## 3A. Extension: precautionary saving

## Microfoundations

Motivations to save in basic rational expectations/permanent income model:

- expected declines in income
- $r>\rho$
$\Rightarrow$ no role for uncertainty on future labor incomes in determining savings: no "precautionary savings"
The role of uncertainty is limited by the assumption of quadratic utility $\Rightarrow$ linear marginal utility, implying:

$$
E u^{\prime}(c)=u^{\prime}(E(c))
$$

$\Rightarrow$ only expected values matter (other characteristics of the distributions of $y$ and $c$, e.g. variance, are irrelevant): an increase in uncertainty (with unchanged expected values) does not cause any reaction by agents ("certainty equivalence" assumption)

If $u^{\prime}(c)$ is a convex function, consumers display "prudent" behaviour, reacting to an increase in uncertainty by increasing savings: precautionary saving

$$
u^{\prime}(c) \quad \text { convex : } u^{\prime \prime \prime}(c)>0
$$

Note: to have risk aversion: $u^{\prime \prime}(c)<0$ (also with quadratic utility)
to have prudence: $u^{\prime \prime \prime}(c)>0$ (with quadratic utility: $u^{\prime \prime \prime}(c)=0$ )

Example: two periods ( $t$ and $t+1$ ), $A_{t}=0$; labor incomes:

$$
\begin{aligned}
y_{t} & =\bar{y} \quad \text { with certainty } \\
y_{t+1} & =\left\{\begin{array}{ll}
y_{t+1}^{H} & \text { with prob. } 1 / 2 \\
y_{t+1}^{L} & \text { with prob. } 1 / 2
\end{array} \text { with } y_{t+1}^{H}>y_{t+1}^{L}\right.
\end{aligned}
$$

No "standard" incentives to save: assume

- $E_{t} y_{t+1}=\bar{y}$
- $r=\rho$ ( $=0$ for simplicity)
F.o.c. :

$$
u^{\prime}\left(c_{t}\right)=E_{t} u^{\prime}\left(c_{t+1}\right)
$$

saving and consumption:

$$
\begin{aligned}
& \text { in } t: \\
& s_{t}=\bar{y}-c_{t} \\
& \text { in } t+1\left.: \begin{array}{c}
c_{t+1}^{H} \\
c_{t+1}^{L}
\end{array}\right\}=(\underbrace{\bar{y}-c_{t}}_{s_{t}})+\left\{\begin{array}{l}
y_{t+1}^{H} \\
y_{t+1}^{L}
\end{array}\right.
\end{aligned}
$$

Using $s_{t}$, f.o.c. becomes:

$$
u^{\prime}\left(\bar{y}-s_{t}\right)=E_{t}\left(u^{\prime}\left(y_{t+1}+s_{t}\right)\right)
$$

- with quadratic utility $u^{\prime \prime \prime}(c)=0$ ( linear marginal utility):

$$
\begin{gathered}
E_{t}\left(u^{\prime}\left(y_{t+1}+s_{t}\right)\right)=u^{\prime}\left(E_{t}\left(y_{t+1}+s_{t}\right)\right)=u^{\prime}(\underbrace{E_{t} y_{t+1}}_{\bar{y}}+s_{t}) \\
\Rightarrow u^{\prime}\left(\bar{y}-s_{t}\right)=u^{\prime}\left(\bar{y}+s_{t}\right) \Rightarrow s_{t}=0
\end{gathered}
$$

- with convex marginal utility $u^{\prime \prime \prime}(c)>0$ and $u^{\prime}\left(E_{t}().\right)<E_{t} u^{\prime}($.$) (Jensen's$ inequality):

$$
\begin{aligned}
& \text { for } s_{t}=0: \quad u^{\prime}\left(c_{t}\right)<E_{t} u^{\prime}\left(c_{t+1}\right) \Rightarrow \text { f.o.c. violated } \\
& \text { for } s_{t}>0: \quad\left\{\begin{array}{cc}
c_{t} \downarrow & \text { and } u^{\prime}\left(c_{t}\right) \uparrow \\
c_{t+1} \uparrow & \text { and } E_{t} u^{\prime}\left(c_{t+1}\right) \downarrow
\end{array}\right\} \Rightarrow \text { f.o.c. holds }
\end{aligned}
$$

## Implications

The precautionary saving motive determines an upward optimal consumption path. The steepness of the path is related to the consumer's degree of prudence.

