PhD Economics
Dynamic Macroeconomics

(F. Bagliano)

September 2024

Notes on: Dynamic models of Investment

General references:
Romer (2012) *Advanced Macroeconomics*, fourth edition, ch 9
Bagliano-Bertola (2007) *Models for dynamic macroeconomics,* ch. 2
Blanchard-Fischer (1989) *Lectures on Macroeconomics*, ch.2, section 4 and ch. 6, section.3

On mathematical methods:
Barro-Sala-i-Martin (1995) *Economic Growth*, Mathematical Appendix

Specific references:
Yoshikawa H. (1980) "On the 'q' theory of investment", *American Economic Review*, 70, 4, 739-743
Hayashi F. (1982) "Tobin's marginal q and average q: a neoclassical interpretation", *Econometrica*, 50, 1, 213-224
Abel A. - Blanchard O.J. (1983) "An intertemporal model of saving and investment", *Econometrica*, 51, 3, 675-692
Caballero R. (1999) "Aggregate Investment", *Handbook of Macroeconomics,* vol. 1B, ch 12

Aims:

1. simple characterization of main determinants of investment spending in a dynamic model of a "representative" firm (under certainty);
2. application of dynamic optimization methods in continuous time.

Topics:

1. Motivation
2. Mathematical methods:
   - dynamic optimization in continuous time: general framework
   - Hamiltonian solution
3. Cost-of-adjustment model of investment demand:
   - forward-looking "q" theory
   - steady-state and dynamics

1. Motivation

Traditional (neoclassical) theory (Jorgenson):
$\rightarrow$ optimization in an essentially static environment with perfectly "flexible" capital

$$\max_K \quad \pi(K, ...) \quad \Rightarrow \quad \text{f.o.c.} \quad \underbrace{\frac{\partial R(K^*, ...)}{\partial K}}_{\text{marg. revenue of capital}} = \underbrace{\left(r + \delta - \frac{\Delta p_K}{p_K}\right) p_K}_{\text{user cost of capital}}$$

given (exogenously) price of capital p_K and its change, interest rate r, depreciation rate δ, product demand conditions and technology;

$$\Rightarrow \quad K^* = K^*\left(r, \delta, \frac{\Delta p_K}{p_K}, ...\right) \quad \text{"desired" capital stock}$$

Then, *ad hoc* assumptions to explain gradual investment over time.

Problems:

- model for "desired" capital: changes in exogenous variables $\Rightarrow$ immediate discrete change in K^* $\rightarrow$ not appropriate to model aggregate dynamics of capital and investment;
- no role for expectations: marginal revenue and user cost expressed in current terms, with no forward-looking behaviour.

$\Rightarrow$ **Model with adjustment costs:**

$$\text{costs of changing } K \rightarrow \begin{cases} \text{model of investment with smooth dynamics for } K; \\ \text{forward-looking behaviour of firms.} \end{cases}$$

2. Dynamic optimization in continuous time (under certainty)

***General set-up*:**

$f(\,t,\, z(t),\, y(t))$: istantaneous objective function

$z(t)$: control variable (flow)

$y(t)$: state variable (stock)

$\dot{y}(t) \equiv \frac{dy(t)}{dt} = g(\,t,\, z(t),\, y(t))$ accumulation constraint (equation of motion)

Set-up of the optimization problem with infinite horizon:

$$\max_{z(t)} \quad L(0) = \int_0^\infty f(\,t,\, z(t),\, y(t))\, \mathrm{e}^{-\rho t}\, dt$$

subject to:

$$\begin{aligned} \dot{y}(t) &= g(\,t,\, z(t),\, y(t)) \\ y(0) &= y_0 \quad \text{(given) and terminal (transversality) condition} \end{aligned}$$

Solution

Form *Lagrangian* with $\mu(t)$ dynamic Lagrange multiplier ("costate variable"):

$$\max \quad \tilde{L}(0) = \int_0^\infty f(\,t,\, z(t),\, y(t))\, \mathrm{e}^{-\rho t}\, dt + \int_0^\infty \mu(t)\ [g(\,t,\, z(t),\, y(t)) - \dot{y}(t)]\ dt$$

To derive f.o.c. use the rule of integration by parts applied to:

$$\int_0^\infty \mu(t)\, \dot{y}(t)\, dt = \lim_{t\to\infty} [\mu(t)\ y(t)] - \mu(0)\ y(0) - \int_0^\infty \dot{\mu}(t)\, y(t)\, dt$$

(from

$$\frac{d\,[\mu(t)\ y(t)]}{dt} = \dot{\mu}(t)\, y(t) + \mu(t)\, \dot{y}(t)$$

integrating from 0 to T (finite):

$$\mu(T)\ y(T) - \mu(0)\ y(0) = \int_0^T \dot{\mu}(t)\, y(t)\, dt + \int_0^T \mu(t)\, \dot{y}(t)\, dt$$

then let $T \to \infty$).

