Notes on: Dynamic models of consumption

General references:
Romer (2012) *Advanced Macroeconomics*, fourth edition, ch. 8

Specific references:
Deaton A. (1992) *Understanding consumption*, especially ch. 3, 4
Aims:

1. analysis of the consumption/saving choice for forward-looking, rational agents in a dynamic, stochastic environment;
2. application of dynamic optimization methods in discrete time.

Topics:

1. optimal consumption choice in the basic dynamic model:
   - basic problem set-up and assumptions;
   - solution and optimal dynamics of consumption/saving;
   - relationship between income, consumption and saving.
2. empirical issues:
   - excess sensitivity to anticipated income changes;
   - excess smoothness to unanticipated income changes.
3. extensions:
   - A: role of precautionary savings;
   - B: joint consumption/asset allocation choices (with stochastic interest rates) → CCAPM model.

1. Intertemporal optimization problem: set-up and solution

Basic framework:

infinite-horizon “representative agent” in an uncertain environment:
- maximizing an intertemporal utility function
- forming rational expectations on future values of relevant variables
At time \( t \) the agent

\[
\max_{\{c_{t+i}; i=0,1,\ldots\}} U(c_t, c_{t+1}, \ldots) \equiv U_t
\]

subject to the budget constraint (with \( i = 0, \ldots, \infty \))

\[
A_{t+i+1} = (1 + r_{t+i}) A_{t+i} + y_{t+i} - c_{t+i},
\]

\( A_{t+i} \): financial wealth at the beginning of period \( t + i \)

\( y_{t+i} \) and \( c_{t+i} \): labor income and consumption at the end of period \( t + i \) (timing convention)

\( r_{t+i} \): real rate of return of financial assets in period \( t + i \), assumed constant: \( r_{t+i} = r \)

**Assumptions on preferences:**

- *intertemporal (time) separability:*

\[
U(c_t, c_{t+1}, \ldots) = v_t(c_t) + v_{t+1}(c_{t+1}) + \ldots
\]

where \( v_{t+i}(c_{t+i}) \equiv \) valuation in \( t \) of utility derived from consumption at \( t + i \) (with \( v' > 0 \) and \( v'' < 0 \) ⇒ “habit formation” and durable goods are ruled out

- *future utility discount of the form*

\[
v_{t+i}(c_{t+i}) = \left( \frac{1}{1 + \rho} \right)^i u(c_{t+i})
\]

where \( \rho > 0 \) is the agent’s rate of time preference

⇒ no “dynamic inconsistency” of preferences

- *expected utility as objective function (with uncertainty):*

\[
U_t = E \left( \sum_{i=0}^{\infty} v_{t+i}(c_{t+i}) \mid I_t \right)
\]

where \( I_t \) is the information set available to the agent at time \( t \)

⇒ jointly with time-separability, this assumption on \( U \) generates an inverse relationship between the elasticity of intertemporal substitution and the degree of risk aversion (two theoretically different concepts)
The agent’s problem becomes:

\[
\max_{\{c_{t+i}, i=0,1,\ldots\}} U_t = E_t \left[ \sum_{i=0}^{\infty} \left( \frac{1}{1+\rho} \right)^i u(c_{t+i}) \right]
\]

subject to the budget constraint (with \(i = 0, \ldots, \infty\)):

\[
A_{t+i+1} = (1 + r)A_{t+i} + y_{t+i} - c_{t+i}, \quad A_t \text{ given}
\]

\(E_t[\cdot] : \text{rational expectation formed at } t \text{ on the information set } I_t \text{ (including } y_t)\)

Note: given a generic variable \(x_{t+i}\) for which \(E_t x_{t+i} = E(x_{t+i} \mid I_t)\), the rational expectations assumption implies that \(E_t(x_{t+i} - E(x_{t+i} \mid I_t)) = 0\) (the forecast error is uncorrelated with variables in the information set \(I_t\))