Let $r=\rho$ and take the f.o.c.

$$
u^{\prime}\left(c_{t}\right)=E_{t} u^{\prime}\left(c_{t+1}\right)
$$

The r.h.s. can be approximated using a second-order Taylor expansion around $c_{t}$ :

$$
E_{t} u^{\prime}\left(c_{t+1}\right) \simeq u^{\prime}\left(c_{t}\right)+E_{t} u^{\prime \prime}\left(c_{t}\right)\left(c_{t+1}-c_{t}\right)+\frac{1}{2} E_{t} u^{\prime \prime \prime}\left(c_{t}\right)\left(c_{t+1}-c_{t}\right)^{2}
$$

f.o.c. becomes

$$
\begin{aligned}
0 & =u^{\prime \prime}\left(c_{t}\right) E_{t}\left(c_{t+1}-c_{t}\right)+\frac{1}{2} u^{\prime \prime \prime}\left(c_{t}\right) E_{t}\left(\left(c_{t+1}-c_{t}\right)^{2}\right) \\
\Rightarrow \quad E_{t}\left(c_{t+1}-c_{t}\right) & =-\frac{1}{2} \frac{u^{\prime \prime \prime}\left(c_{t}\right)}{u^{\prime \prime}\left(c_{t}\right)} E_{t}\left(\left(c_{t+1}-c_{t}\right)^{2}\right)
\end{aligned}
$$

dividing both sides by $c_{t}$

$$
E_{t}\left(\frac{c_{t+1}-c_{t}}{c_{t}}\right)=-\frac{1}{2} \frac{c_{t} u^{\prime \prime \prime}\left(c_{t}\right)}{u^{\prime \prime}\left(c_{t}\right)} E_{t}\left(\left(\frac{c_{t+1}-c_{t}}{c_{t}}\right)^{2}\right)
$$

and defining the coefficient of "relative prudence" $p \equiv-\frac{c_{t} u^{\prime \prime}\left(c_{t}\right)}{u^{\prime \prime}\left(c_{t}\right)}$ :

$$
\begin{array}{lcc}
E_{t}\left(\frac{c_{t+1}-c_{t}}{c_{t}}\right) & =\frac{1}{2} p & E_{t}\left(\left(\frac{c_{t+1}-c_{t}}{c_{t}}\right)^{2}\right) \\
\text { expected consumption } \\
\text { growth rate } & & \text { measure of consumption } \\
\text { variability due to uncertainty }
\end{array}
$$

Important special case: CRRA utility function $u\left(c_{t}\right)=\frac{c_{t}^{1-\gamma}-1}{1-\gamma}$ and $r \neq \rho$

$$
\begin{aligned}
& \text { f.o.c. } \quad c_{t}^{-\gamma}=\frac{1+r}{1+\rho} E_{t}\left(c_{t+1}^{-\gamma}\right) \quad \Rightarrow 1=\frac{1+r}{1+\rho} E_{t}\left[\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}\right] \\
& \text { in logs } \quad 0=(r-\rho)+\log E_{t}\left[\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}\right]
\end{aligned}
$$

Assume the following: $\Delta \log c_{t+1} \sim N\left(E_{t} \Delta \log c_{t+1}, \sigma_{c}^{2}\right)$ and make use of the property of lognormal distributions:

$$
\log E(x)=E(\log x)+\frac{1}{2} \operatorname{var}(\log x)
$$

here $x=\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}$ with $\log x=-\gamma \Delta \log c_{t+1} \sim N\left(-\gamma E_{t} \Delta \log c_{t+1}, \gamma^{2} \sigma_{c}^{2}\right)$ :

$$
\log E_{t}\left[\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}\right]=-\gamma E_{t}\left(\Delta \log c_{t+1}\right)+\frac{1}{2} \gamma^{2} \sigma_{c}^{2}
$$

in f.o.c.:

$$
\begin{aligned}
& 0=(r-\rho)-\gamma E_{t}\left(\Delta \log c_{t+1}\right)+\frac{1}{2} \gamma^{2} \sigma_{c}^{2} \\
& \Rightarrow E_{t}\left(\Delta \log c_{t+1}\right)=\frac{1}{\gamma}(r-\rho)+\frac{\gamma}{2} \sigma_{c}^{2} \\
& \begin{array}{c}
\text { precautionary } \\
\text { saving effect }
\end{array}
\end{aligned}
$$