Lagrangian becomes:

$$\max \quad \tilde{L}(0) = \int_0^\infty \left[f(\, t,\, z(t),\, y(t))\, \mathrm{e}^{-\rho t} + \mu(t)\, g(\, t,\, z(t),\, y(t)) \right] \, dt + \int_0^\infty \dot{\mu}(t)\, y(t)\, dt + \mu(0)\ y(0)$$

imposing $\lim_{t\to\infty} \mu(t)\ y(t) = 0$.

F.o.c.:

$$\begin{aligned}
\frac{\partial \tilde{L}}{\partial\, z} &= 0 \Rightarrow \quad \frac{\partial\, f(.)}{\partial\, z(t)}\, \mathrm{e}^{-\rho t} + \mu(t)\, \frac{\partial\, g(.)}{\partial\, z(t)} = 0 \\
\frac{\partial \tilde{L}}{\partial\, y} &= 0 \Rightarrow \quad \frac{\partial\, f(.)}{\partial\, y(t)}\, \mathrm{e}^{-\rho t} + \mu(t)\, \frac{\partial\, g(.)}{\partial\, y(t)} + \dot{\mu}(t) = 0 \\
\frac{\partial \tilde{L}}{\partial\, \mu} &= 0 \Rightarrow \quad \dot{y}(t) = g(\, t,\, z(t),\, y(t))
\end{aligned}$$

and $\quad \lim_{t\to\infty} \mu(t)\ y(t) = 0$, $y(0) = y_0$.

Hamiltonian solution procedure

Define the (present value) *Hamiltonian*:

$$H(t) = [f(\, t,\, z(t),\, y(t))\ + \lambda(t)\, g(\, t,\, z(t),\, y(t))]\ \mathrm{e}^{-\rho t}$$

where $\lambda(t)$ is in current value terms:

$$\mu(t) = \lambda(t)\, \mathrm{e}^{-\rho t}$$

The f.o.c. are:

$$\frac{\partial H}{\partial\, z} = 0 \Rightarrow \qquad \frac{\partial\, f(.)}{\partial\, z(t)}\, \mathrm{e}^{-\rho t} + \lambda(t)\, \mathrm{e}^{-\rho t}\, \frac{\partial\, g(.)}{\partial\, z(t)} = 0$$

$$-\frac{\partial H}{\partial\, y} = \frac{d\,[\lambda(t)\, \mathrm{e}^{-\rho t}]}{dt} \Rightarrow \qquad -\left(\frac{\partial\, f(.)}{\partial\, y(t)}\, \mathrm{e}^{-\rho t} + \underbrace{\lambda(t)\, \mathrm{e}^{-\rho t}}_{\mu(t)}\, \frac{\partial\, g(.)}{\partial\, y(t)} \right) = \underbrace{\dot{\lambda}(t)\, \mathrm{e}^{-\rho t} - \rho \lambda(t)\, \mathrm{e}^{-\rho t}}_{\dot{\mu}(t)}$$

$$\frac{\partial H}{\partial\, [\lambda(t)\, \mathrm{e}^{-\rho t}]} = \dot{y} \Rightarrow \qquad \dot{y}(t) = g(\, t,\, z(t),\, y(t))$$

$$\lim_{t\to\infty} \lambda(t)\, \mathrm{e}^{-\rho t}\ y(t) = 0 \quad \text{and} \quad y(0) = y_0.$$

3. Dynamic, cost-of-adjustment model of investment demand

Objective function of "representative" firm with infinite horizon under certainty:

$$F(t) = R\left(t, K(t), N(t)\right) - p_K(t)\, G\left(I(t), K(t)\right) - w(t)N(t)$$

$F(t)$: cash flow at time t

$K(t)$: capital stock used in production at time $t \rightarrow$ "predetermined" variable

$R(.)$: revenue function (depending on technology and product demand conditions), with $R_K > 0$, $R_N > 0$, $R_{KK} < 0$, $R_{NN} < 0$

$N(t)$: labour $\rightarrow$ perfectly "flexible" input (only wage costs, no adjustment costs)