The one-period budget constraint can be used to derive the \textit{intertemporal budget constraint} (by repeated forward substitution of \(A_{t+i}\)):

\[
\frac{1}{1+r} \sum_{i=0}^{j-1} \left( \frac{1}{1+r} \right)^i c_{t+i} + \left( \frac{1}{1+r} \right)^j A_{t+j} = \frac{1}{1+r} \sum_{i=0}^{j-1} \left( \frac{1}{1+r} \right)^i y_{t+i} + A_t
\]

\(A\) is allowed to be negative (no liquidity constraints), but debt cannot grow at a rate larger than \(r\) (no Ponzi-game condition):

\[
\lim_{j \to \infty} \left( \frac{1}{1+r} \right)^j A_{t+j} \geq 0
\]

Therefore, letting \(j \to \infty\):

\[
\frac{1}{1+r} \sum_{i=0}^{\infty} \left( \frac{1}{1+r} \right)^i c_{t+i} = \frac{1}{1+r} \sum_{i=0}^{\infty} \left( \frac{1}{1+r} \right)^i y_{t+i} + A_t
\]

\(\text{present value of consumption flows} \quad \text{present value of labor income flows} \quad \text{initial financial wealth}\)

The \textit{intertemporal budget constraint} must hold also in expectation:

\[
\frac{1}{1+r} \sum_{i=0}^{\infty} \left( \frac{1}{1+r} \right)^i E_t c_{t+i} = \frac{1}{1+r} \sum_{i=0}^{\infty} \left( \frac{1}{1+r} \right)^i E_t y_{t+i} + A_t \quad (*)
\]

\[
= H_t + A_t
\]
**Solution:**

from the maximization (given $A_t$ and the terminal condition on financial wealth):

$$
\max_{A_{t+i}} U_t = E_t \sum_{i=0}^{\infty} \left( \frac{1}{1 + \rho} \right)^i u ((1 + r)A_{t+i} - A_{t+i+1} + y_{t+i})
$$

\[ \Rightarrow \text{ f.o.c. } E_t u'(c_{t+i}) = \frac{1 + r}{1 + \rho} E_t u'(c_{t+i+1}) \]

for $i = 0$ :

$$
u'(c_t) = \frac{1 + r}{1 + \rho} E_t u'(c_{t+1}) \quad \text{Euler equation}
$$

→ optimal dynamic path for marginal utility of consumption. Given $u'' < 0$ this implies:

$$
c_{t+1} > c_t \quad \text{if } r > \rho \\
c_{t+1} < c_t \quad \text{if } r < \rho \\
c_{t+1} = c_t \quad \text{if } r = \rho
$$

By how much consumption changes in response to $r - \rho$ (⇒ intertemporal substitution) depends on the shape of the marginal utility function, measured by $-\frac{u''(c)}{u'(c)}$.

**Example** (with certainty): CRRA (constant relative risk aversion) utility function

$$
u(c_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} \quad \gamma \equiv -\frac{cu''(c)}{u'(c)} > 0
$$

for which $u'(c) = c^{-\gamma}$ and $\gamma$ is the coefficient of relative risk aversion. The Euler equation is:

$$
c_t^{-\gamma} = \frac{1 + r}{1 + \rho} c_{t+1}^{-\gamma} \quad \Rightarrow \left( \frac{c_{t+1}}{c_t} \right)^\gamma = \frac{1 + r}{1 + \rho}
$$

Taking logs and using $\log(1 + r) \simeq r$ and $\log(1 + \rho) \simeq \rho$ :

$$
\Delta \log c_{t+1} = \frac{1}{\gamma} (r - \rho)
$$

$\gamma$ : (relative) risk aversion $\iff \frac{1}{\gamma}$ : (elasticity of) intertemp. substitution
Level and dynamics of optimal consumption:

defining
\[ u'(c_{t+1}) - E_t u'(c_{t+1}) = \eta_{t+1} \]
we get:
\[ u'(c_{t+1}) = \frac{1 + \rho}{1 + r} u'(c_t) + \eta_{t+1}. \]
Assuming \( r = \rho \), the stochastic process governing the dynamics of marginal utility is:
\[ u'(c_{t+1}) = u'(c_t) + \eta_{t+1} \]
with \( E_t \eta_{t+1} = 0 \) (by rational expectations).

