3A. Extension: precautionary saving

Microfoundations

Motivations to save in basic rational expectations/permanent income model:

- expected declines in income
- $r > \rho$

$\Rightarrow$ no role for uncertainty on future labor incomes in determining savings: no "*precautionary savings*"

The role of uncertainty is limited by the assumption of *quadratic utility* $\Rightarrow$ *linear marginal utility*, implying:

$$E\, u'(c) = u'\left(E\,(c)\right)$$

> $\Rightarrow$ only expected values matter (other characteristics of the distributions of y and c, e.g. variance, are irrelevant): an increase in uncertainty (with unchanged expected values) does **not** cause any reaction by agents ("*certainty equivalence*" assumption)

If $u'(c)$ is a *convex* function, consumers display "*prudent*" behaviour, reacting to an increase in uncertainty by increasing savings: *precautionary saving*

$$u'(c) \quad \text{convex}: \; u'''(c) > 0$$

Note: to have *risk aversion*: $u''(c) < 0$ (also with quadratic utility)
to have *prudence*: $u'''(c) > 0$ (with quadratic utility: $u'''(c) = 0$)

Example: two periods (t and $t+1$), $A_t = 0$; labor incomes:

$$y_t = \bar{y} \quad \text{with certainty}$$

$$y_{t+1} = \begin{cases} y_{t+1}^H & \text{with prob. } 1/2 \\ y_{t+1}^L & \text{with prob. } 1/2 \end{cases} \qquad \text{with } y_{t+1}^H > y_{t+1}^L$$

No "standard" incentives to save: assume

- $E_t\, y_{t+1} = \bar{y}$
- $r = \rho$ ($= 0$ for simplicity)

F.o.c. :

$$u'(c_t) = E_t\, u'(c_{t+1})$$

saving and consumption:

$$\text{in } t \quad : \quad s_t = \bar{y} - c_t$$

$$\text{in } t+1 \quad : \quad \left.\begin{array}{c} c_{t+1}^H \\ c_{t+1}^L \end{array}\right\} = \underbrace{(\bar{y} - c_t)}_{s_t} + \left\{\begin{array}{c} y_{t+1}^H \\ y_{t+1}^L \end{array}\right.$$

Using s_t, f.o.c. becomes:

$$u'(\bar{y} - s_t) = E_t\left(u'\left(y_{t+1} + s_t\right)\right)$$

- with quadratic utility $u'''(c) = 0$ (linear marginal utility):

$$\begin{aligned} E_t\left(u'\left(y_{t+1} + s_t\right)\right) &= u'\left(E_t(y_{t+1} + s_t)\right) = u'(\underbrace{E_t\, y_{t+1}}_{\bar{y}} + s_t) \\ \Rightarrow u'(\bar{y} - s_t) &= u'(\bar{y} + s_t) \quad \Rightarrow \quad s_t = 0 \end{aligned}$$

- with convex marginal utility $u'''(c) > 0$ and $u'(E_t\,(.)) < E_t\, u'(.)$ (Jensen's inequality):

$$\text{for } s_t = 0 : \quad u'(c_t) < E_t\, u'\left(c_{t+1}\right) \quad \Rightarrow \quad \text{f.o.c. violated}$$

$$\text{for } s_t > 0 : \quad \left\{\begin{array}{ccc} c_t \downarrow & \text{and} & u'(c_t) \uparrow \\ c_{t+1} \uparrow & \text{and} & E_t\, u'\left(c_{t+1}\right) \downarrow \end{array}\right\} \Rightarrow \quad \text{f.o.c. holds}$$

Implications

The precautionary saving motive determines an upward optimal consumption path. The steepness of the path is related to the consumer's degree of *prudence*.

