growth of $k$ (i.e. $\dot{k}/k > 0$)? if so, under what condition?

3. Consider an economy with the following aggregate production function (with constant labour input $\bar{L}$):
$$Y(t) = F(K(t), \bar{L}) = A_0 \bar{L} + 2B\sqrt{K(t)\bar{L}}$$
The capital stock depreciates at a constant (instantaneous) rate $\delta$. Consumers maximize a CRRA utility function $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$ (with $c \equiv C/\bar{L}$) over an infinite horizon with a discount rate $\rho$.

a) Find the per capita production function $f(k)$ with $k \equiv \frac{K}{L}$ and set up the dynamic optimization problem of the representative consumer specifying the capital accumulation constraint. Derive the optimal path for consumption. Provide a clear economic explanation of consumer's behaviour;

b) obtain and plot the stationary equations for consumption and capital ($\dot{C} = 0$ e $\dot{K} = 0$), with a brief explanation of their shape. Comment on the dynamics of $c$ and $k$ outside the steady state and find the stable (convergent) path;

c) suppose that at time $t_0$ there is an *unexpected permanent* increase in $A$ from $A_0$ to $A_1 > A_0$. Find (and plot) the new steady state of the economy and the dynamic adjustment paths of $c$ and $k$. Provide an explanation for your results;

d) in a decentralized market economy (with perfect competition in factor markets) what is the effect of the increase in $A$ on the new steady state levels of the interest rate $r$ and of the wage rate $w$?

e) What happens if the increase in $A$ is only *temporary* (from $t_0$ to $t_1$)? Obtain the dynamic adjustment path of consumption and capital in this case, providing an economic explanation.

4. Consider an economy with the following aggregate production function (with constant labour input $\bar{L}$):

$$Y(t) = F(K(t), \bar{L}) = a\bar{L} + bK(t)^{\alpha}\bar{L}^{1-\alpha} \qquad \alpha < 1$$

where $a, b > 0$. The capital stock depreciation rate is $\delta_0$. Consumers maximize the following intertemporal utility function:

$$U = \int_0^{\infty} u(c(t))\, e^{-\rho t}\, dt \qquad \text{with} \qquad u(c) = 1 - \frac{1}{c} \quad \text{and} \quad c \equiv \frac{C}{\bar{L}}$$

a) Set up the consumer's maximization problem and find the optimal rate of growth of consumption;

b) write the stationary equations for consumption and the capital stock ($\dot{c} = 0$ and $\dot{k} = 0$) and plot the convergent optimal (saddle)path;

c) starting from an initial steady state equilibrium, consider an increase (unexpected and permanent) of the depreciation rate $\delta$ from $\delta_0$ to $\delta_1 > \delta_0$. Explain the shifts of the stationary curves and plot the optimal dynamic adjustment of $c$ and $k$ to the new steady state.