To derive implications for the dynamics of consumption we assume quadratic utility → linear marginal utility:
\[ u(c) = c - \frac{b}{2} c^2 \rightarrow u'(c) = 1 - bc \]
obtaining the random walk model for consumption (Hall 1978):
\[ c_{t+1} = c_t + u_{t+1} \Rightarrow E_t c_{t+1} = c_t \]
with \( u_{t+1} \equiv -\frac{1}{b} \eta_{t+1} \Rightarrow E_t u_{t+1} = 0 \)
⇒ the best forecast in \( t \) of consumption in \( t + 1 \) is simply current consumption \( c_t \)
(the change in consumption \( u_{t+1} \) is orthogonal to any variable in the information set used to form \( E_t c_{t+1} \)).

To derive the consumption function use the intertemporal budget constraint (*) and note that
\[ E_t c_{t+1} = E_t c_{t+2} = ... = E_t c_{t+i} = ... = c_t \]
\[ \Rightarrow \frac{1}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i E_t c_{t+i} = \frac{1}{r} c_t \]
\[ \Rightarrow c_t = \frac{r}{\underline{(H_t + A_t)}} \equiv y_t^P \]
\( r(H_t + A_t) \) is the return on total wealth (annuity value), i.e. “permanent income” → current consumption is equal to permanent income.
To give economic content to the (unforecastable) change in consumption $u_{t+1}$, note:

$$u_{t+1} = c_{t+1} - c_t = c_{t+1} - E_t c_{t+1} = y^P_{t+1} - E_t y^P_{t+1}$$

$$y^P_{t+1} - E_t y^P_{t+1} = r \left( H_{t+1} - E_t H_{t+1} \right) + r \left( A_{t+1} - E_t A_{t+1} \right)$$

taking expectations of the one-period budget constraint in $t$:

$$E_t A_{t+1} = (1 + r) E_t A_t + E_t y_t - E_t c_t = (1 + r) A_t + y_t - c_t = A_{t+1}$$

To construct the “surprise” in human wealth, using $E_t E_{t+1}(\cdot) = E_t (\cdot)$:

$$H_{t+1} - E_t H_{t+1} = \frac{1}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i E_{t+1} y_{t+1+i} - \frac{1}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i E_t \left( E_{t+1} y_{t+1+i} \right)$$

$$= \frac{1}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i \left( E_{t+1} - E_t \right) y_{t+1+i}$$

present value of revision in expectations of future labor incomes

Therefore:

$$y^P_{t+1} = y^P_t + r \frac{1}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i \left( E_{t+1} - E_t \right) y_{t+1+i} \quad (** \quad \text{present value of revision in expectations of future labor incomes})$$

$$\Rightarrow c_{t+1} = c_t + u_{t+1}$$

**Optimal savings:**

optimal consumption choice has implications for savings and financial wealth accumulation

Define *disposable income*:

$$y^D_t = r A_t + y_t$$

and *saving* (using optimal consumption choice):

$$s_t = y^D_t - c_t = y^D_t - y^P_t = y_t - r H_t$$

transitory income
→ financial wealth $A$ is increased (decreased) when current income is higher (lower) than the annuity value of human wealth: buffer role for financial assets

\[ 
\Rightarrow s_t = y_t - \frac{r}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i E_t y_{t+i} \\
= \frac{1}{1 + r} y_t - \left( \frac{1}{1 + r} - \left( \frac{1}{1 + r} \right)^2 \right) E_t y_{t+1} - \\
- \left( \left( \frac{1}{1 + r} \right)^2 - \left( \frac{1}{1 + r} \right)^3 \right) E_t y_{t+2} + ... \\
= - \sum_{i=1}^{\infty} \left( \frac{1}{1 + r} \right)^i E_t \Delta y_{t+i} \\
\text{where } \Delta y_{t+i} = y_{t+i} - y_{t+i-1}
\]

→ “saving for a rainy day” behavior: financial asset accumulation when labor income is expected to fall