Let $r = \rho$ and take the f.o.c.

$$u'(c_t) = E_t\, u'(c_{t+1})$$

The r.h.s. can be approximated using a second-order Taylor expansion around c_t:

$$E_t\, u'(c_{t+1}) \simeq u'(c_t) + E_t\, u''(c_t)\,(c_{t+1} - c_t) + \frac{1}{2} E_t\, u'''(c_t)\,(c_{t+1} - c_t)^2$$

f.o.c. becomes

$$\begin{aligned} 0 &= u''(c_t)\, E_t\,(c_{t+1} - c_t) + \frac{1}{2}\, u'''(c_t)\, E_t\, \left((c_{t+1} - c_t)^2\right) \\ \Rightarrow \quad E_t\,(c_{t+1} - c_t) &= -\frac{1}{2}\, \frac{u'''(c_t)}{u''(c_t)}\, E_t\, \left((c_{t+1} - c_t)^2\right) \end{aligned}$$

dividing both sides by c_t

$$E_t \left(\frac{c_{t+1} - c_t}{c_t} \right) = -\frac{1}{2}\, \frac{c_t\, u'''(c_t)}{u''(c_t)}\, E_t \left(\left(\frac{c_{t+1} - c_t}{c_t} \right)^2 \right)$$

and defining the coefficient of "*relative prudence*" $p \equiv -\frac{c_t\, u'''(c_t)}{u''(c_t)}$:

$$\underbrace{E_t \left(\frac{c_{t+1} - c_t}{c_t} \right)}_{\text{expected consumption growth rate}} = \frac{1}{2}\; p \quad \underbrace{E_t \left(\left(\frac{c_{t+1} - c_t}{c_t} \right)^2 \right)}_{\text{measure of consumption variability due to uncertainty}}$$

Important special case: CRRA utility function $u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$ and $r \neq \rho$

$$\begin{aligned} \text{f.o.c.} \quad c_t^{-\gamma} &= \frac{1+r}{1+\rho} E_t\, \left(c_{t+1}^{-\gamma}\right) \quad \Rightarrow 1 = \frac{1+r}{1+\rho}\, E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \\ \text{in logs} \quad 0 &= (r - \rho) + \log\; E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \end{aligned}$$

Assume the following: $\Delta \log\, c_{t+1} \sim N(E_t\, \Delta \log\, c_{t+1}, \sigma_c^2)$ and make use of the property of lognormal distributions:

$$\log\; E\,(x) = E\,(\log\, x) + \frac{1}{2}\, \text{var}\,(\log\, x)$$

here $x = \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$ with $\log\, x = -\gamma\,\Delta\log\, c_{t+1} \sim N(-\gamma\, E_t\,\Delta\log\, c_{t+1}, \gamma^2\,\sigma_c^2)$:

$$\log\, E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = -\gamma\, E_t\,(\,\Delta\log\, c_{t+1}) + \frac{1}{2}\gamma^2\,\sigma_c^2$$

in f.o.c.:

$$\begin{aligned} 0 &= (r-\rho) - \gamma\, E_t\,(\,\Delta\log\, c_{t+1}) + \frac{1}{2}\gamma^2\,\sigma_c^2 \\ \Rightarrow E_t\,(\,\Delta\log\, c_{t+1}) &= \frac{1}{\gamma}(r-\rho)\; + \underbrace{\frac{\gamma}{2}\sigma_c^2}_{\substack{\text{precautionary} \\ \text{saving effect}}} \end{aligned}$$

3B. Extension: consumption and asset allocation

Basic Consumption Capital Asset Pricing Model (CCAPM)

Many financial assets with stochastic returns

n assets with uncertain returns r^j $(j = 1, ..., n)$

A^j_{t+i} : stock of asset j held at the beginning of period $t+i$

$A_{t+i} = \sum_{j=1}^n A^j_{t+i}$: stock of financial wealth

r^j_{t+i+1} : return on asset j in period $t+i$ *not known* at the beginning of $t+i$
$\Rightarrow\; A^j_{t+i+1} = (1 + r^j_{t+i+1})\, A^j_{t+i}$

Problem:

$$\max_{\left\{c_{t+i}, A^j_{t+i}\right\}} \quad U_t = E_t \sum_{i=0}^{\infty}\left(\frac{1}{1+\rho}\right)^i u\,(c_{t+i})$$

subject to

$$\sum_{j=1}^n A^j_{t+i+1} = \sum_{j=1}^n (1 + r^j_{t+i+1})\, A^j_{t+i} + y_{t+i} - c_{t+i} \qquad (i = 0, ...\infty)$$