$I(t)$: investment at time $t \rightarrow$ changes K entailing costs given by: $p_K(t)\, G(I(t), K(t))$

$G\left(I(t), K(t)\right)$: (physical) investment costs with $G_I > 0$, $G_{II} > 0$ (convex function in I) and

$$\left.\begin{array}{ll} G\left(0, K(t)\right) = 0 & \forall K(t) \\ G_I\left(0, K(t)\right) = 1 & \forall K(t) \end{array}\right\} \Rightarrow \left\{\begin{array}{r} \text{if } I = 0: \text{ no costs} \\ \text{if } I > 0: \text{ unit investment cost } > p_K \\ \text{if } I < 0: \text{ unit investment "revenue" } < p_K \end{array}\right.$$

Consequences on firm's behaviour:

- graduality in investment/disinvestment;
- investments followed by disinvestments are costly $\rightarrow$ investments are (partly) irreversible.

Accumulation constraint:
in discrete time

$$K(t + \Delta t) = K(t) + I(t)\Delta t - \delta K(t)\Delta t$$

in continuous time

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{K(t + \Delta t) - K(t)}{\Delta t} &= I(t) - \delta K(t) \\ \Rightarrow \quad \dot{K}\,(t) &= I(t) - \delta K(t) \end{aligned}$$

From the equation of motion, K can be expressed as the result of past investment decisions:

$$\left[\dot{K}\,(t) + \delta K(t)\right] \mathrm{e}^{\delta t} = I(t)\, \mathrm{e}^{\delta t}$$

$$\int_{t_0}^{T} \left[\dot{K}(t) + \delta K(t)\right] \mathrm{e}^{\delta t}\, dt = \int_{t_0}^{T} I(t)\, \mathrm{e}^{\delta t}\, dt$$

$$K(t)\, \mathrm{e}^{\delta t}\Big|_{t_0}^{T} = \int_{t_0}^{T} I(t)\, \mathrm{e}^{\delta t}\, dt$$

$$K(T)\, \mathrm{e}^{\delta T} - K(t_0)\, \mathrm{e}^{\delta t_0} = \int_{t_0}^{T} I(t)\, \mathrm{e}^{\delta t}\, dt$$

$$K(T) = K(t_0)\, \mathrm{e}^{-\delta(T-t_0)} + \int_{t_0}^{T} I(t)\, \mathrm{e}^{-\delta(T-t)}\, dt$$

letting $t_0 \to -\infty$:

$$K(T) = \int_{-\infty}^{T} I(t)\, \mathrm{e}^{-\delta(T-t)}\, dt \quad .$$

Firm's dynamic optimization problem

$$\max_{I(t),N(t),K(t)} \quad V(0) = \int_{0}^{\infty} \underbrace{\left[R\left(t, K(t), N(t)\right) - p_K(t)\, G\left(I(t), K(t)\right) - w(t)N(t)\right]}_{F(t)} \mathrm{e}^{-\int_0^t r(s)ds}\, dt$$

subject to:

$$\begin{aligned} \dot{K}(t) &= I(t) - \delta K(t) \\ K(0) &= K_0 \quad \text{(given)} \quad \text{and transversality condition} \end{aligned}$$

Solution

Hamiltonian:

$$H(t) = \left\{\left[R\left(t, K(t), N(t)\right) - p_K(t)\, G\left(I(t), K(t)\right) - w(t)N(t)\right] + \lambda(t)\ \left[I(t) - \delta K(t)\right]\right\}\ \mathrm{e}^{-\int_0^t r(s)ds}$$

f.o.c.:

$$\frac{\partial H}{\partial N} = 0 \Rightarrow \qquad \frac{\partial\, R(.)}{\partial\, N(t)} = w(t)$$

$$\frac{\partial H}{\partial I} = 0 \Rightarrow \qquad p_K(t)\, \frac{\partial\, G(.)}{\partial\, I(t)} = \lambda(t)$$

$$-\frac{\partial H}{\partial K} = \frac{d\left[\lambda(t)\ \mathrm{e}^{-\int_0^t r(s)ds}\right]}{dt} \Rightarrow \qquad -\left(\frac{\partial\, R(.)}{\partial\, K(t)} - p_K(t)\, \frac{\partial\, G(.)}{\partial\, K(t)} - \delta\, \lambda(t)\right)\ \mathrm{e}^{-\int_0^t r(s)ds} =$$

$$= \dot{\lambda}(t)\ \mathrm{e}^{-\int_0^t r(s)ds} - r(t)\, \lambda(t)\ \mathrm{e}^{-\int_0^t r(s)ds}$$

$$\Rightarrow\ r(t)\, \lambda(t) = \left(\frac{\partial\, R(.)}{\partial\, K(t)} - p_K(t)\, \frac{\partial\, G(.)}{\partial\, K(t)} - \delta\, \lambda(t)\right) + \dot{\lambda}(t)$$

$$\dot{K}\,(t) = I(t) - \delta K(t)$$

$$\lim_{t\to\infty} \qquad \lambda(t)\, \mathrm{e}^{-\int_0^t r(s)ds}\, K(t) = 0 \quad , \quad K(0) = K_0$$

