PhD Economics
Dynamic Macroeconomics

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Notes on: Dynamic models of consumption

General references:

Romer (2012) *Advanced Macroeconomics*, fourth edition, ch. 8

Bagliano F.C.-Bertola G. (2007) *Models for dynamic macroeconomics,* ch. 1

Blanchard O.J. and S. Fischer (1989) *Lectures on Macroeconomics*, sections 6.2 and 10.1

Specific references:

Deaton A. (1992) *Understanding consumption,* especially ch. 3, 4

Attanasio O. (1999) "Consumption demand", in *Handbook of Macroeconomics*, vol. 1B, ch. 11

Campbell J. (1999) "Asset prices, consumption, and the business cycle", in *Handbook of Macroeconomics*, vol. 1C, ch. 19

Carroll C. (2001) "A theory of the consumption function, with and without liquidity constraints", *Journal of Economic Perspectives*, vol. 15(3) ("graduate students version" NBER wp 8387)

Angeletos et al. (2001) "The hyperbolic consumption model: calibration, simulation and empirical evaluation", *Journal of Economic Perspectives*, vol. 15(3)

Meghir C. (2004) "A retrospective on Friedman's theory of permanent income", *Economic Journal*, vol. 114, June

Attanasio O. and G. Weber (2010) "Consumption and saving: models of intertemporal allocation snd their implications for public policy", *Journal of Economic Literature*, vol. 48(3).

Aims:

1. analysis of the consumption/saving choice for forward-looking, rational agents in a dynamic, stochastic environment;
2. application of dynamic optimization methods in discrete time.

Topics:

1. optimal consumption choice in the basic dynamic model:
   - basic problem set-up and assumptions;
   - solution and optimal dynamics of consumption/saving;
   - relationship between income, consumption and saving.
2. empirical issues:
   - *excess sensitivity* to anticipated income changes;
   - *excess smoothness* to unanticipated income changes.
3. extensions:
   - A: role of precautionary savings;
   - B: joint consumption/asset allocation choices (with stochastic interest rates) $\rightarrow$ CCAPM model.

1. Intertemporal optimization problem: set-up and solution

Basic framework:

infinite-horizon "representative agent" in an uncertain environment:

- maximizing an intertemporal utility function
- forming rational expectations on future values of relevant variables

At time t the agent

$$\max_{\{c_{t+i};\, i=0,1,\ldots\}} U(c_t, c_{t+1}, \ldots) \equiv U_t$$

subject to the budget constraint (with $i = 0, \ldots, \infty$)

$$A_{t+i+1} = (1 + r_{t+i})A_{t+i} + y_{t+i} - c_{t+i},$$

A_{t+i} : financial wealth at the beginning of period $t + i$
y_{t+i} and c_{t+i} : *labor* income and consumption at the end of period $t + i$ (timing convention)
r_{t+i} : real rate of return of financial assets in period $t + i$, assumed constant: $r_{t+i} = r$

Assumptions on preferences:

- *intertemporal* (time) *separability:*

  $$U\ (c_t, c_{t+1}, ...) = v_t\,(c_t) + v_{t+1}\,(c_{t+1}) + ...$$

  where $v_{t+i}\,(c_{t+i}) \equiv$ valuation in t of utility derived from consumption at $t+i$ (with $v' > 0$ and $v'' < 0$) $\Rightarrow$ "habit formation" and durable goods are ruled out

- future utility *discount* of the form

  $$v_{t+i}\,(c_{t+i}) = \left(\frac{1}{1+\rho}\right)^i u\,(c_{t+i})$$

  where $\rho > 0$ is the agent's rate of time preference
  $\Rightarrow$ no "dynamic inconsistency" of preferences

- *expected utility* as objective function (with uncertainty):

  $$U_t = E\left(\sum_{i=0}^{\infty} v_{t+i}\,(c_{t+i}) \mid I_t\right)$$

  where I_t is the information set available to the agent at time t
  $\Rightarrow$ jointly with time-separability, this assumption on U generates an inverse relationship between the *elasticity of intertemporal substitution* and the *degree of risk aversion (*two theoretically different concepts)