Solution:

f.o.c. (for $i = 0$) $$u'(c_t) = \frac{1}{1+\rho} E_t \left[(1 + r^j_{t+1})\, u'(c_{t+1}) \right] \qquad (j = 1, ..., n)$$

$$\Rightarrow \quad 1 = E_t \left[(1 + r^j_{t+1}) \quad \underbrace{\frac{1}{1+\rho} \frac{u'(c_{t+1})}{u'(c_t)}}_{\text{stochastic discount factor}} \right]$$

$$\Rightarrow \quad 1 = E_t \left[(1 + r^j_{t+1})\, M_{t+1} \right]$$

where $M_{t+1} \equiv \frac{1}{1+\rho} \frac{u'(c_{t+1})}{u'(c_t)}$ SDF (marginal rate of intertemporal substitution)

Theoretical implications

Using the property:

$$E_t \left[(1 + r^j_{t+1})\, M_{t+1} \right] = E_t\ (1 + r^j_{t+1})\, E_t(M_{t+1}) + \mathrm{cov}_t\, (r^j_{t+1}, M_{t+1})$$

we get:

$$E_t\ (1 + r^j_{t+1}) = \frac{1}{E_t(M_{t+1})} \left[1 - \mathrm{cov}_t\, (r^j_{t+1}, M_{t+1}) \right] \tag{CCAPM 1}$$

If one of the assets is riskless, with certain return r^0:

$$1 + r^0_{t+1} = \frac{1}{E_t(M_{t+1})} \tag{CCAPM 2}$$

Combining *CCAPM 1* and *CCAPM 2* :

$$\underbrace{E_t\ r^j_{t+1} - r^0_{t+1}}_{\text{equity premium}} = -(1 + r^0_{t+1})\, \mathrm{cov}_t\, (r^j_{t+1}, M_{t+1}) \tag{CCAPM 3}$$

Example: assuming "power utility" (CRRA), for each asset j we get:

$$1 = E_t \left[(1 + r^j_{t+1}) \frac{1}{1+\rho} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]$$

In logs:

$$0 = -\rho + \log\ E_t \left[(1 + r^j_{t+1}) \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]$$

Distributional assumption: growth rate of consumption and asset returns are (conditionally) jointly lognormally distributed. For generic random variables x and y:

$$\log E_t\left(x_{t+1}y_{t+1}\right) = E_t\left(\log\left(x_{t+1}y_{t+1}\right)\right) \quad + \quad \frac{1}{2} \qquad \underbrace{\mathrm{var}_t\left(\log\left(x_{t+1}y_{t+1}\right)\right)}_{E_t[\log(x_{t+1}y_{t+1}) - E_t(\log(x_{t+1}y_{t+1}))^2]}$$

Using this property:

$$\log E_t\left[(1+r_{t+1}^j)\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = E_t\left(r_{t+1}^j - \gamma\,\Delta\log c_{t+1}\right) + \frac{1}{2}\Sigma_j$$

where

$$\Sigma_j = E\left[\left((r_{t+1}^j - \gamma\,\Delta\log c_{t+1}) - E_t\left(r_{t+1}^j - \gamma\,\Delta\log c_{t+1}\right)\right)^2\right]$$

(the expectation is not conditional on t if we assume conditional homoscedasticity of asset returns and consumption)

$$\Rightarrow \quad E_t\, r_{t+1}^j = \gamma\, E_t\,\Delta\,\log c_{t+1} + \rho - \frac{1}{2}\,\Sigma_j$$

Note:

(i) the f.o.c. can be expressed as Euler equations:

$$E_t\,\Delta\log\,c_{t+1} = \frac{1}{\gamma}\,(E_t\,r^j_{t+1} - \rho) + \frac{1}{2\gamma}\,\Sigma_j$$

(ii) the f.o.c. is a relation between the expected consumption growth rate and the expected returns on *all* assets, given by $\frac{1}{\gamma}$.

(iii) calculating Σ_j:

$$\begin{aligned}
\Sigma_j \quad &= \quad E\left[\left((r^j_{t+1} - E_t\,r^j_{t+1}) - \gamma\,(\Delta\log\,c_{t+1} - E_t\,\Delta\log\,c_{t+1})\right)^2\right] \\
&= \underbrace{E\,\left[(r^j_{t+1} - E_t r^j_{t+1})^2\right]}_{\sigma^2_j} + \gamma^2 \underbrace{E\,\left[(\Delta\,\log c_{t+1} - E_t\,\Delta\,\log c_{t+1})^2\right]}_{\sigma^2_c} \\
&\quad -2\gamma \underbrace{E\,\left[(r^j_{t+1} - E_t r^j_{t+1})\,(\Delta\,\log c_{t+1} - E_t\,\Delta\,\log c_{t+1})\right]}_{\sigma_{jc}} \\
&\equiv \quad \sigma^2_j + \gamma^2\,\sigma^2_c - 2\,\gamma\,\sigma_{jc}
\end{aligned}$$