## 3B. Extension: consumption and asset allocation

## Basic Consumption Capital Asset Pricing Model (CCAPM)

Many financial assets with stochastic returns

$$
\begin{aligned}
& n \text { assets with uncertain returns } r^{j}(j=1, \ldots, n) \\
& A_{t+i}^{j}: \text { stock of asset } j \text { held at the beginning of period } t+i \\
& A_{t+i}=\sum_{j=1}^{n} A_{t+i}^{j}: \text { stock of financial wealth } \\
& r_{t+i+1}^{j}: \text { return on asset } j \text { in period } t+i \text { not known at the beginning of } t+i \\
& \Rightarrow A_{t+i+1}^{j}=\left(1+r_{t+i+1}^{j}\right) A_{t+i}^{j}
\end{aligned}
$$

Problem:

$$
\max _{\left\{c_{t+i}, A_{t+i}^{j}\right\}} U_{t}=E_{t} \sum_{i=0}^{\infty}\left(\frac{1}{1+\rho}\right)^{i} u\left(c_{t+i}\right)
$$

subject to

$$
\sum_{j=1}^{n} A_{t+i+1}^{j}=\sum_{j=1}^{n}\left(1+r_{t+i+1}^{j}\right) A_{t+i}^{j}+y_{t+i}-c_{t+i} \quad(i=0, \ldots \infty)
$$

## Solution:

$$
\begin{array}{ll}
\text { f.o.c. }(\text { for } i=0) & u^{\prime}\left(c_{t}\right)=\frac{1}{1+\rho} E_{t}\left[\left(1+r_{t+1}^{j}\right) u^{\prime}\left(c_{t+1}\right)\right] \quad(j=1, \ldots, n) \\
\Rightarrow & 1=E_{t}[\left(1+r_{t+1}^{j}\right) \underbrace{\frac{1}{1+\rho} \frac{u^{\prime}\left(c_{t+1}\right)}{u^{\prime}\left(c_{t}\right)}}_{\text {stochastic discount factor }}] \\
\Rightarrow & 1=E_{t}\left[\left(1+r_{t+1}^{j}\right) M_{t+1}\right]
\end{array}
$$

where $M_{t+1} \equiv \frac{1}{1+\rho} \frac{u^{\prime}\left(c_{t+1}\right)}{u^{\prime}\left(c_{t}\right)}$ SDF (marginal rate of intertemporal substitution)
Theoretical implications
Using the property:

$$
E_{t}\left[\left(1+r_{t+1}^{j}\right) M_{t+1}\right]=E_{t}\left(1+r_{t+1}^{j}\right) E_{t}\left(M_{t+1}\right)+\operatorname{cov}_{t}\left(r_{t+1}^{j}, M_{t+1}\right)
$$

we get:

$$
\begin{equation*}
E_{t}\left(1+r_{t+1}^{j}\right)=\frac{1}{E_{t}\left(M_{t+1}\right)}\left[1-\operatorname{cov}_{t}\left(r_{t+1}^{j}, M_{t+1}\right)\right] \tag{CCAPM1}
\end{equation*}
$$

If one of the assets is riskless, with certain return $r^{0}$ :

$$
\begin{equation*}
1+r_{t+1}^{0}=\frac{1}{E_{t}\left(M_{t+1}\right)} \tag{CCAPM2}
\end{equation*}
$$

Combining CCAPM 1 and CCAPM 2 :

$$
\begin{equation*}
\underbrace{E_{t} r_{t+1}^{j}-r_{t+1}^{0}}_{\text {equity premium }}=-\left(1+r_{t+1}^{0}\right) \operatorname{cov}_{t}\left(r_{t+1}^{j}, M_{t+1}\right) \tag{CCAPM3}
\end{equation*}
$$