The agent's problem becomes:

$$\max_{\{c_{t+i},\, i=0,1,\dots\}} U_t = E_t \left[\sum_{i=0}^{\infty} \left(\frac{1}{1+\rho} \right)^i u(c_{t+i}) \right]$$

subject to the budget constraint (with $i = 0, \dots, \infty$):

$$A_{t+i+1} = (1+r)A_{t+i} + y_{t+i} - c_{t+i}, \qquad A_t \quad \text{given}$$

$E_t[\cdot]$: *rational* expectation formed at t on the information set I_t (including y_t)

> Note: given a generic variable x_{t+i} for which $E_t x_{t+i} = E(x_{t+i} \mid I_t)$, the rational expectations assumption implies that $E_t\left(x_{t+i} - E(x_{t+i} \mid I_t)\right) = 0$ (the forecast error is uncorrelated with variables in the information set I_t)

The one-period budget constraint can be used to derive the *intertemporal budget constraint* (by repeated forward substitution of A_{t+i}):

$$\frac{1}{1+r} \sum_{i=0}^{j-1} \left(\frac{1}{1+r} \right)^i c_{t+i} + \left(\frac{1}{1+r} \right)^j A_{t+j} = \frac{1}{1+r} \sum_{i=0}^{j-1} \left(\frac{1}{1+r} \right)^i y_{t+i} + A_t$$

A is allowed to be negative (no liquidity constraints), but debt cannot grow at a rate larger than r (*no Ponzi-game condition*):

$$\lim_{j \to \infty} \left(\frac{1}{1+r} \right)^j A_{t+j} \geq 0$$

Therefore, letting $j \to \infty$:

$$\underbrace{\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i c_{t+i}}_{\text{present value of consumption flows}} = \underbrace{\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i y_{t+i}}_{\text{present value of labor income flows}} + \underbrace{A_t}_{\text{initial financial wealth}}$$

The *intertemporal budget constraint* must hold also in expectation:

$$\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i E_t\, c_{t+i} = \underbrace{\frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r} \right)^i E_t\, y_{t+i}}_{\text{human wealth}} + A_t \qquad (*)$$

$$= \quad H_t \quad + \quad A_t$$

Solution:

from the maximization (given A_t and the terminal condition on financial wealth):

$$\max_{A_{t+i}} U_t = E_t \sum_{i=0}^{\infty} \left(\frac{1}{1+\rho}\right)^i u\left((1+r)A_{t+i} - A_{t+i+1} + y_{t+i}\right)$$

$$\Rightarrow \quad \text{f.o.c.} \quad E_t\, u'(c_{t+i}) = \frac{1+r}{1+\rho} E_t\, u'(c_{t+i+1})$$

for $i = 0$:

$$u'(c_t) = \frac{1+r}{1+\rho} E_t u'(c_{t+1}) \qquad \text{Euler equation}$$

$\rightarrow$ optimal dynamic path for *marginal utility* of consumption. Given $u'' < 0$ this implies:

$$\begin{array}{lcll} c_{t+1} & > & c_t & \text{if} \quad r > \rho \\ c_{t+1} & < & c_t & \text{if} \quad r < \rho \\ c_{t+1} & = & c_t & \text{if} \quad r = \rho \end{array}$$

By how much consumption changes in response to $r - \rho$ ($\Rightarrow$ *intertemporal substitution*) depends on the shape of the marginal utility function, measured by $-\frac{u''(c)}{u'(c)}$

example (with certainty): CRRA (constant relative risk aversion) utility function

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} \qquad \gamma \equiv -\frac{cu''(c)}{u'(c)} > 0$$

for which $u'(c) = c^{-\gamma}$ and γ is the coefficient of relative *risk aversion*. The Euler equation is:

$$c_t^{-\gamma} = \frac{1+r}{1+\rho} c_{t+1}^{-\gamma} \quad \Rightarrow \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \frac{1+r}{1+\rho}$$

Taking logs and using $\log(1+r) \simeq r$ and $\log(1+\rho) \simeq \rho$:

$$\Delta \log c_{t+1} = \frac{1}{\gamma}(r - \rho)$$

γ : (relative) *risk aversion* $\quad \Leftrightarrow \quad$ $\frac{1}{\gamma}$: (elasticity of) *intertemp. substitution*