The three basic CCAPM relations become:

$$E_t\,r^j_{t+1} = \gamma\,E_t\,\Delta\,\log c_{t+1} + \rho - \frac{\sigma^2_j}{2} - \frac{\gamma^2\,\sigma^2_c}{2} + \gamma\,\sigma_{jc} \qquad \text{(CCAPM 1)}$$

For the riskfree asset $\sigma_{jc} = \sigma^2_j = 0$:

$$r^0_{t+1} = \gamma\,E_t\,\Delta\,\log c_{t+1} + \rho - \frac{\gamma^2\,\sigma^2_c}{2} \qquad \text{(CCAPM 2)}$$

Premia on risky assets:

$$\underbrace{(E_t\,r^j_{t+1} - r^0_{t+1}) + \frac{\sigma^2_j}{2}}_{\text{adjusted excess return}} = \gamma\,\sigma_{jc} \qquad \text{(CCAPM 3)}$$

Empirical evidence and puzzles

Selected *facts* (mainly for US but confirmed in cross-countries studies):

1. "high" real average return on stocks (US S&P index: 8.62% per year 1950-1999)
2. "low" real average return on "riskless" bonds $\rightarrow$ riskfree real rate (US T-bills: 1.99% per year 1950-1999)
3. positive and very smooth real consumption growth (for non-durables & services: 2% annual growth rate with 1.1% standard deviation)
4. relatively low correlation between consumption growth and stock returns (0.22 at quarterly horizon)

Implications for the interpretation of the joint behavior of asset returns and consumption:

$\Rightarrow$ (1) and (2) imply a high expected excess return on stocks (*high equity premium*)

theory: **CCAPM** suggests an explanation in terms of:

(i) *covariance* between consumption and stock returns, σ_{jc}

(ii) degree of agents' *risk aversion,* γ

but:

$\Rightarrow$ (3) and (4) imply that σ_{jc} is low, so a *very high* γ is needed to generate the observed premium:

$\rightarrow$ **equity premium puzzle**

Is the hypothesis of a very high γ consistent with facts (1), (2) and (3) ?

with high risk aversion:

$\rightarrow$ strong incentive to transfer purchasing power to periods of low expected consumption levels

→ given consumption growth (see fact (3)) there should be a tendency for consumers to borrow heavily in capital markets, generating an upward pressure on (the general level of) interest rates,

but:

⇒ the relatively low observed interest rate (fact (2)) implies that the consumers' *intertemporal rate of time preference* is very low (even "negative": agents are very "patient"). Only extremely low rates of time preference could reconcile consumption growth with low interest rates:

→ **riskfree rate puzzle**

New directions: more general specification of preferences

General insight: to explain the high equity premium, additional variables are needed in the utility function affecting marginal utility -and the stochastic discount factor- in a *non-separable* way. For a generic variable z:

$$E(r) - r^0 = \frac{-c\, u_{cc}}{u_c} \operatorname{cov}(r, \Delta \log c) + \frac{z u_{cz}}{u_c} \operatorname{cov}(r, z)$$

One possibility is: (external) **habit formation** in consumers' behavior (Campbell-Cochrane, *JPE* 1999) → introduce *time non-separability*

- intuition: people get accustomed to a standard of living and a decline in consumption after some time of high consumption (i.e. a recession) hurts more in utility terms
- extension of utility function:

$$u(c_t, x_t) = u(c_t - x_t) = \frac{(c_t - x_t)^{1-\gamma} - 1}{1 - \gamma}$$

  where $x \equiv$ level of "habit" and γ is the power parameter (*not* capturing risk aversion). The relation between the current level of consumption and "habit" is captured by the *surplus consumption ratio* $s_t = \frac{c_t - x_t}{c_t}$ so that:

$$u_c(s_t c_t) = (s_t c_t)^{-\gamma} \Rightarrow \frac{-c_t u_{cc}}{u_c} \equiv \eta_t = \frac{\gamma}{s_t}$$

  risk aversion ("curvature" of marginal utility) η higher than power parameter and time-varying according to the surplus ratio: people are more risk averse when consumption falls towards habit