Simplified case

$$F(t) = R\left(K(t), N(t)\right) - p_K\, G\left(I(t)\right) - wN(t)$$

r, w, p_K constant.
F.o.c. become:

$$\frac{\partial\, R(.)}{\partial\, N(t)} = w \quad \Rightarrow \quad N(t) = n\left(w, K(t)\right)$$

$$p_K\, \frac{\partial\, G(.)}{\partial\, I(t)} = \lambda(t)$$

$$r\, \lambda(t) = \frac{\partial\, R(.)}{\partial\, K(t)} - \delta\, \lambda(t) + \dot{\lambda}(t)$$

Define:

$$q\,(t) \equiv \frac{\lambda\,(t)}{p_K}$$

$$\Rightarrow \frac{\partial\, G(.)}{\partial\, I(t)} = q\,(t)$$

since $G_I > 0$ and $G_{II} > 0 \rightarrow G_I$ invertible

$$\Rightarrow I\left(t\right) = \iota\left(q\left(t\right)\right) \quad \text{with} \quad \iota' \equiv \frac{dI}{dq} = \frac{1}{G_{II}} > 0$$

Using definition of $q(t)$ and $\dot{q}(t) = \frac{\dot{\lambda}(t)}{p_K}$, f.o.c. are expressed as:

$$\begin{aligned}
\frac{\partial R(.)}{\partial N(t)} &= w \quad \Rightarrow \quad N(t) = n\left(w, K(t)\right) \\
\frac{\partial G(.)}{\partial I(t)} &= q\left(t\right) \\
r\, q(t) &= \frac{1}{p_K}\frac{\partial R(.)}{\partial K(t)} - \delta\, q(t) + \dot{q}(t) \\
\dot{K}\left(t\right) &= \iota\left(q\left(t\right)\right) - \delta K(t)
\end{aligned}$$

$\Rightarrow$ system of two differential equations in q and K :

$$\begin{cases} \dot{q}(t) = \left(r+\delta\right) q(t) - \frac{1}{p_K}\frac{\partial R(K(t),\, n(w,K(t)))}{\partial K(t)} \\ \\ \dot{K}\left(t\right) = \iota\left(q\left(t\right)\right) - \delta K(t) \end{cases}$$

Qualitative analysis of steady state and dynamic properties

Stationary loci for q and K :

- $\dot{q}(t) = 0$

$$\Rightarrow \quad q = \frac{1}{r+\delta}\,\frac{1}{p_K}\,\frac{\partial R\left(K,\, n(w,K)\right)}{\partial K}$$

  slope:

$$\left.\frac{dq}{dK}\right|_{\dot{q}=0} = \frac{1}{r+\delta}\,\frac{1}{p_K}\underbrace{\left(\underset{(-)}{\frac{\partial^2 R\left(.\right)}{\partial K^2}} + \underset{(+)}{\frac{\partial^2 R\left(.\right)}{\partial K\, \partial N}}\,\underset{(+)}{\frac{\partial n}{\partial K}}\right)}_{(-)\text{ by assumption}} < 0$$

- $\dot{K}(t) = 0$

$$\Rightarrow \quad \iota\,(q\,) = \delta K$$

  slope:

$$\left.\frac{dq}{dK}\right|_{\dot{K}=0} = \frac{\delta}{\iota\,'} > 0$$

Linearizing the system around the steady state (q_{ss}, K_{ss}):

$$\begin{pmatrix} \dot{q} \\ \dot{K} \end{pmatrix} = \begin{pmatrix} r+\delta & -\frac{1}{p_K}\frac{d}{dK}\left(\frac{\partial R(.)}{\partial K}\right) \\ \iota\,' & -\delta \end{pmatrix} \begin{pmatrix} q - q_{ss} \\ K - K_{ss} \end{pmatrix}$$