Example: assuming "power utility" (CRRA), for each asset $j$ we get:

$$
1=E_{t}\left[\left(1+r_{t+1}^{j}\right) \frac{1}{1+\rho}\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}\right]
$$

In logs:

$$
0=-\rho+\log E_{t}\left[\left(1+r_{t+1}^{j}\right)\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}\right]
$$

Distributional assumption: growth rate of consumption and asset returns are (conditionally) jointly lognormally distributed. For generic random variables $x$ and $y$ :

$$
\log E_{t}\left(x_{t+1} y_{t+1}\right)=E_{t}\left(\log \left(x_{t+1} y_{t+1}\right)\right) \quad+\frac{1}{2} \underbrace{\operatorname{var}_{t}\left(\log \left(x_{t+1} y_{t+1}\right)-E_{t}\left(\log \left(x_{t+1} y_{t+1}\right)\right)^{2}\right]}_{E_{t}[\log }
$$

Using this property:

$$
\log E_{t}\left[\left(1+r_{t+1}^{j}\right)\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}\right]=E_{t}\left(r_{t+1}^{j}-\gamma \Delta \log c_{t+1}\right)+\frac{1}{2} \Sigma_{j}
$$

where

$$
\Sigma_{j}=E\left[\left(\left(r_{t+1}^{j}-\gamma \Delta \log c_{t+1}\right)-E_{t}\left(r_{t+1}^{j}-\gamma \Delta \log c_{t+1}\right)\right)^{2}\right]
$$

(the expectation is not conditional on $t$ if we assume conditional homoscedasticity of asset returns and consumption)

$$
\Rightarrow \quad E_{t} r_{t+1}^{j}=\gamma E_{t} \Delta \log c_{t+1}+\rho-\frac{1}{2} \Sigma_{j}
$$

Note:
(i) the f.o.c. can be expressed as Euler equations:

$$
E_{t} \Delta \log c_{t+1}=\frac{1}{\gamma}\left(E_{t} r_{t+1}^{j}-\rho\right)+\frac{1}{2 \gamma} \Sigma_{j}
$$

(ii) the f.o.c. is a relation between the expected consumption growth rate and the expected returns on all assets, given by $\frac{1}{\gamma}$.
(iii) calculating $\Sigma_{j}$ :

$$
\begin{aligned}
\Sigma_{j}= & E\left[\left(\left(r_{t+1}^{j}-E_{t} r_{t+1}^{j}\right)-\gamma\left(\Delta \log c_{t+1}-E_{t} \Delta \log c_{t+1}\right)\right)^{2}\right] \\
= & \underbrace{E\left[\left(r_{t+1}^{j}-E_{t} r_{t+1}^{j}\right)^{2}\right]}_{\sigma_{j}^{2}}+\gamma^{2} \underbrace{E\left[\left(\Delta \log c_{t+1}-E_{t} \Delta \log c_{t+1}\right)^{2}\right]}_{\sigma_{c}^{2}} \\
& -2 \gamma \underbrace{E\left[\left(r_{t+1}^{j}-E_{t} r_{t+1}^{j}\right)\left(\Delta \log c_{t+1}-E_{t} \Delta \log c_{t+1}\right)\right]}_{\sigma_{j c}} \\
\equiv & \sigma_{j}^{2}+\gamma^{2} \sigma_{c}^{2}-2 \gamma \sigma_{j c}
\end{aligned}
$$

The three basic CCAPM relations become:

$$
\begin{equation*}
E_{t} r_{t+1}^{j}=\gamma E_{t} \Delta \log c_{t+1}+\rho-\frac{\sigma_{j}^{2}}{2}-\frac{\gamma^{2} \sigma_{c}^{2}}{2}+\gamma \sigma_{j c} \tag{CCAPM1}
\end{equation*}
$$

For the riskfree asset $\sigma_{j c}=\sigma_{j}^{2}=0$ :

$$
\begin{equation*}
r_{t+1}^{0}=\gamma E_{t} \Delta \log c_{t+1}+\rho-\frac{\gamma^{2} \sigma_{c}^{2}}{2} \tag{CCAPM2}
\end{equation*}
$$