**Consumption, savings and current income:**

theory predicts:

\[ y_t \rightarrow y_t^P \rightarrow c_t \]

depends on

revisions of future

expected labor incomes

To derive the link between current income and consumption assume a stochastic process generating labor income (first-order autoregression):

\[ y_{t+1} = \lambda y_t + (1 - \lambda) \overline{y} + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0 \]

where $0 \leq \lambda \leq 1$ and $\overline{y}$ is the unconditional mean of the process. Realizations of the “innovation” component $\varepsilon_{t+1}$ at $t + 1$ cause revisions in expected values of
future labor incomes; e.g. for $y_{t+2}$:

$$E_{t+1}y_{t+2} = \lambda y_{t+1} + (1 - \lambda)\bar{y}$$

$$= \lambda [\lambda y_t + (1 - \lambda)\bar{y} + \varepsilon_{t+1}] + (1 - \lambda)\bar{y}$$

$$= \lambda^2 y_t + (1 + \lambda)(1 - \lambda)\bar{y} + \lambda\varepsilon_{t+1}$$

$$E_t y_{t+2} = \lambda^2 y_t + (1 + \lambda)(1 - \lambda)\bar{y}$$

$$\Rightarrow E_{t+1}y_{t+2} - E_t y_{t+2} = \lambda\varepsilon_{t+1}$$

In general:

$$E_{t+1}y_{t+i+1} - E_t y_{t+i+1} = \lambda^i \varepsilon_{t+1} \quad \forall i \geq 0$$

Using these revisions in expected future incomes in (**) yields the change in consumption:

$$c_{t+1} = c_t + \left(\frac{r}{1+r-\lambda}\right)\varepsilon_{t+1}$$

Given an innovation in current income $\varepsilon_{t+1}$, permanent income and consumption change by $\left(\frac{r}{1+r-\lambda}\right)\varepsilon_{t+1}$. Transitory income amounts to

$$\varepsilon_{t+1} - \frac{r}{1+r-\lambda}\varepsilon_{t+1} = \frac{1 - \lambda}{1+r-\lambda}\varepsilon_{t+1}$$

which is saved and added to the existing financial wealth.

The magnitude of the response of consumption to an innovation in income depends crucially on the degree of persistence of the stochastic process for $y$, captured by $\lambda$: e.g.

- $\lambda = 0$ (no persistence): innovation is entirely transitory. Human wealth evaluated at the beginning of period $t+1$ changes by $\frac{1}{1+r}\varepsilon_{t+1}$ and

$$c_{t+1} = c_t + \left(\frac{r}{1+r}\right)\varepsilon_{t+1}$$

- $\lambda = 1$: innovation has permanent effects on income (random walk) $y_{t+1} = y_t + \varepsilon_{t+1}$. Human wealth changes by $\frac{\varepsilon_{t+1}}{r}$ and

$$c_{t+1} = c_t + \varepsilon_{t+1}$$
2. Empirical issues

Basic implication of rational expectations/permanent income theory (with quadratic utility):

→ change in consumption orthogonal to variables in the agents’ information set: e.g. in equation

\[ \Delta c_{t+1} = \alpha \Delta y_t + e_{t+1} \]

theory implies : \( \alpha = 0 \)

if past incomes are in the agents’ information set at time \( t \): orthogonality test (Hall 1978)

More recently, two main lines of empirical research:

(a) test if consumption reacts to changes in current income as predicted by theory: \( \rightarrow \) excess sensitivity

(b) test if consumption reacts to innovations (i.e. unanticipated changes) in income as predicted by theory: \( \rightarrow \) excess smoothness

**Excess sensitivity**

Test procedure (Flavin 1981) based on two equations:

Euler equation \( \Rightarrow \) \( c_{t+1} = c_t + u_{t+1} \)

stochastic process for income \( \Rightarrow \) \( \Delta y_{t+1} = \mu + \lambda \Delta y_t + \varepsilon_{t+1} \)

According to theory:
realization of \( \varepsilon_{t+1} \) \( \rightarrow \) revision in expectations of future incomes \( \rightarrow \) change in current consumption \( c_{t+1} \)