Determinant of matrix of derivatives (evaluated at steady state):

$$-\delta(r+\delta) + \iota\,' \frac{1}{p_K}\frac{d}{dK}\left(\frac{\partial R(.)}{\partial K}\right) < 0 \quad \Rightarrow \quad \text{“saddlepoint” stability}$$

Forward-looking interpretation of λ and q

Solving “forward” the dynamic equation

$$\dot{\lambda}(t) - (r+\delta)\,\lambda(t) = -\frac{\partial R(.)}{\partial K(t)}$$

$$\left[\dot{\lambda}(t) - (r+\delta)\,\lambda(t)\right]\, \mathrm{e}^{-(r+\delta)\,t} = -\frac{\partial R(.)}{\partial K(t)}\,\mathrm{e}^{-(r+\delta)\,t}$$

$$\int_{t_0}^{T} \underbrace{\left[\dot{\lambda}(t) - (r+\delta)\,\lambda(t)\right]}_{\frac{d}{dt}\left(\lambda(t)\,\mathrm{e}^{-(r+\delta)\,t}\right)} \mathrm{e}^{-(r+\delta)\,t} dt = -\int_{t_0}^{T} \frac{\partial R(.)}{\partial K(t)}\,\mathrm{e}^{-(r+\delta)\,t} dt$$

$$\lambda(T)\,\mathrm{e}^{-(r+\delta)T} - \lambda(t_0)\,\mathrm{e}^{-(r+\delta)\,t_0} = -\int_{t_0}^{T} \frac{\partial R(.)}{\partial K(t)}\,\mathrm{e}^{-(r+\delta)\,t} dt$$

Letting $T \to \infty$ with $\lim_{T\to\infty} \lambda(T)\,\mathrm{e}^{-(r+\delta)T} = 0$

$$\Rightarrow \quad \lambda(t_0)\,\mathrm{e}^{-(r+\delta)\,t_0} = \int_{t_0}^{\infty} \frac{\partial R(.)}{\partial K(t)}\,\mathrm{e}^{-(r+\delta)\,t} dt$$

$$\lambda(t_0) = \int_{t_0}^{\infty} \frac{\partial R(.)}{\partial K(t)} \, \mathrm{e}^{-(r+\delta)\,(t-t_0)} dt$$

and

$$q(t_0) = \int_{t_0}^{\infty} \frac{1}{p_K} \frac{\partial R(.)}{\partial K(t)} \, \mathrm{e}^{-(r+\delta)\,(t-t_0)} dt$$

Marginal q ***and average*** q

If $R(.)$ and $G(.)$ are linearly homogeneous in K, N and I, K respectively:

$$R\,(\alpha K, \alpha N) = \alpha\, R\,(K, N) \quad \text{and} \quad G\,(\alpha I, \alpha K) = \alpha\, G\,(I, K)$$

the Euler theorem holds:

$$R\,(K, N) = \ R_K\, K + R_N\, N \quad \text{and} \quad G\,(I, K) = \ G_I\, I + G_K\, K$$

so that the cash flow function $F(t)$ becomes.

$$\begin{aligned}
F(t) &= R\,(K(t), N(t)) - p_K\, G\,(I(t), K(t)) - w\, N(t) \\
&= \underbrace{(R_K\, K + R_N\, N)}_{R(.)} - p_K\, \underbrace{(G_I\, I + G_K\, K)}_{G(.)} - w\, N \\
\text{since } (R_N - w)\, N &= 0 \quad \text{by f.o.c. (along an optimal path)} \\
&= \underbrace{(R_K - p_K\, G_K)}_{\substack{(r+\delta)\,\lambda - \dot{\lambda} \\ \text{by f.o.c.}}} K - \underbrace{p_K\, G_I}_{\substack{\lambda \\ \text{by f.o.c.}}} \underbrace{I}_{\substack{\dot{K} + \delta K \\ \text{by f.o.c.}}} \\
\Rightarrow \quad F(t) &= r\,\lambda(t)\, K(t) - \dot{\lambda}(t)\, K(t) - \lambda(t)\, \dot{K}(t)
\end{aligned}$$

This is equivalent to:

$$\begin{aligned}
\mathrm{e}^{-rt}\, F(t) &= \mathrm{e}^{-rt}\, r\,\lambda(t)\, K(t) - \mathrm{e}^{-rt}\, \dot{\lambda}(t)\, K(t) - \mathrm{e}^{-rt}\, \lambda(t)\, \dot{K}(t) \\
\text{or} \quad \mathrm{e}^{-rt}\, F(t) &= \frac{d}{dt} \left(-\mathrm{e}^{-rt}\, \lambda(t)\, K(t) \right)
\end{aligned}$$