Premia on risky assets:

$$
\begin{equation*}
\underbrace{\left(E_{t} r_{t+1}^{j}-r_{t+1}^{0}\right)+\frac{\sigma_{j}^{2}}{2}}_{\text {adjusted excess return }}=\gamma \sigma_{j c} \tag{CCAPM3}
\end{equation*}
$$

## Empirical evidence and puzzles

Selected facts (mainly for US but confirmed in cross-countries studies):

1. "high" real average return on stocks (US S\&P index: $8.62 \%$ per year 19501999)
2. "low" real average return on "riskless" bonds $\rightarrow$ riskfree real rate (US Tbills: $1.99 \%$ per year 1950-1999)
3. positive and very smooth real consumption growth (for non-durables \& services: $2 \%$ annual growth rate with $1.1 \%$ standard deviation)
4. relatively low correlation between consumption growth and stock returns (0.22 at quarterly horizon)

Implications for the interpretation of the joint behavior of asset returns and consumption:
$\Rightarrow$ (1) and (2) imply a high expected excess return on stocks (high equity premium)
theory: CCAPM suggests an explanation in terms of:
(i) covariance between consumption and stock returns, $\sigma_{j c}$
(ii) degree of agents' risk aversion, $\gamma$

## but:

$\Rightarrow$ (3) and (4) imply that $\sigma_{j c}$ is low, so a very high $\gamma$ is needed to generate the observed premium:

## $\rightarrow$ equity premium puzzle

Is the hypothesis of a very high $\gamma$ consistent with facts (1), (2) and (3)?
with high risk aversion:
$\rightarrow$ strong incentive to transfer purchasing power to periods of low expected consumption levels
$\rightarrow$ given consumption growth (see fact (3)) there should be a tendency for consumers to borrow heavily in capital markets, generating an upward pressure on (the general level of) interest rates,

## but:

$\Rightarrow$ the relatively low observed interest rate (fact (2)) implies that the consumers' intertemporal rate of time preference is very low (even "negative": agents are very "patient"). Only extremely low rates of time preference could reconcile consumption growth with low interest rates:

## $\rightarrow$ riskfree rate puzzle

## New directions: more general specification of preferences

General insight: to explain the high equity premium, additional variables are needed in the utility function affecting marginal utility -and the stochastic discount factor- in a non-separable way. For a generic variable $z$ :

$$
E(r)-r^{0}=\frac{-c u_{c c}}{u_{c}} \operatorname{cov}(r, \Delta \log c)+\frac{z u_{c z}}{u_{c}} \operatorname{cov}(r, z)
$$

One possibility is: (external) habit formation in consumers' behavior (CampbellCochrane, JPE 1999) $\rightarrow$ introduce time non-separability

- intuition: people get accustomed to a standard of living and a decline in consumption after some time of high consumption (i.e. a recession) hurts more in utility terms
- extension of utility function:

$$
u\left(c_{t}, x_{t}\right)=u\left(c_{t}-x_{t}\right)=\frac{\left(c_{t}-x_{t}\right)^{1-\gamma}-1}{1-\gamma}
$$

where $x \equiv$ level of "habit" and $\gamma$ is the power parameter (not capturing risk aversion). The relation between the current level of consumption and "habit" is captured by the surplus consumption ratio $s_{t}=\frac{c_{t}-x_{t}}{c_{t}}$ so that:

$$
u_{c}\left(s_{t} c_{t}\right)=\left(s_{t} c_{t}\right)^{-\gamma} \Rightarrow \frac{-c_{t} u_{c c}}{u_{c}} \equiv \eta_{t}=\frac{\gamma}{s_{t}}
$$

risk aversion ("curvature" of marginal utility) $\eta$ higher than power parameter and time-varying according to the surplus ratio: people are more risk averse when consumption falls towards habit