Level and dynamics of optimal consumption:

defining

$$u'(c_{t+1}) - E_t u'(c_{t+1}) \equiv \eta_{t+1}$$

we get:

$$u'(c_{t+1}) = \frac{1+\rho}{1+r} u'(c_t) + \eta_{t+1}.$$

Assuming $r = \rho$, the *stochastic process* governing the dynamics of *marginal utility is:*

$$u'(c_{t+1}) = u'(c_t) + \eta_{t+1}$$

with $E_t \eta_{t+1} = 0$ (by rational expectations)
To derive implications for the dynamics of *consumption* we assume
quadratic utility $\rightarrow$ *linear marginal utility:*

$$u(c) = c - (b/2)c^2 \quad \rightarrow \quad u'(c) = 1 - bc$$

obtaining the **random walk** model for consumption (Hall 1978):

$$c_{t+1} = c_t + u_{t+1} \quad \Rightarrow \quad E_t c_{t+1} = c_t$$

with $u_{t+1} \equiv -\frac{1}{b}\eta_{t+1} \Rightarrow E_t u_{t+1} = 0$
$\Rightarrow$ the best forecast in t of consumption in $t+1$ is simply current consumption c_t (the change in consumption u_{t+1} is *orthogonal* to any variable in the information set used to form $E_t c_{t+1}$)
To derive the *consumption function* use the intertemporal budget constraint (*) and note that

$$E_t c_{t+1} = E_t c_{t+2} = \ldots = E_t c_{t+i} = \ldots = c_t$$

$$\Rightarrow \quad \frac{1}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i E_t \, c_{t+i} = \frac{1}{r} c_t$$

$$\Rightarrow \quad c_t = \underbrace{r\ (H_t + A_t)}_{\text{permanent income}} \equiv y_t^P$$

$r(H_t + A_t)$ is the return on total wealth (*annuity value*), i.e. "permanent income"
$\rightarrow$ current consumption is equal to permanent income

To give economic content to the (unforecastable) change in consumption u_{t+1}, note:

$$u_{t+1} = c_{t+1} - c_t = c_{t+1} - E_t\, c_{t+1} = y^P_{t+1} - E_t\, y^P_{t+1}$$

$$y^P_{t+1} - E_t\, y^P_{t+1} = r\,(H_{t+1} - E_t\, H_{t+1}) + r\,\underbrace{(A_{t+1} - E_t\, A_{t+1})}_{=0}$$

taking expectations of the one-period budget constraint in t :

$$E_t\, A_{t+1} = (1+r)E_t\, A_t + E_t\, y_t - E_t\, c_t = (1+r)\, A_t + y_t - c_t = A_{t+1}$$

To construct the "surprise" in human wealth, using $E_t\, E_{t+1}(.) = E_t\,(.)$:

$$\begin{aligned} H_{t+1} - E_t\, H_{t+1} &= \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_{t+1}\, y_{t+1+i} - \frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t\,(E_{t+1}\, y_{t+1+i}) \\ &= \underbrace{\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i (E_{t+1} - E_t)\; y_{t+1+i}}_{\text{present value of revision in expectations of future labor incomes}} \end{aligned}$$

Therefore:

$$y^P_{t+1} = y^P_t + \underbrace{r\,\frac{1}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i (E_{t+1} - E_t)\; y_{t+1+i}}_{u_{t+1}} \qquad (**)$$

$$\Rightarrow c_{t+1} = c_t + u_{t+1}$$

Optimal savings:

optimal consumption choice has implications for savings and financial wealth accumulation

Define *disposable income*:

$$y^D_t = r\, A_t + y_t$$

and *saving* (using optimal consumption choice):

$$s_t \equiv y^D_t - c_t = \underbrace{y^D_t - y^P_t}_{\text{transitory income}} = y_t - r\, H_t$$

$\rightarrow$ financial wealth A is increased (decreased) when current income is higher (lower) than the annuity value of human wealth: *buffer* role for financial assets

$$\begin{aligned}
\Rightarrow s_t &= y_t - \frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t y_{t+i} \\
&= \frac{1}{1+r}y_t - \left(\frac{1}{1+r} - \left(\frac{1}{1+r}\right)^2\right) E_t y_{t+1} - \\
&\quad - \left(\left(\frac{1}{1+r}\right)^2 - \left(\frac{1}{1+r}\right)^3\right) E_t y_{t+2} + ... \\
&= -\sum_{i=1}^{\infty}\left(\frac{1}{1+r}\right)^i E_t \Delta y_{t+i} \qquad \text{where} \quad \Delta y_{t+i} = y_{t+i} - y_{t+i-1}
\end{aligned}$$