Integrating:

$$
\begin{aligned}
V(0) &= \int_0^\infty \mathrm{e}^{-rt}\, F(t)\, dt = \left[-\mathrm{e}^{-rt}\, \lambda(t)\, K(t)\right]_0^\infty \\
\Rightarrow \quad V(0) &= \lambda(0)\, K(0)
\end{aligned}
$$

using $\lim_{t\to\infty} \mathrm{e}^{-\,rt}\, \lambda(t)\, K(t) = 0$

$$\Rightarrow \quad \lambda(0) = V(0)/\, K(0)$$

Then, for every time t:

$$q(t) \equiv \frac{\lambda(t)}{p_K} = \frac{V(t)}{p_K\, K(t)}$$

$\Rightarrow$ marginal and average q coincide.

Problems

1. Consider a firm with capital as the only factor of production. Its revenues at time t are $R(K(t))$ if installed capital is $K(t)$. The accumulation constraint has the usual form, $\dot{K}(t) = I(t) - \delta K(t)$, and the cost of investing $I(t)$ is a function $G(I(t))$ that does not depend on installed capital (for simplicity, $p_k \equiv 1$).

• (a) Suppose the firm aims at maximizing the present discounted value at rate r of its cash flows, $F(t)$. Express cash flows in terms of the functions $R(\cdot)$ and $G(\cdot)$, derive the relevant first-order conditions, and characterize the solution graphically making specific assumptions as to the derivatives of $R(\cdot)$ and $G(\cdot)$.

(b) Characterize the solution under more specific assumptions: suppose revenues are a linear function of installed capital, $R(K) = \alpha K$, and let the investment cost function be quadratic, $G(I) = I + bI^2$. Derive and interpret an expression for the steady-state capital stock: what happens if $\delta = 0$?

2. A firm's production function is

$$Y(t) = \alpha\sqrt{K(t)} + \beta\sqrt{L(t)},$$

and its product is sold at a given price, normalized to unity. Factor L is not subject to adjustment costs, and is paid w per unit time. Factor K obeys the accumulation constraint

$$\dot{K}(t) = I(t) - \delta K(t)$$

and the cost of investing I is

$$G(I) = I + \frac{\gamma}{2}I^2$$

per unit time (we let $p_k = 1$). The firm maximizes the present discounted value at rate r of its cash flows.

(a) Write the Hamiltonian for this problem, derive and discuss briefly the first-order conditions, and draw a diagram to illustrate the solution.

(b) Analyze graphically the effects of an increase in δ (faster depreciation of installed capital) and give an economic interpretation of the adjustment trajectory.

3. As a function of installed capital K, a firm's revenues are given by

$$R(K) = K - \frac{1}{2}K^2.$$

The usual accumulation constraint has $\delta = 0.25$, so $\dot{K} = I - 0.25K$. Investing I costs $p_k G(I) = p_k \left(I + \frac{1}{2}I^2\right)$. The firm maximizes the present discounted value at rate $r = 0.25$ of its cash flows.

(a) Write the first-order conditions of the dynamic optimization problem, and characterize the solution graphically supposing that $p_k = 1$ (constant).

(b) Starting from the steady state of the $p_k = 1$ case, show the effects of a 50% subsidy of investment (so that p_k is halved).

(c) Discuss the dynamics of optimal investment if at time $t = 0$, when p_k is halved, it is also announced that at some future time $T > 0$ the interest rate will be tripled, so that $r(t) = 0.75$ for $t \geq T$.

4. The revenue flow of a firm is given by

$$R(K, N) = 2K^{1/2}N^{1/2}$$

where N is a freely adjustable factor, paid a wage $w(t)$ at time t; K is accumulated according to

$$\dot{K} = I - \delta K$$

and an investment flow I costs

$$G(I) = \left(I + \frac{1}{2}I^2\right)$$

(note that $p_k = 1$, hence $q = \lambda$).

(a) Write the first-order conditions for maximization of present discounted (at rate r) value of cash flows over an infinite planning horizon.

(b) Given r e δ constant, write an expression for $\lambda(0)$ in terms of $w(t)$, the function describing the time path of wages.

(c) Evaluate that expression in the case where $w(t) = \bar{w}$ is constant, and characterize the solution graphically.

(d) How could the problem be modified so that investment is a function of the average value of capital (that is, of Tobin's *average* q)?