- implication for *equity premium*:

$$\text{f.o.c.} \quad 1 = E_t \left[(1 + r^j_{t+1}) \, \frac{1}{1+\rho} \left(\frac{s_{t+1}}{s_t} \right)^{-\gamma} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \qquad j = 1, ...n$$

  with distributional assumptions:

$$\left(E_t \; r^j_{t+1} - r^0_{t+1} \right) + \frac{\sigma^2_j}{2} = \eta_t \, \sigma_{jc}$$

  higher risk aversion may explain high premium even with low consumption-return covariance

- implication for *riskfree rate*. A higher risk aversion does not imply a higher riskfree interest rate ("riskfree rate puzzle") due to a strong precautionary savings effect:

$$r^0_{t+1} = \gamma \, E_t \, \Delta \log c_{t+1} + \rho - \frac{1}{2} \left(\frac{\gamma}{\bar{s}} \right)^2 \sigma^2_c$$

Problems

1. Assume the following general stochastic process for labor income:

$$y_{t+1} = \lambda\, y_t + (1-\lambda)\,\bar{y} + \varepsilon_{t+1}$$

and consider the two polar cases: (i) $\lambda = 0$ and (ii) $\lambda = 1$. Calculate the effect of an income innovation ε_{t+1} on *savings* in $t+1$ (s_{t+1}) and on savings and *disposable income* in subsequent periods (s_{t+i} and y^D_{t+i} for $i \geq 2$).

2. With reference to the same stochastic process for labor income

$$y_{t+1} = \lambda\, y_t + (1-\lambda)\,\bar{y} + \varepsilon_{t+1}$$

check that the *consumption function* (expressing c_t as a function of A_t, y_t and $\bar{y}$) has the following form:

$$c_t = r\,A_t + \frac{r}{1+r-\lambda}\,y_t + \frac{1-\lambda}{1+r-\lambda}\,\bar{y}$$

Comment on the two particular cases: $\lambda = 0$ and $\lambda = 1$.

3. Using the basic version of the rational expectations/permanent income model with quadratic utility and $r = \rho$, assume that labor income is generated by the following stochastic process:

$$y_{t+1} = \bar{y} + \varepsilon_{t+1} - \delta\,\varepsilon_t \qquad (\delta > 0)$$

where $\bar{y}$ is the mean value of income and $E_t \varepsilon_{t+1} = 0$.

   (a) Discuss the impact of an increase in mean income $\bar{y}$ ($\Delta\bar{y} > 0$) on agent's *permanent income*, *consumption* and *savings;*

   (b) now suppose that, in period $t+1$ only, a positive innovation in income occurs: $\varepsilon_{t+1} > 0$. In all past periods income has been equal to its mean level: $y_{t-i} = \bar{y}$ for $i = 0, ...\infty$. Find the change in consumption between t and $t+1$ (Δc_{t+1}) as a function of ε_{t+1}, providing an economic intuition for your result.

   (c) with reference to question (b), discuss what happens to *savings* in periods $t+1$ and $t+2$.

4. Consider the optimization problem of a consumer living only two periods with consumption levels c_1 and c_2 and labor incomes y_1 and y_2. The utility function $u(c)$ has the following form (with $a, b > 0$):

$$u(c) = \begin{cases} a\,c - \frac{b}{2}\,c^2 & \text{for} \quad c < \frac{a}{b} \\ \frac{a^2}{2b} & \text{for} \quad c \geq \frac{a}{b} \end{cases}$$

(a) Plot marginal utility u' as a function of consumption c;

(b) assume $r = \rho = 0$, $y_1 = \frac{a}{b}$ (with certainty) and

$$y_2 = \begin{cases} \frac{a}{b} + \sigma & \text{with probability } 1/2 \\ \frac{a}{b} - \sigma & \text{with probability } 1/2 \end{cases}$$

Solve the consumer's expected utility maximization problem (writing down the first order condition linking c_1 and c_2), find the values of c_1 and c_2 with $\sigma = 0$, and discuss the effect of $\sigma > 0$ on c_1.