$\rightarrow$ "*saving for a rainy day*" behavior: financial asset accumulation when labor income is expected to fall

Consumption, savings and current income:

theory predicts:

$$y_t \quad \underset{\substack{\uparrow \\ \text{depends on} \\ \text{revisions of future} \\ \text{expected labor incomes}}}{\longrightarrow} \quad y_t^P \quad \rightarrow \quad c_t$$

To derive the link between current income and consumption assume a *stochastic process* generating labor income (first-order autoregression):

$$y_{t+1} = \lambda y_t + (1-\lambda)\overline{y} + \varepsilon_{t+1}, \qquad E_t\varepsilon_{t+1} = 0$$

where $0 \leq \lambda \leq 1$ and $\overline{y}$ is the unconditional mean of the process. Realizations of the "innovation" component ε_{t+1} at $t+1$ cause revisions in expected values of future labor incomes; e.g. for y_{t+2}:

$$\begin{aligned}
E_{t+1}y_{t+2} &= \lambda y_{t+1} + (1-\lambda)\bar{y} \\
&= \lambda\left[\lambda y_t + (1-\lambda)\overline{y} + \varepsilon_{t+1}\right] + (1-\lambda)\bar{y} \\
&= \lambda^2 y_t + (1+\lambda)(1-\lambda)\bar{y} + \lambda\varepsilon_{t+1}
\end{aligned}$$

$$E_t y_{t+2} = \lambda^2 y_t + (1+\lambda)(1-\lambda)\bar{y}$$

$$\Rightarrow E_{t+1}y_{t+2} - E_t y_{t+2} = \lambda\varepsilon_{t+1}$$

In general:

$$E_{t+1}y_{t+1+i} - E_t y_{t+1+i} = \lambda^i\varepsilon_{t+1} \qquad \forall i \geq 0$$

Using these revisions in expected future incomes in (**) yields the change in consumption:

$$c_{t+1} = c_t + \left(\frac{r}{1+r-\lambda}\right)\varepsilon_{t+1}$$

Given an innovation in current income ε_{t+1}, permanent income and consumption change by $\left(\frac{r}{1+r-\lambda}\right)\varepsilon_{t+1}$. Transitory income amounts to

$$\varepsilon_{t+1} - \frac{r}{1+r-\lambda}\varepsilon_{t+1} = \frac{1-\lambda}{1+r-\lambda}\varepsilon_{t+1}$$

which is saved and added to the existing financial wealth.

The magnitude of the response of consumption to an innovation in income depends crucially on the degree of *persistence* of the stochastic process for y, captured by λ: e.g.

- $\lambda = 0$ (no persistence): innovation is entirely transitory. Human wealth -evaluated at the beginning of period $t+1$- changes by $\frac{1}{1+r}\varepsilon_{t+1}$and

$$c_{t+1} = c_t + \left(\frac{r}{1+r}\right)\varepsilon_{t+1}$$

- $\lambda = 1$: innovation has permanent effects on income (random walk) $y_{t+1} = y_t + \varepsilon_{t+1}$. Human wealth changes by $\frac{\varepsilon_{t+1}}{r}$ and

$$c_{t+1} = c_t + \varepsilon_{t+1}$$

2. Empirical issues

Basic implication of rational expectations/permanent income theory (with quadratic utility):

$\rightarrow$ change in consumption orthogonal to variables in the agents' information set: e.g. in equation

$$\Delta c_{t+1} = \alpha \Delta y_t + e_{t+1}$$

$$\text{theory implies} \quad : \quad \alpha = 0$$

if past incomes are in the agents' information set at time t : *orthogonality test* (Hall 1978)

More recently, two main lines of empirical research:

(a) test if consumption reacts to changes in current income as predicted by theory: $\rightarrow$ *excess sensitivity*

(b) test if consumption reacts to *innovations* (i.e. unanticipated changes) in income as predicted by theory: $\rightarrow$ *excess smoothness*

Excess sensitivity

Test procedure (Flavin 1981) based on two equations:

$$\text{Euler equation} \quad \Rightarrow \quad c_{t+1} = c_t + u_{t+1}$$

$$\text{stochastic process for income} \quad \Rightarrow \quad \Delta\, y_{t+1} = \mu + \lambda\, \Delta\, y_t + \varepsilon_{t+1}$$