\( c_{t+1} = c_t + \theta \varepsilon_{t+1} \)

where \( \theta \) measures the effect of income innovations on permanent income

\( \Rightarrow \) if reaction of \( c \) is larger than \( \theta \varepsilon \) : “excess sensitivity” of consumption to current income

Empirically:

\[ \Delta c_{t+1} = \beta \Delta y_{t+1} + \theta \varepsilon_{t+1} + v_{t+1} \]
$v_{t+1}$: effect on consumption due to news about $y^P$ not included in current income

According to theory: $\beta = 0$ (the signalling effect of current income is captured by $\theta \varepsilon$); if $\beta > 0$: *excess sensitivity* to current income ⇒ current consumption reacts also to *anticipated* changes in income.

Link with orthogonality test:

$$
\Delta c_{t+1} = \beta \mu + \beta \lambda \Delta y_t + (\theta + \beta) \varepsilon_{t+1} + v_{t+1}
$$

$\beta = 0$ implies $\alpha = 0$ in Hall’s test.

Notes:

- Flavin’s test more “powerful”: excess sensitivity can be detected even when $\alpha = 0$ in Hall’s test ($\Delta y_t$ not good predictor of $\Delta y_{t+1}$)
- Flavin’s test yields a quantitative measure of excess sensitivity to current income: $\beta$ (0.36 in original US estimates)

**Excess smoothness:**

Original insight of permanent income theory (in Friedman’s version):

⇒ consumption shows lower variability (is “smoother”) than current income because it is related to *permanent* income, presumably less volatile.

With rational expectations changes in permanent income are precisely related to innovations in current income → importance of *persistence* in the stochastic process generating income.

Empirically:

(aggregate) income is a *non-stationary* variable: an innovation in income at time $t$ has permanent effect on the level of $y$

⇒ implication: permanent income and consumption should display greater (not smaller) variability than current income.

*Example:* let income follow the stochastic process (AR(2)):

$$
\Delta y_{t+1} = \mu + \lambda \Delta y_t + \varepsilon_{t+1}
$$

⇒ $y_{t+1} = \mu + (1 + \lambda) y_t - \lambda y_{t-1} + \varepsilon_{t+1}$
For a general stochastic ARMA (autoregressive moving average) process of the form

$$a(L) y_{t+1} = \mu + b(L) \varepsilon_{t+1}$$

where $a(L)$ and $b(L)$ are polynomial in the lag operator ($L^i x_t = x_{t-i}$):

$$a(L) = a_0 + a_1 L + a_2 L^2 + ...$$
$$b(L) = b_0 + b_1 L + b_2 L^2 + ...$$

the following property holds:

$$\frac{r}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i (E_{t+1} - E_t) y_{t+1+i} = \frac{r}{1 + r} \sum_{i=0}^{\infty} \left( \frac{1}{1 + r} \right)^i b_i \varepsilon_{t+1}$$

For the above income process:

$$a(L) = 1 - (1 + \lambda)L + \lambda L^2$$
$$b(L) = 1$$

$$\Delta c_{t+1} = y_{t+1}^P - y_t^P = \frac{r}{1 + r} \frac{1}{1 + \lambda} \left( 1 + \frac{1}{1 + \lambda} \right) \varepsilon_{t+1}$$

$$\Rightarrow \Delta c_{t+1} = \frac{1 + r}{1 + r - \lambda} \varepsilon_{t+1}$$

$$\Rightarrow \text{if } \lambda > 0 \text{ then } \Delta y_{t+1}^P = \Delta c_{t+1} > \varepsilon_{t+1} \text{ and}$$

$$\frac{\sigma_{\Delta c}}{\sigma_{\Delta c}} = \frac{1 + r}{1 + r - \lambda} \sigma_\varepsilon > \sigma_\varepsilon \quad \text{consumption variability}$$

$$\text{current income variability}$$

but in the data (US) typically $\sigma_{\Delta c} < \sigma_\varepsilon \Rightarrow \text{excess smoothness}$