- implication for equity premium:

$$
\text { f.o.c. } \quad 1=E_{t}\left[\left(1+r_{t+1}^{j}\right) \frac{1}{1+\rho}\left(\frac{s_{t+1}}{s_{t}}\right)^{-\gamma}\left(\frac{c_{t+1}}{c_{t}}\right)^{-\gamma}\right] \quad j=1, \ldots n
$$

with distributional assumptions:

$$
\left(E_{t} r_{t+1}^{j}-r_{t+1}^{0}\right)+\frac{\sigma_{j}^{2}}{2}=\eta_{t} \sigma_{j c}
$$

higher risk aversion may explain high premium even with low consumptionreturn covariance

- implication for riskfree rate. A higher risk aversion does not imply a higher riskfree interest rate ("riskfree rate puzzle") due to a strong precautionary savings effect:

$$
r_{t+1}^{0}=\gamma E_{t} \Delta \log c_{t+1}+\rho-\frac{1}{2}\left(\frac{\gamma}{\bar{s}}\right)^{2} \sigma_{c}^{2}
$$

## Problems

1. Assume the following general stochastic process for labor income:

$$
y_{t+1}=\lambda y_{t}+(1-\lambda) \bar{y}+\varepsilon_{t+1}
$$

and consider the two polar cases: (i) $\lambda=0$ and (ii) $\lambda=1$. Calculate the effect of an income innovation $\varepsilon_{t+1}$ on savings in $t+1\left(s_{t+1}\right)$ and on savings and disposable income in subsequent periods ( $s_{t+i}$ and $y_{t+i}^{D}$ for $i \geq 2$ ).
2. With reference to the same stochastic process for labor income

$$
y_{t+1}=\lambda y_{t}+(1-\lambda) \bar{y}+\varepsilon_{t+1}
$$

check that the consumption function (expressing $c_{t}$ as a function of $A_{t}, y_{t}$ and $\bar{y}$ ) has the following form:

$$
c_{t}=r A_{t}+\frac{r}{1+r-\lambda} y_{t}+\frac{1-\lambda}{1+r-\lambda} \bar{y}
$$

Comment on the two particular cases: $\lambda=0$ and $\lambda=1$.
3. Using the basic version of the rational expectations/permanent income model with quadratic utility and $r=\rho$, assume that labor income is generated by the following stochastic process:

$$
y_{t+1}=\bar{y}+\varepsilon_{t+1}-\delta \varepsilon_{t} \quad(\delta>0)
$$

where $\bar{y}$ is the mean value of income and $E_{t} \varepsilon_{t+1}=0$.
(a) Discuss the impact of an increase in mean income $\bar{y}(\Delta \bar{y}>0)$ on agent's permanent income, consumption and savings;
(b) now suppose that, in period $t+1$ only, a positive innovation in income occurs: $\varepsilon_{t+1}>0$. In all past periods income has been equal to its mean level: $y_{t-i}=\bar{y}$ for $i=0, \ldots \infty$. Find the change in consumption between $t$ and $t+1\left(\Delta c_{t+1}\right)$ as a function of $\varepsilon_{t+1}$, providing an economic intuition for your result.
(c) with reference to question (b), discuss what happens to savings in periods $t+1$ and $t+2$.
4. Consider the optimization problem of a consumer living only two periods with consumption levels $c_{1}$ and $c_{2}$ and labor incomes $y_{1}$ and $y_{2}$. The utility function $u(c)$ has the following form (with $a, b>0$ ):

$$
u(c)=\left\{\begin{array}{c}
a c-\frac{b}{2} c^{2} \text { for } c<\frac{a}{b} \\
\frac{a^{2}}{2 b} \text { for } c \geq \frac{a}{b}
\end{array}\right.
$$

(a) Plot marginal utility $u^{\prime}$ as a function of consumption $c$;
(b) assume $r=\rho=0, y_{1}=\frac{a}{b}$ (with certainty) and

$$
y_{2}= \begin{cases}\frac{a}{b}+\sigma & \text { with probability } 1 / 2 \\ \frac{a}{b}-\sigma & \text { with probability } 1 / 2\end{cases}
$$

Solve the consumer's expected utility maximization problem (writing down the first order condition linking $c_{1}$ and $c_{2}$ ), find the values of $c_{1}$ and $c_{2}$ with $\sigma=0$, and discuss the effect of $\sigma>0$ on $c_{1}$.