According to theory:
realization of ε_{t+1} $\rightarrow$ revision in expectations of future incomes $\rightarrow$ change in current consumption c_{t+1}

$$c_{t+1} = c_t + \theta\, \varepsilon_{t+1}$$

where θ measures the effect of income innovations on permanent income
$\Rightarrow$ if reaction of c is larger than $\theta\varepsilon$: "*excess sensitivity*" of consumption to current income

Empirically:

$$\Delta\, c_{t+1} = \beta\, \Delta\, y_{t+1} + \theta\, \varepsilon_{t+1} + v_{t+1}$$

v_{t+1} : effect on consumption due to news about y^P not included in current income
According to theory: $\beta = 0$ (the signalling effect of current income is captured by $\theta\varepsilon$); if $\beta > 0$: *excess sensitivity* to current income $\Rightarrow$ current consumption reacts also to *anticipated* changes in income
Link with orthogonality test:

$$\Delta\, c_{t+1} \;=\; \beta\,\mu + \beta\,\lambda\,\Delta\, y_t + (\theta + \beta)\,\varepsilon_{t+1} + v_{t+1}$$

$$\beta \;=\; 0 \quad \text{implies} \quad \alpha = 0 \text{ in Hall's test}$$

Notes:

- Flavin's test more "powerful": excess sensitivity can be detected even when $\alpha = 0$ in Hall's test ($\Delta\, y_t$ not good predictor of $\Delta\, y_{t+1}$)
- Flavin's test yields a quantitative measure of excess sensitivity to current income: β (0.36 in original US estimates)

Excess smoothness:

Original insight of permanent income theory (in Friedman's version):
$\Rightarrow$ consumption shows lower variability (is "smoother") than current income because it is related to *permanent* income, presumably less volatile
With rational expectations changes in permanent income are precisely related to innovations in current income $\rightarrow$ importance of *persistence* in the stochastic process generating income

Empirically:

> (aggregate) income is a *non-stationary* variable: an innovation in income at time t has permanent effect on the level of y

$\Rightarrow$ implication: permanent income and consumption should display *greater* (not smaller) variability than current income

Example: let income follow the stochastic process (AR(2)):

$$\Delta\, y_{t+1} \;=\; \mu + \lambda\,\Delta\, y_t + \varepsilon_{t+1}$$

$$\Rightarrow y_{t+1} \;=\; \mu + (1+\lambda) y_t - \lambda y_{t-1} + \varepsilon_{t+1}$$

For a general stochastic ARMA (autoregressive moving average) process of the form

$$a(L)\, y_{t+1} = \mu + b(L)\, \varepsilon_{t+1}$$

where $a(L)$ and $b(L)$ are polynomial in the lag operator ($L^i x_t = x_{t-i}$):

$$\begin{aligned} a(L) &= a_0 + a_1 L + a_2 L^2 + ... \\ b(L) &= b_0 + b_1 L + b_2 L^2 + ... \end{aligned}$$

the following property holds:

$$\underbrace{\frac{r}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i (E_{t+1} - E_t)\; y_{t+1+i}}_{\text{change in permanent income due to } \varepsilon_{t+1}} = \frac{r}{1+r} \frac{\sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i b_i}{\sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i a_i} \varepsilon_{t+1}$$

For the above income process:

$$\begin{aligned} a(L) &= 1 - (1+\lambda)L + \lambda L^2 \\ b(L) &= 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow \Delta\, c_{t+1} &= y^P_{t+1} - y^P_t = \frac{r}{1+r} \frac{1}{1 - \frac{1}{1+r}(1+\lambda) + \left(\frac{1}{1+r}\right)^2 \lambda} \varepsilon_{t+1} \\ \Rightarrow \Delta\, c_{t+1} &= \frac{1+r}{1+r-\lambda} \varepsilon_{t+1} \end{aligned}$$

$\Rightarrow$ if $\lambda > 0$ then $\Delta\, y^P_{t+1} = \Delta\, c_{\,t+1} > \varepsilon_{t+1}$ and

$$\underset{\substack{\text{consumption} \\ \text{variability}}}{\sigma_{\Delta c}} = \frac{1+r}{1+r-\lambda} \sigma_\varepsilon > \underset{\substack{\text{current income} \\ \text{variability}}}{\sigma_\varepsilon}$$

but in the data (US) typically $\sigma_{\Delta c} < \sigma_\varepsilon \Rightarrow$ *excess smoothness*